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THE ARCSINE LAW IN THE ANALOGUE OF WIENER SPACE

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Abstract. In this note, we prove the theorems in analogue of Wiener space corresponding to the second and the third arcsine laws in the concrete Wiener space [1, 2] and we show that our results are exactly same to the concrete Wiener case when the initial condition gives the Dirac measure at the origin.

1. Introduction

The study of Brownian motion started about two hundred years ago, and the study to Wiener measure space which based on Brownian motion in mathematical way started around one hundred years ago. Wiener measure space has been studied widely and profoundly by many mathematicians and mathematical physical scientists. In 1939, Levy proved a beautiful Theorem, say the first arcsine law in the concrete Wiener space[3], that proportion of time $m_{\delta_0}(T_+(x) \leq t) = \frac{2}{\pi}\arcsin\sqrt{t}$ for $t \in [0,1]$, where $T_+(x) = m_L(\{s \in [0,1] | x(s) > 0\})$. Since then, one proved the second and the third arcsine laws in the concrete Wiener space that $m_{\delta_0}(L(x) \leq t) = m_{\delta_0}(M_1(x) \leq t) = \frac{2}{\pi}\arcsin\sqrt{t}$ for $t \in [0,1]$ where $L(x) = \sup_{x(s)=0} s$ and $M_1(x) = \sup_{0 \leq s \leq 1} x(s)$[1, 2, 4]. In 2002, the author and Dr.Im presented the definition of the analogue of Wiener space, a kind of the generalization of the concrete Wiener space, and its properties[5].

In this note, we will prove the theorems in the analogue of Wiener space, corresponding to the second and the third arcsine laws in the concrete Wiener space.

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2. The definitions and the basics properties of analogue of Wiener space

In this section, we introduce some definitions of the analogue of Wiener space and investigate the basic properties of it which are needed to understand the next section.

Throughout in this note, let T be a positive real number, let $C[0,T]$ be the space of all continuous functions on a closed interval $[0,T]$ with the supremum norm $||x||_\infty = \sup_{t\in[0,T]}|x(t)|$ and let ϕ be a probability Borel measure on $\mathbb{R}$. We define the analogue of Wiener measure m_ϕ on $C[0,T]$ as follows. Let $\vec{t} = (t_0, t_1, t_2, \cdots, t_n)$ with $0 = t_0 < t_1 < t_2 < \cdots < t_n \le T$. Let $J_{\vec{t}} : C[0,T] \to \mathbb{R}^{n+1}$ be a function with $J_{\vec{t}}(x) = (x(t_0), x(t_1), x(t_2), \cdots, x(t_n))$ and let B_j $(j = 0,1,2,\cdots,n)$ be in $\mathcal{B}(\mathbb{R})$. The subsets $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called an interval and let $\mathcal{M}$ be the smallest σ-algebra contains all intervals. For an interval $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$, let $\omega_\phi(J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)) = \int_{B_0}\int_{B_1}\cdots\int_{B_n} W(n+1;\vec{t};u_0,u_1,\cdots,u_n)du_{n+1}du_n\cdots du_1 d\phi(u_0)$ where $W(n+1;\vec{t};u_0,u_1,\cdots,u_n)$ $= (\prod_{j=1}^n \frac{1}{\sqrt{2\pi(t_j-t_{j-1})}})\exp(-\frac{1}{2}\sum_{j=1}^n \frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}})$. Let m_ϕ be the Borel measure on $C[0,T]$ such that for all I, $\omega_\phi(I) = m_\phi(I)$, this measure is called the analogue of Wiener measure on $C[0,T]$.

When ϕ is a Dirac measure δ_0 at the origon in $\mathbb{R}$, m_ϕ is the concrete Wiener measure.

From [5], we can find the following lemma.

Lemma 2.1. *Under the notations in above, if $f : \mathbb{R}^{n+1}\to\mathbb{R}$ is a Borel measureable fuction then the following equality holds.*

$$\int_{C[0,T]} f(x(t_0), x(t_1), x(t_2), \cdots, x(t_n))dm_\phi(x)$$
$$= \int_{\mathbb{R}^{n+1}} f(u_0, u_1, u_2, \cdots, u_n)W(n+1;\vec{t};u_0,u_1,\cdots,u_n)$$
$$(((\prod_{j=1}^n m_L)\times\phi)((u_1,u_2,\cdots,u_n),u_0))$$

where if one side integral exists, the both sides integral exist and the two values are same.

Remark 2.2.

(1) *For a Borel subset B of $C[0,T]$, $m_\phi(B) = \int_\mathbb{R} m_{\delta_u}(B)d\phi(u)$.*
(2) *m_ϕ has no atoms.*

(3) *For* $0 \le s_1 \le s_2 \le s_3 \le s_4 \le T$, $x(s_2) - x(s_1)$ *and* $x(s_4) - x(s_3)$ *are stochastically independent.*

For x in $C[0,T]$, t in $[0,T]$ and a real number b, we let $T_b(x) = \inf_{x(t)=b} t$, $L(x) = \sup_{x(t)=0} t$, $M_t(x) = \sup_{0\le s\le t} x(s)$ and $\theta_t(x) = \sup_{x(s)=M_t(x)} s$. Here, T_b is called the first time hits b, L is called the last time of the last zero before T and M_t is called the running maximum.

A random variable X is said to have the arcsine distribution if it is supported on $[0,T]$ with the cumulative density function $F(t) = \frac{2}{\pi}\arcsin\sqrt{\frac{t}{T}}$(or probability density function $f(t) = F'(t) = \frac{2}{\pi\sqrt{t(T-t)}}$).

Lemma 2.3. (The reflection principle in the analogue of Wiener space) *For a real number* a *and* t *in* $(0,T]$,

$$m_\phi(x(t) < a) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\int_{2u_0-a}^{+\infty}\exp(-\frac{(u-u_0)^2}{2t})dud\phi(u_0).$$

Proof. By the change of variables theorem, we have

$$m_{\delta_{u_0}}(x(t) < a) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a}\exp(-\frac{(u-u_0)^2}{2t})du$$

$$= \frac{1}{\sqrt{2\pi t}}\int_{2u_0-a}^{+\infty}\exp(-\frac{(u-u_0)^2}{2t})du = m_{\delta_{u_0}}(x(t) > 2u_0 - a).$$

So, we obtain our equality. □

Remark 2.4. *If* ϕ *is symmetric, that is,* $\phi(B) = \phi(-B)$ *for all Borel subset* B *in* $\mathbb{R}$, $m_\phi(x(t) < a) = m_\phi(x(t) > -a)$.

Lemma 2.5. *For a real number* b *and* t *in* $(0,T]$,

$$m_\phi(T_b(x) < t) = \sqrt{\frac{2}{\pi}}\{\int_{-\infty}^{b}(\int_{(b-u_0)/\sqrt{t}}^{+\infty}\exp(-\frac{v^2}{2})dv)d\phi(u_0)$$

$$+\int_{b}^{+\infty}(\int_{-\infty}^{(b-u_0)/\sqrt{t}}\exp(-\frac{v^2}{2})dv)d\phi(u_0)\} + \phi(\{b\}).$$

Proof. When $b = u_0$, $m_{\delta_{u_0}}(T_b(x) < t) = m_{\delta_{u_0}}(x(0) = u_0) = 1$. If $b < u_0$, by the symmetry of Brownian motion and the intermediate value theorem, we have

$$\begin{aligned} m_{\delta_{u_0}}(T_b(x) < t) &= m_{\delta_{u_0}}(T_b(x) < t, x(t) > b) + m_{\delta_{u_0}}(T_b(x) < t, x(t) < b) \\ &= 2m_{\delta_{u_0}}(T_b(x) < t, x(t) < b) \\ &= 2m_{\delta_{u_0}}(x(t) < b) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{(b-u_0)/\sqrt{t}} \exp(-\frac{v^2}{2})dv. \end{aligned}$$

By the essentially similar method in above, if $b > u_0$,

$$m_{\delta_{u_0}}(T_b(x) < t) = \sqrt{\frac{2}{\pi}} \int_{(b-u_0)/\sqrt{t}}^{+\infty} \exp(-\frac{v^2}{2})dv.$$

Hence, we have our conclusion. □

Remark 2.6.

(1) *When* $b \neq u_0$, $\frac{\partial}{\partial b} m_{\delta_{u_0}}(T_b(x) < t) = sgn(u_0 - b)\sqrt{\frac{2}{\pi t}} \exp\left(-\frac{(b-u_0)^2}{2t}\right)$.

(2) $\frac{\partial}{\partial t} m_{\delta_{u_0}}(T_b(x) < t) = \frac{1}{\sqrt{2\pi t^3}} |u_0 - b| \exp(-\frac{(b-u_0)^2}{2t})$ *for* t *in* $(0, T]$. *So,* $m_{\delta_{u_0}}(T_b(x) > t) = \int_t^{+\infty} \frac{1}{\sqrt{2\pi s^3}} |u_0 - b| \exp(-\frac{(b-u_0)^2}{2s})ds$.

(3) *If* $\phi = \delta_0$ *then* $m_\phi(T_b(x) < t) = \sqrt{\frac{2}{\pi}} \int_{b/\sqrt{t}}^{+\infty} \exp(-\frac{v^2}{2})dv$, $\frac{\partial}{\partial t} m_\phi(T_b(x) < t) = \frac{|b|}{\sqrt{2\pi t^3}} \exp(-\frac{b^2}{2t})$. *This is exactly same to the results in the concrete Wiener case.*

Form [1, 2], we know that for $b > 0$, $0 < t < T$ and a Borel subset B in $\mathbb{R}$,

$$m_{\delta_0}(T_b(x) \leq t, x(t) \in B) = \frac{1}{\sqrt{2\pi t}} \int_{2b-B} \exp(-\frac{v^2}{2t})dv.$$

Hence, if $u_0 \leq b$, we have

$$m_{\delta_0}(T_b(x) \leq t, x(t) \leq a) = \frac{1}{\sqrt{2\pi t}} \int_{2b-a}^{+\infty} \exp(-\frac{(u-u_0)^2}{2t})du.$$

Lemma 2.7. *Let* a *and* b *be two real numbers with* $a \leq b$ *and* $b > 0$. *Then* $m_\phi(M_t(x) \geq b, x(t) < a) = \frac{1}{\sqrt{2\pi t}} \{ \int_b^{+\infty} \int_{-\infty}^a \exp(-\frac{(u-u_0)^2}{2t}) du d\phi(u_o) + \int_{-\infty}^b \int_{2b-a}^{+\infty} \exp(-\frac{(u-u_0)^2}{2t}) du d\phi(u_o)\}$ *and* $m_\phi(M_t(x) \leq b, x(t) < a) = \frac{1}{\sqrt{2\pi t}} \{ \int_{-\infty}^b \int_{-\infty}^a \exp(-\frac{(u-u_0)^2}{2t}) du d\phi(u_o) - \int_{-\infty}^b \int_{2b-a}^{+\infty} \exp(-\frac{(u-u_0)^2}{2t}) du d\phi(u_o)\}$.

Proof. If $u_0 \geq b$, by the intermediate value theorem, $m_{\delta_{u_0}}(M_t(x) \geq b, x(t) < a) = m_{\delta_{u_0}}(x(t) < a)$. If $u_0 < b$, $m_{\delta_{u_0}}(M_t(x) \leq b, x(t) < a) = m_{\delta_{u_0}}(T_b(x) \leq b, x(t) < a)$. So, we obtain our equality. □

Remark 2.8.

(1) $m_\phi(M_t(x) \leq b, x(t) < a) = m_\phi(x(t) < a) - m_\phi(M_t(x) \geq b, x(t) > a) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{b}\{\int_{-\infty}^{a} \exp(-\frac{(u-u_0)^2}{2t})du\}d\phi(u_0) - \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a}\{\int_{2b-a}^{+\infty} \exp(-\frac{(u-u_0)^2}{2t})du\}\ d\phi(u_0)$.

(2) *For* $b > 0$, $m_{\delta_0}(M_t(x) \leq b, x(t) < a) = \frac{1}{\sqrt{2\pi t}}(\int_{-\infty}^{a} \exp(-\frac{u^2}{2t})du - \int_{2b-a}^{+\infty} \exp(-\frac{u^2}{2t})du)$. *So,* $\frac{\partial}{\partial b} m_{\delta_0}(M_t(x) \leq b, x(t) < a) = \sqrt{\frac{2}{\pi t}} \exp(-\frac{(2b-a)^2}{2t})$ *and* $\frac{\partial^2}{\partial b \partial a} m_{\delta_0}(M_t(x) \leq b, x(t) < a) = \frac{\sqrt{2}(2b-a)}{\sqrt{\pi t^3}} \exp(-\frac{(2b-a)^2}{2t})$. *This is exactly same to the result in the concrete Wiener case.*

(3) *If* $u_0 > b$, *then* $m_{\delta_{u_0}}(M_t(x) \leq b, x(t) < a) = 0$. *So,* $\frac{\partial^2}{\partial b \partial a} m_{\delta_{u_0}}(M_t(x) \leq b, x(t) < a) = 0$ *and if* $u_0 \leq b$ *then from Remark(2) in above,* $\frac{\partial^2}{\partial b \partial a} m_{\delta_{u_0}}(M_t(x) \leq b, x(t) < a) = \frac{\sqrt{2}(2b-a-u_0)}{\sqrt{\pi t^3}} \exp(-\frac{(2b-a-u_0)^2}{2t})$.

(4) *If* $u_0 > b$, *then* $m_{\delta_{u_0}}(M_t(x) > b, x(t) < a) = m_{\delta_{u_0}}(x(t) < a)$. *So,* $\frac{\partial^2}{\partial b \partial a} m_{\delta_{u_0}}(M_t(x) > b, x(t) < a) = 0$ *and if* $u_0 \leq b$ *then* $\frac{\partial^2}{\partial b \partial a} m_{\delta_{u_0}}(M_t(x) > b, x(t) < a) = \frac{\partial^2}{\partial b \partial a} m_{\delta_{u_0}}(M_t(x) > b, M_t(x) - x(t) > b - a) = \frac{\sqrt{2}(a+b-u_0)}{\sqrt{\pi t^3}} \exp(-\frac{(a+b-u_0)^2}{2t})$.

3. The second and the third arcsine laws in analogue of Wiener space

In this section, we prove the second and the third arcsine laws in analogue of Wiener space which are main theorems in this notes.

Let $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp(-\frac{u^2}{2})du$. Then the following two equalities are known facts :

$$\int_0^{+\infty} \exp(-\frac{(u+\alpha)^2}{2\sigma^2})du = \sqrt{2\pi}\sigma\Phi(-\frac{\alpha}{\sigma})$$

and

$$\int_0^{+\infty} u\exp(-\frac{(u+\alpha)^2}{2\sigma^2}du = \sigma^2\exp(-\frac{\alpha^2}{2\sigma^2}) - \sqrt{2\pi}\alpha\sigma\Phi(-\frac{\alpha}{\sigma}).$$

THEOREM 3.1. (The Second Arcsine Laws in Analogue of Wiener Space) *Let* $\alpha = \frac{tu_0}{s+t}$ *and* $\sigma = \sqrt{\frac{st}{s+t}}$. *For* $0 < s < T$,

$$m_\phi(L(x) \le s) = \int_{-\infty}^{+\infty}\int_{T-s}^{+\infty} \frac{1}{\pi\sqrt{st^3}}(\sigma^2 \exp(-\frac{\alpha^2}{2\sigma^2}) + \alpha\sigma\int_0^{\alpha/\sigma} \exp(-\frac{u^2}{2})du)$$
$$\exp(-\frac{{u_0}^2}{2(s+t)})dtd\phi(u_0).$$

Proof. From Remark (3) in Lemma 2.3, we have

$$m_{\delta_u}(T_0(x) > t) = \frac{1}{\sqrt{2\pi t^3}}\int_t^{+\infty} \frac{1}{\sqrt{2\pi s^3}}|u|\exp(-\frac{u^2}{2s})ds$$

.

Hence, by the Fubini Theorem, for u_0 in $\mathbb{R}$,

$$\begin{aligned} m_{\delta_{u_0}}(L(x) \le s) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi s}}\exp(-\frac{(u_1-u_0)^2}{2s})m_{\delta_{u_1}}(T_0(x) > T-s)du_1 \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi s}}\exp(-\frac{(u_1-u_0)^2}{2s})\{\int_{T-s}^{+\infty} \frac{1}{\sqrt{2\pi t^3}}|u_1|\exp(-\frac{{u_1}^2}{2t})dt\}du_1 \\ &= \int_{T-s}^{+\infty} \frac{1}{\pi\sqrt{st^3}}(\int_{-\infty}^{+\infty} |u_1|\exp(-\frac{u_1-\alpha^2}{2\sigma})\exp(-\frac{{u_0}^2}{2(s+t)})du_1)dt. \end{aligned}$$

Using the equality in above of this theorem,

$$m_{\delta_{u_0}}(L(x) \le s) = \int_{T-s}^{+\infty} \frac{1}{2\pi\sqrt{st^3}}(\int_{-\infty}^{+\infty} (\sigma^2 \exp(-\frac{\alpha^2}{2\sigma^2}) + \alpha\sigma\int_0^{\alpha/\sigma} \exp(-\frac{u^2}{2})$$
$$du)\exp(-\frac{{u_0}^2}{2(s+t)})dt.$$

Therefore, we have our equality. □

Remark. If $\phi = \delta_0$, then putting $u = \frac{s}{t+s}$,

$$m_\phi(L(x) \le s) = \int_{T-s}^{+\infty} \frac{\sigma^2}{\pi\sqrt{st^3}} = \frac{\sqrt{s}}{\pi}\int_{T-s}^{+\infty} \frac{1}{\sqrt{t}(s+t)}dt = \frac{1}{\pi}\int_0^{s/T} \frac{1}{\sqrt{u(1-u)}}$$
$$du = \frac{2}{\pi}\arcsin\sqrt{\frac{s}{T}}.$$

This is exactly same to the results in concrete Wiener case.

THEOREM 3.2. *(The Third Arcsine Laws in Analogue of Wiener Space) Let* $0 < s < t < T$ *with* $t + s < T$. *Then*

$$m_\phi(\theta_t(x) \le s) = \int_{-\infty}^{+\infty} (\int_0^s \frac{1}{\pi\sqrt{r(t-r)}}\exp(-\frac{{u_0}^2}{2r})dr)d\phi(u_0).$$

Proof. Let $X_t(x) = x(t+s) - x(s)$ and $N_u(x) = \max_{0\le v\le u} X(v)$ for $0 < u \le T$. Then

$$N_{t-s}(x) = (\max_{0\le u\le t-s} x(u+s) - x(s)) = (\max_{s\le u\le t} x(u)) - x(s)$$

and the following are equivalent :

$$\text{(a) } \theta_t(x) \le s \quad \text{(b)} M_s(x) = M_t(x) \quad \text{(c)} M_s(x) \ge N_{t-s}(x) + x(s).$$

Hence, for u_0 in $\mathbb{R}$, $b \ge 0$ and $0 < s < t < T$ with $t+s < T$, by Remark (3) in Lemma 2.1,

$$m_{\delta_{u_0}}(M_t(x) \le b, \theta_t(x) \le s) = m_{\delta_{u_0}}(M_s(x) \le b, M_s(x) - x(s) \ge N_{t-s}(x))$$
$$= m_{\delta_{u_0}}(M_s(x) - u_0 \le b - u_0, M_s(x) - x(s) > c, c \ge N_{t-s}(x))\frac{m_{\delta_{u_0}}(c > N_{t-s}(x))}{m_{\delta_{u_0}}(c > N_{t-s}(x))}$$
$$= m_{\delta_{u_0}}(M_s(x) \le b, M_s(x) - x(s) > c) m_{\delta_{u_0}}(c > N_{t-s}(x)).$$

Therefore, from Lemma 2.4 and the Fubini theorem,

$$m_{\delta_{u_0}}(M_t(x) \le b, \theta_t(x) \le s) = \int_0^{+\infty}[\int_0^{+\infty}\{\int_0^{+\infty}\frac{\partial^2}{\partial b\partial h} m_{\delta_{u_0}}(M_s(x) \le b, N_s(x)$$
$$- x(s) > h)\frac{\partial}{\partial c} m_{\delta_{u_0}}(c > N_{t-s}(x) dh\}dc]db = \int_0^{+\infty}[\int_0^{+\infty}\{\int_c^{+\infty}\frac{2(b+h-u_0)}{\pi\sqrt{s^3(t-s)}}$$
$$\exp(-\frac{(b+h-u_0)^2}{2s} - \frac{c^2}{2(t-s)})dh\}dc]db = \int_0^{+\infty}[\int_0^{+\infty}\frac{2}{\pi\sqrt{s(t-s)}}\exp$$
$$(-\frac{(b+c-u_0)^2}{2s} - \frac{c^2}{2(t-s)})dc]db = (\int_0^{+\infty}\frac{2}{\pi\sqrt{t}}\exp(-\frac{(u-u_0)^2}{2t})$$
$$du(\int_{(b-u_0)\sqrt{(t-s)/st}}^{+\infty}\exp(-\frac{u^2}{2})du)).$$

$$\text{So, } m_{\delta_{u_0}}(\theta_t(x) \le s) = \int_0^s(\int_0^{+\infty}\frac{\partial^2}{\partial b\partial r} m_{\delta_{u_0}}(M_t(x) \le b, \theta_t(x) \le r)db)dr$$
$$= \int_0^s(\int_0^{+\infty}\frac{(b-u_0)}{\pi\sqrt{(t-r)r^3}}\exp(-\frac{(b-u_0)^2}{2r})db)dr$$
$$= \int_0^s\frac{1}{\pi\sqrt{(t-r)r}}\exp(-\frac{{u_0}^2}{2r})dr.$$

Therefore, we obtain our equality □

REMARK 3.3. *If $\phi = \delta_0$, then*

$$m_\phi(\theta_t(x) \le s) = \int_0^s \frac{1}{\pi\sqrt{r(t-r)}}\exp(-\frac{{u_0}^2}{2r})dr = \frac{2}{\pi}\arcsin\sqrt{\frac{s}{t}}.$$

This is exactly same to the results in the concrete Wiener case.

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