JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **28**, No. 4, November 2015
http://dx.doi.org/10.14403/jcms.2015.28.4.603

G'_p-SPACES FOR MAPS AND HOMOLOGY DECOMPOSITIONS

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Abstract. For a map $p : X \to A$, we define and study a concept of G'_p-space for a map, which is a generalized one of a G'-space. Any G'-space is a G'_p-space, but the converse does not hold. In fact, CP^2 is a G'_δ-space, but not a G'-space. It is shown that X is a G'_p-space if and only if $G^n(X, p, A) = H^n(X)$ for all n. We also obtain some results about G'_p-spaces and homology decompositions for spaces. As a corollary, we can obtain a dual result of Haslam's result about G-spaces and Postnikov systems.

1. Introduction

The Gottlieb groups $G_n(X)$ of a space X have been defined by Gottlieb in [3, 4]. A space X is called a *G-space* if $G_n(X) = \pi_n(X)$ for all n. It is well known [4] that any H-space is a G-space, but the converse does not hold. On the other hand, Haslam in [6] introduced the dual Gottlieb groups $G^n(X)$ of a space X and the concepts of G'-spaces. A space X is called *G'-space* if $G^n(X) = H^n(X)$ for all n. It is known [6] that any co-H-space is a G'-space, but the converse does not hold. Moreover, Haslam studied relationships between Postnikov systems and G-spaces. In [6], He showed that if X is a G-space, then each X_n is G-space and all the k invariants k_X^{n+2} are G-primitive, and if X_{n-1} is a G-space and the k-invariants k_X^{n+1} is G-primitive, then X_n is a G-space.

In 1959, Eckmann and Hilton [2] introduced a dual concept of Postnikov system as follows; *A homology decomposition of* X consists of a sequence of spaces and maps $\{X_n, q_n, i_n\}$ satisfying (1) $q_n : X_n \to X$ induces an isomorphism $(q_n)_* : H_i(X_n) \to H_i(X)$ for $i \leq n$, (2) $i_n : X_n \to$

Received September 16, 2015; Accepted October 26, 2015.
2010 Mathematics Subject Classification: Primary 55P45, 55P35.
Key words and phrases: G^f-spaces for maps, Postnikov systems.
This work was supported by Hannam University Research Fund during research year 2014.

X_{n+1} is a cofibration with cofiber $M(H_{n+1}(X),n)$(a Moore space of type $(H_{n+1}(X),n)$), and (3) $q_n \sim q_{n+1} \circ i_n$. It is known by [7] that if X is a 1-connected space having the homotopy type of CW complex, then there is a homology decomposition $\{X_n, q_n, i_n\}$ of X such that $i_n : X_n \to X_{n+1}$ is the principal cofibration induced from $\iota : M(H_{n+1}(X),n) \to cM(H_{n+1}(X),n)$ by a map $\kappa'_n : M(H_{n+1}(X),n) \to X_n$ which is called the dual Postnikov invariants. A space X is called a *rational space* [14] if X is a 1-connected space having homotopy type of a CW-complex such that for each $n > 0$, $H_n(X, \mathbb{Z})$ is a finite dimensional vector space over $\mathbb{Q}$. It is well known [14] that if X and A are rational spaces and $p : X \to A$ is a based map, then there exist homology decompositions $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ for X and A respectively and induced maps $\{p_n : X_n \to A_n\}$ satisfying :

(1) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} M(H_{n+1}(X),n) & \xrightarrow{\tilde{p}_*} & M(H_{n+1}(A),n) \\ {\scriptstyle k'_n(X)}\big\downarrow & & {\scriptstyle k'_n(A)}\big\downarrow \\ X_n & \xrightarrow{p_n} & A_n, \end{array}$$

that is, $(k'_n(A), k'_n(X)) : \tilde{p}_\# \to p_n$ is a map,

(2) $p_{n+1} : X_{n+1} \to A_{n+1}$ given by $p_{n+1} = \bar{p_n}$ satisfying commute diagram

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ {\scriptstyle i_n(=\iota_{k'_n(X)})}\big\downarrow & & {\scriptstyle i'_n(=\iota_{k'_n(A)})}\big\downarrow \\ X_{n+1} & \xrightarrow{p_{n+1}} & A_{n+1}, \end{array}$$

(3) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ {\scriptstyle q_n}\big\downarrow & & {\scriptstyle q'_n}\big\downarrow \\ X & \xrightarrow{p} & A. \end{array}$$

For a map $p : X \to A$, *the dual Gottlieb sets $G^n(X,p,A)$ of a map* $p : X \to A$, which are generalized of dual Gottlieb groups $G^n(X)$, are defined in [20]. In general, $G^n(X) \subset G^n(X,p,A) \subset H^n(X)$ for any map $p : X \to A$.

In this paper, for a map $p : X \to A$, we define and study a concept of G'_p-space for a map, which is a generalized one of a G'-space. Any G'-space is a G'_p-space, but the converse does not hold. In fact, CP^2

is a G'_δ-space, but not a G'-space. It is shown that X is a G'_p-space if and only if $G^n(X,p,A) = H^n(X)$ for all n. It is clear that any co-H^p-space is a co-T^p-space and any co-T^p-space is a G'_p-space. Moreover, we show that X is a G'-space if and only if for any space A and any map $p : X \to A$, X is a G'_p-space for a map $p : X \to A$. We can obtain the following results about G'_p-spaces and homology decompositions which are dual generalizations of the above Haslam's results about G-spaces and Postnikov systems. Let X and A be rational spaces and $p : X \to A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively. (1) If X is a G'_p-space for a map $p : X \to A$, then each X_n is G'_{p_n}-space and the all pairs of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are G'_{p_n}-primitive. (2) If X_n is a G'_{p_n}-space and the pair of k'-invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ is G'_{p_n}-primitive, then X_n is a G'_{p_n}-space. As a corollary, we can obtain a result for G'-spaces as follows. Let X be rational space and $\{X_n, q_n, i_n\}$ homology decomposition for X. (1) If X is a G'-space, then each X_n is G'-space and all the k' invariants $k'_n(X)$ are G'-primitive. (2) If X_{n-1} is a G'-space and the k'-invariants $k'_n(X)$ is G'-primitive, then X_n is a G'-space.

Throughout this paper, space means a space of the homotopy type of connected locally finite CW complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by $*$. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta\colon X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$, the folding map $\nabla\colon X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. ΣX denote the reduced suspension of X and ΩX denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$and $\tau(1_{\Sigma X})$ respectively.

2. G'_p-spaces for maps

Let $p : X \to A$ be a map. A based map $f : X \to B$ is called *p-cocyclic* [13] if there is a map $\theta : X \to A \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X & \xrightarrow{\theta} & A\vee B \\ \Delta\downarrow & & j\downarrow \\ X\times X & \xrightarrow{(p\times f)} & A\times B, \end{array}$$

where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. We call such a map θ a *coassociated map* of a p-cocyclic map f.

In the case $p = 1_X : X \to X$, $f : X \to B$ is called *cocyclic* [16]. Clearly any cocyclic map is a p-cocyclic map and also $f : X \to B$ is p-cocyclic iff $p : X \to A$ is f-cocyclic. The *dual Gottlieb set* $DG(X, p, A; B)$ *for a map* $p : X \to A$ [20] is the set of all homotopy classes of p-cocyclic maps from X to B. In the case $p = 1_X : X \to X$, we called such a set $DG(X, 1, X; B)$ the *dual Gottlieb set* [16] denoted $DG(X; B)$, that is, the dual Gottlieb set is exactly same with the dual Gottlieb set for the identity map. We denote $DG(X, p, A; K(\pi, n))$ by $G^n(X, p, A; \pi)$ and $DG(X, p, A; K(\mathbb{Z}, n))$ by $G^n(X, p, A)$, $DG(X; K(\mathbb{Z}, n))$ by $G^n(X)$. Haslam [6] introduced and studied the *coevaluation subgroups* $G^n(X; \pi)$ of $H^n(X; \pi)$. $G^n(X; \pi)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, n)$. A space X is called [6] a G'-space if $G^n(X) = H^n(X)$ for all n. The next proposition is an immediate consequence from the definition.

Proposition 2.1. [22]

(1) *For any maps $g : X \to A$, $h : A \to B$ and any space C, $DG(X, g, A; C) \subset DG(X, hg, B; C)$.*
(2) *$DG(X, B) = DG(X, 1_X, X;\ B) \subset DG(X, g, A; B) \subset DG(X, *, A; B) = [X, B]$ for any spaces X, A and B.*
(3) *$DG(X, B) = \cap\{DG(X, g, A; B) | g : X \to A$ is a map and A is a space$\}$.*
(4) *If $h : A \to B$ is a homotopy equivalence, then $DG(X, g, A; C) = DG(X, hg, B; c)$.*
(5) *For any map $k : Y \to X$, $k^*(DG(X, g, A; B)) \subset DG(Y, gk, A; B)$.*
(6) *For any map $k : Y \to X$, $k^*(DG(X; B)) \subset DG(Y, k, X; B)$.*
(7) *For any map $s : B \to C$, $s_*(DG(X, g, A; B)) \subset DG(X, g, A; C)$.*

In general, $DG(X; B) \subset DG(X, p, A; B) \subset [X, B]$ for any map $p : X \to B$ and any space B. It is known [20] that for any n, $G^n(S^n \times S^n; \mathbb{Z}) \neq G^n(S^n \times S^n, p_1, S^n; \mathbb{Z}) \neq H^n(S^n \times S^n; \mathbb{Z})$.

A based map $g : X \to A$ is called *weakly cocyclic* [18] if $g^*(H^n(X)) \subset G^n(X)$ for all n. Any cocyclic map is an weakly cocyclic map, but the converse does not holds. It is known [6] that RP^2 is a G'-space, but not co-H-space. Thus we know that the identity map 1_{RP^2} is an weakly cocyclic map, but not cocyclic map.

Proposition 2.2. [18] *X is a G'-space if and only if $e' : X \to \Omega\Sigma X$ is weakly cocyclic.*

Definition 2.3. *Let $p : X \to A$ be a based map. A based map $g : X \to B$ is called an weakly p-cocyclic if $g^*(H^n(B)) \subset G^n(X, p, A)$ for all n.*

The next proposition is an immediate consequence from the definition.

Proposition 2.4. (1) *If $g : X \to B$ is an weakly cocyclic map and $\theta : B \to C$ is an arbitrary map, then $\theta g : X \to C$ is weakly cocyclic.*
(2) *For a map $p : X \to A$, any weakly cocyclic map $g : X \to B$ is weakly p-cocyclic.*
(3) *For a map $p : X \to A$, if $g : X \to B$ is an weakly p-cocyclic map and $\theta : B \to C$ is an arbitrary map, then $\theta g : X \to C$ is weakly p-cocyclic.*

The following proposition says that co-H-spaces are completely characterized by the dual Gottlieb sets.

Proposition 2.5. [11] *X is a co-H-space if and only if $DG(X, B) = [X, B]$ for any space B.*

A space X is called [22] a *co-H^p-space for a map $p : X \to A$* if there is a map $\theta : X \to X \vee A$ such that $j\theta \sim (1 \times p)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map, that is, $1_X : X \to X$ is p-cocyclic.

Proposition 2.6. [22] *X is a co-H^p-space for a map $p : X \to A$ if and only if $DG(X, p, A; B) = [X, B]$ for any space B.*

A space X is called a *co-T-space* [18] if $e' : X \to \Omega\Sigma X$ is cocyclic. The following proposition says that co-T-spaces are completely characterized by the dual Gottlieb sets.

Proposition 2.7. [18] *X is a co-T-space if and only if $DG(X, \Omega B) = [X, \Omega B]$ for any space B.*

A space X is called [23] a *co-T^p-space for a map $p : X \to A$* if there is a map $\theta : X \to \Omega\Sigma X \vee A$ such that $j\theta \sim (e' \times p)\Delta$, where

$j : \Omega\Sigma X \vee A \to \Omega\Sigma X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map, that is, $e' : X \to \Omega\Sigma X$ is p-cocyclic.

PROPOSITION 2.8. [23] *X is a co-T^p-space for a map $p : X \to A$ if and only if $DG(X, p, A; \Omega B) = [X, \Omega B]$ for any space B.*

It is clear, from Proposition 2.1(2) and the above propositions, that any co-T-space is a co-T^p-space for any map $p : X \to A$.

DEFINITION 2.9. *A space X is called a G'_p-space for a map $p : X \to A$ if $e' : X \to \Omega\Sigma X$ is weakly p-cocyclic.*

The following theorem says that a G'_p-space can be characterized by the dual Gottlieb sets for a map $p : X \to A$.

THEOREM 2.10. *X is a G'_p-space for a map $p : X \to A$ if and only if $G^n(X, p, A) = H^n(X)$ for all n.*

Proof. Suppose that X is a G'_p-space for a map $p : X \to A$. Let $g : X \to K(\mathbb{Z}, n) = K(\mathbb{Z}, n+1)$ be any map. Since $g = \Omega\tau^{-1}(g)e' : X \to \Omega K(\mathbb{Z}, n+1)$ and $e' : X \to \Omega\Sigma X$ is weakly p-cocyclic, $g : X \to K(\mathbb{Z}, n)$ is weakly p-cocyclic. On the other hand, suppose that $G^n(X, p, A) = H^n(X)$ for all n. Since $1_X : X \to X$ is weakly p-cocyclic, we know that the map $e' = e'1_X$ is weakly p-cocyclic and X is a G'_p-space for a map $p : X \to A$. □

Since $G^n(X) = DG(X; K(\mathbb{Z}, n)) \subset DG(X, p, A; K(\mathbb{Z}, n)) = G^n(X, p, A) \subset [X, K(\mathbb{Z}, n)] = H^n(X)$, any G'-space is a G'_p-space for any map $p : X \to A$.

Moreover, we can easily obtain, from the fact $K(\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}, n+1)$ and Proposition 2.6 and Proposition 2.8, the following corollary.

COROLLARY 2.11. *Any co-H^p-space is a co-T^p-space and any co-T^p-space is a G'_p-space.*

COROLLARY 2.12. *Let X be a G'_r-space for a map $r : X \to A$.*

(1) *If $r : X \to A$ has a right homotopy inverse $i : A \to X$, then A is a G'-space.*
(2) *If $r : X \to A$ has a left homotopy inverse $i : A \to X$, then X is a G'-space.*

Proof. (1) It is sufficient to show that $H^n(A) = G^n(A)$ for all n. Since X is a G'_r-space, $H^n(X) = G^n(X, r, A)$. Moreover, we know, from the fact $ri \sim 1 : A \to A$, that $i^* : H^n(X) \to H^n(A)$ is an epimorphism. Thus we have, from Proposition 2.1(5), that $H^n(A) = i^*(H^n(X)) =$

$i^*(G^n(X,r,A)) \subset G^n(A,ri,A) = G^n(A)$ and A is a G'-space. (2) We show that $H^n(X) \subset G^n(X)$ for all n. We obtain, from Proposition 2.1(1), that $H^n(X) = G^n(X,r,A) \subset G^n(X,ir,X) = G^n(X,1,X) = G^n(X)$. Thus we know that X is a G'-space. □

From the above corollary, we know that if X dominates A and X is a G'-space, then A is also a G'-space. Moreover, we can clearly obtain, from Proposition 2.1(2),(3), the following corollary.

Corollary 2.13. *X is a G'-space if and only if for any space A and any map $p : X \to A$, X is a G'_p-space for a map $p : X \to A$.*

It is well known fact [20, Theorem 2.8] that $p : X \to A$ is a cocyclic map if and only if $DG(X,p,A;B) = [X,B]$ for any space B. It is also known [7, Proposition 15.8] that for any cofibration sequence $B \xrightarrow{i} E \xrightarrow{q} F \xrightarrow{\delta} \Sigma B \to \cdots$, $\delta : F \to \Sigma B$ is cocyclic. Thus we have that for any cofibration sequence $B \xrightarrow{i} E \xrightarrow{q} F \xrightarrow{\delta} \Sigma B \to \cdots$, $G^n(F,\delta,\Sigma B;\pi) = H^n(F;\pi)$ for all n. The cuplength, $cup(X)$, [12] is the length of the longest nontrivial product in the reduced cohomology $\tilde{H}^*(X)$. Let R be a ring. let $P^n(X;R) = \{\alpha \in H^n(X;R) | \beta \cup \alpha = 0$ for all $\beta \in \tilde{H}^*(X,R)\}$. Then it is known [6] that $G^n(X;R) \subset P^n(X;R)$ for all n and R. Now we have an example which is a G'_p-space, but not a G'-space.

Example 2.14. *Consider the complex projective space CP^2. There is a cofibration of the unitary groups $U(2,1) \xrightarrow{i} U(3) \xrightarrow{q} CP^2$ [17]. From the cofibration sequence $U(2,1) \xrightarrow{i} U(3) \xrightarrow{q} CP^2 \xrightarrow{\delta} \Sigma U(2,1) \to \cdots$, we know that $G^n(CP^2,\delta,\Sigma U(2,1)) = H^n(CP^2)$ for all n and CP^2 is a G'_δ-space. However, it is known [12] that $cup(CP^2) = 2$. Thus, from the the above fact of that $G^n(X) \subset P^n(X)$, we know that CP^2 is not a G'-space.*

3. G'_p-spaces for maps and homology decompositions

Given maps $p : X \to A$, $p' : X' \to A'$, let $(s,r) : p' \to p$ be a map from p' to p, that is, the following diagram is commutative;

$$\begin{array}{ccc} X' & \xrightarrow{p'} & A' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{p} & A. \end{array}$$

It is a well known fact that $Y \xrightarrow{\iota} cY \to \Sigma Y$ is a cofibration, where $\iota(y) = [y, 1]$. Let $i_r : X \to C_r$ be the cofibration induced by $r : X' \to X$ from $\iota_{X'} : X' \to cX'$. Let $i_s : A \to C_s$ be the cofibration induced by $s : A' \to A$ from $\iota_{A'} : A' \to cA'$. Then there is a map $\bar{p} : C_t \to C_s$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{p} & A \\ {\scriptstyle i_r}\downarrow & & {\scriptstyle i_s}\downarrow \\ C_r & \xrightarrow{\bar{p}} & C_s, \end{array}$$

where $C_r = cX' \amalg X/[x', 1] \sim r(x')$, and $C_s = cA' \amalg A/[a', 1] \sim s(a')$, $\bar{p} : C_r \to C_s$ is given by $\bar{p}([x', t]) = [p'(x'), t]$ if $[x', t] \in cX'$ and $\bar{p}(x) = p(x)$ if $x \in X$, $i_r(x) = x$, $i_s(a) = a$.

Definition 3.1. *Let X be a G_p'-space for a map $p : X \to A$. A map $(s, r) : p' \to p$ is called a G_p'-primitive if for each map $g : \Omega\Sigma X \to K(\mathbb{Z}, m)$, m arbitrary, there is a map $G : X \to A \vee K(\mathbb{Z}, m)$ such that $jG \sim (p \times g \circ e_X')\Delta$ and $(i_s \vee 1)Gr \sim * : X' \to C_s \vee K(\mathbb{Z}, m)$, where $j : A \vee K(\mathbb{Z}, m) \to A \times K(\mathbb{Z}, m)$ is the inclusion and $e_X' : X \to \Omega\Sigma X$ is the adjoint functor image, $\tau(1_{\Sigma X})$, of $1_{\Sigma X}$.*

The following lemmas are standard.

Lemma 3.2. *Let $f : X \to B$ be a map. Then there is a map $h : C_r \to B$ such that $hi_r = f$ if and only if $fr \sim *$.*

Lemma 3.3. [19] *Let $g_t : C_r \to B_t (t = 1, 2)$ and $g : C_r \to B_1 \vee B_2$ be maps such that $p_t jgi_r \sim g_t i_r (t = 1, 2)$, where $j : B_1 \vee B_2 \to B_1 \times B_2$ is the inclusion and $p_t : B_1 \times B_2 \to B_t, t = 1, 2$ are projections. Then there is a map $h : C_r \to B_1 \vee B_2$ such that $gi_r = hi_r$ and $p_t j'h \sim g_t (t = 1, 2)$.*

Theorem 3.4. *If X is a G_p'-space for a map $p : X \to A$ and $(s, r) : p' \to p$ is G_p'-primitive, then C_r is a $G_{\bar{p}}'$-space for a map $\bar{p} : C_r \to C_s$.*

Proof. It is sufficient to show that $H^m(C_r) \subset G^m(C_r, \bar{p}, C_s)$ for all m. For each m, let $f : C_r \to K(\mathbb{Z}, m) = \Omega K(\mathbb{Z}, m+1)$ be any map. Then clearly we have the following homotopy commutative diagram;

$$\begin{array}{ccccc} X & \xrightarrow{i_r} & C_r & \xrightarrow{f} & \Omega K(\mathbb{Z}, m+1) \\ {\scriptstyle e_X'}\downarrow & & {\scriptstyle e_{C_r}'}\downarrow & & \| \\ \Omega\Sigma X & \xrightarrow{\Omega\Sigma i_r} & \Omega\Sigma C_r & \xrightarrow{\Omega\tau^{-1}(f)} & \Omega K(\mathbb{Z}, m+1). \end{array}$$

Since $(s,r) : p' \to p$ is a G'_p-primitive, for a map $\Omega\Sigma i_r \circ \Omega\tau^{-1}(f) : \Omega\Sigma X \to K(\mathbb{Z},m)$, there is a map $G : X \to A \vee K(\mathbb{Z},m)$ such that $jG \sim (p \times \Omega\Sigma i_r \circ \Omega\tau^{-1}(f) \circ e'_X)\Delta$ and $(i_s \vee 1)Gr \sim * : X' \to C_s \vee K(\mathbb{Z},m)$, where $j : A \vee K(\mathbb{Z},m) \to A \times K(\mathbb{Z},m)$ is the inclusion and $e'_X : X \to \Omega\Sigma X$ is the adjoint functor image, $\tau(1_{\Sigma X})$, of $1_{\Sigma X}$. From Lemma 3.2, there is an extending $G' : C_r \to C_s \vee K(\mathbb{Z},m)$ of $(i_s \vee 1) \circ G : X \to C_s \vee K(\mathbb{Z},m)$, that is, $G' \circ i_r = (i_s \vee 1) \circ G$. Then we have that $p_1 jG' i_r = p_1 j(i_s \vee 1)G = p_1(i_s \times 1)jG \sim p_1(i_s \times 1)(p \times \Omega\Sigma i_r \circ \Omega\tau^{-1}(f) \circ e')\Delta = i_s \circ p \sim \bar{p} \circ i_r$ and $p_2 jG' i_r = p_2 j(i_s \vee 1)G = p_2(i_s \times 1)jG \sim p_2(i_s \times 1)(p \times \Omega\Sigma i_r \circ \Omega\tau^{-1}(f) \circ e')\Delta \sim \Omega\Sigma i_r \circ \Omega\tau^{-1}(f) \circ e'_X \sim f \circ i_r$. Thus we have, from Lemma 3.3, that there is a map $\bar{G} : C_r \to C_s \vee K(\mathbb{Z},m)$ such that $\bar{G} i_r = G' i_r = (i_s \vee 1)G$ and $p_1 j' \bar{G} \sim \bar{p}$ and $p_2 j' \bar{G} \sim f$. Thus we know that $f : C_r \to K(\mathbb{Z},m)$ is $\bar{p}$-cocyclic and C_r is a $G'_{\bar{p}}$-space for a map $\bar{p} : C_r \to C_s$. □

In 1959, Eckmann and Hilton [2] introduced a dual concept of Postnikov system as follows; *A homology decomposition of* X consists of a sequence of spaces and maps $\{X_n, q_n, i_n\}$ satisfying (1) $q_n : X_n \to X$ induces an isomorphism $(q_n)_* : H_i(X_n) \to H_i(X)$ for $i \leq n$, (2) $i_n : X_n \to X_{n+1}$ is a cofibration with cofiber $M(H_{n+1}(X), n)$(a Moore space of type $(H_{n+1}(X), n)$), (3) $q_n \sim q_{n+1} \circ i_n$. It is known by [7] that if X be a 1-connected space having the homotopy type of CW complex, then there is a homology decomposition $\{X_n, q_n, i_n\}$ of X such that $i_n : X_n \to X_{n+1}$ is the principal cofibration induced from $\iota : M(H_{n+1}(X), n) \to cM(H_{n+1}(X), n)$ by a map $\kappa'_n : M(H_{n+1}(X), n) \to X_n$ which is called the dual Postnikov invariants. A space X is called a *rational space* [14] if X is a 1-connected space having homotopy type of a CW-complex such that for each $n > 0$, $H_n(X, \mathbb{Z})$ is a finite dimensional vector space over $\mathbb{Q}$. It is well known [14] that if X and A are rational spaces and $p : X \to A$ is a based map, then there exist homology decompositions $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ for X and A respectively and induced maps $\{p_n : X_n \to A_n\}$ satisfying

(1) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} M(H_{n+1}(X), n) & \xrightarrow{\tilde{p}_*} & M(H_{n+1}(A), n) \\ {\scriptstyle k'_n(X)}\downarrow & & {\scriptstyle k'_n(A)}\downarrow \\ X_n & \xrightarrow{p_n} & A_n, \end{array}$$

that is, $(k'_n(A), k'_n(X)) : \tilde{p}_\# \to p_n$ is a map,

(2) $p_{n+1} : X_{n+1} \to A_{n+1}$ given by $p_{n+1} = \bar{p_n}$ satisfying commute diagram

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ {\scriptstyle i_n(=\iota_{k'_n(X)})}\downarrow & & {\scriptstyle i'_n(=\iota_{k'_n(A)})}\downarrow \\ X_{n+1} & \xrightarrow{p_{n+1}} & A_{n+1}, \end{array}$$

(3) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ {\scriptstyle q_n}\downarrow & & {\scriptstyle q'_n}\downarrow \\ X & \xrightarrow{p} & A. \end{array}$$

THEOREM 3.5. *Let X and A be rational spaces and $p : X \to A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively.*

(1) *If X is a G'_p-space for a map $p : X \to A$, then each X_n is G'_{p_n}-space and the all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are G'_{p_n}-primitive.*
(2) *If X_n is a G'_{p_n}-space and the pair of k'-invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ is G'_{p_n}-primitive, then X_n is a G'_{p_n}-space.*

Proof. (1) Let $f : X_n \to K(\mathbb{Z}, m) = \Omega K(\mathbb{Z}, m+1)$ be any map. Since $(q_n)_* : H_i(X_n) \to H_i(X)$ for $i \leq n$ and $H_i(X_n) = 0$ for $i > n$, there is a map $f' : X \to K(\mathbb{Z}, m)$ such that $f'q_n \sim f$. Since X is a G'_p-space for a map $p : X \to A$, there is a map $G : X \to A \vee K(\mathbb{Z}, m)$ such that $jG \sim (p \times f')\Delta$, where $j : A \vee K(\mathbb{Z}, m) \to A \times K(\mathbb{Z}, m)$ is the inclusion. Let $\{B_n, q''_n, i''_n\}$ be a homology decomposition for $K(\mathbb{Z}, m)$. Then $\{A_n \vee B_n, q'_n \vee q''_n, i'_n \vee i''_n\}$ is a homology decomposition for $A \vee K(\mathbb{Z}, m)$. Then we have, by Toomer's result [15,Theorem 4], that there are families of maps $p_n : X_n \to A_n$ and $G_n : X_n \to A_n \vee B_n$ such that $i'_n p_n = p_{n+1} i_n$ and $q'_n p_n \sim p q_n$, and $(i'_n \vee i''_n) G_n = G_{n+1} i_n$ and $(q'_n \vee q''_n) G_n \sim G q_n$ for $n = 2, 3, \cdots$ respectively, and $k'_n(A)\tilde{p}_* \sim p_n k'_n(X) : M(H_{n+1}(X), n) \to A_n$ and $(k'_n(A) \vee k'_n(K(\mathbb{Z}, m))\tilde{G}_* \sim G_n k'_n(X) : M(H_{n+1}(X), n) \to A_n \vee B_n$, where $k'_n(A) : M(H_{n+1}(A), n) \to A_n$, $k'_n(X) : M(H_{n+1}(X), n) \to X_n$ and $k'_n(K(\mathbb{Z}, m)) : M(H_{n+1}(K(\mathbb{Z}, m)), n) \to B_n$ are k'-invariants of A, X and $K(\mathbb{Z}, m)$ respectively, $\tilde{p}_* : M(H_{n+1}(X), n) \to M(H_{n+1}(A), n)$ and $\tilde{G}_* : M(H_{n+1}(X), n) \to M(H_{n+1}(A \vee K(\mathbb{Z}, m)), n) \approx M(H_{n+1}(A \oplus H_{n+1}(K(\mathbb{Z}, m)), n) \approx M(H_{n+1}(A), n) \vee M(H_{n+1}(K(\mathbb{Z}, m)), n)$ are the induced maps by $p : X \to A$ and $G : X \to A \vee K(\mathbb{Z}, m)$ respectively. Consider the composition $G' = (1 \vee q''_n) G_n : X_n \to A_n \vee K(\mathbb{Z}, m)$. It is clear that $p_1 jG' = p_1 jG_n \sim p_n$ and $p_2 jG' = p_2 j(1 \vee q''_n) G_n \sim p_2 jGq_n =$

$f'q_n \sim f$. Thus $[f] \in G^m(X_n, p_n, A_n)$ and X_n is a G'_{p_n}-space for a map $p_n : X_n \to A_n$. Moreover, to show that $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ is G'_{p_n}-primitive, let $g'' : \Omega\Sigma X_n \to K(\mathbb{Z}, m)$ be any map and m arbitrary. Since $g''e'_{X_n} : X_n \to K(\mathbb{Z}, m) \in H^m(X_n)$ is a map, by taking $f = g''e'_{X_n}$ from the above proof, we have a map $G' = (1 \vee q''_n)G_n : X_n \to A_n \vee K(\mathbb{Z}, m)$ such that $jG' \sim (p_n \times g''e'_{X_n})\Delta : X_n \to A_n \times K(\mathbb{Z}, m)$, where $j : A_n \vee K(\mathbb{Z}, m) \to A_n \times K(\mathbb{Z}, m)$ is the inclusion. Since $(i'_n \vee 1)G' = (i'_n \vee 1)(1 \vee q''_n)G_n = (i'_n \vee q''_n)G_n \sim (1 \vee q''_{n+1})G_{n+1}i_n : X_n \to A_{n+1} \vee K(\mathbb{Z}, m)$, $(i'_n \vee 1)G' : X_n \to A_{n+1} \vee K(\mathbb{Z}, m)$ has an extending $(1 \vee q''_{n+1})G_{n+1} : X_{n+1} \to A_{n+1} \vee K(\mathbb{Z}, m)$ and $(i'_n \vee 1)G'k'_n(X) \sim *$. Thus $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are G'_{p_n}-primitive.

(2) It follows from Theorem 3.4. □

Taking $p = 1_X,\ p' = 1_{X'},\ s = r$, we can obtain the following corollary which is a dual result of Haslam's results about G-spaces and Postnikov systems [6].

Corollary 3.6. *Let X be rational space and $\{X_n, q_n, i_n\}$ homology decomposition for X.*

1. *If X is a G'-space, then each X_n is G'-space and all the k' invariants $k'_n(X)$ are G'-primitive.*
2. *If X_{n-1} is a G'-space and the k'-invariants $k'_n(X)$ is G'-primitive, then X_n is a G'-space.*

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