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THE EXISTENCE OF WARPING FUNCTIONS ON RIEMANNIAN WARPED PRODUCT MANIFOLDS WITH FIBER MANIFOLDS OF CLASS (A)

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Abstract. In this paper, we prove the existence and the nonexistence of warping functions on Riemannian warped product manifolds with some prescribed scalar curvatures according to the fiber manifolds of class (A).

1. Introduction

One of the basic problems in the differential geometry is studying the set of curvature functions which a given manifold possesses.

The well-known problem in differential geometry is that of whether there exists a warping function of warped metric with some prescribed scalar curvature function. One of the main methods of studying differential geometry is by the existence and the nonexistence of a warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds. In order to study these kinds of problems, we need some analytic methods in differential geometry.

For Riemannian manifolds, warped products have been useful in producing examples of spectral behavior, examples of manifolds of negative curvature (cf. [2, 3, 4, 5, 7, 13, 14]), and also in studying L_2-cohomology (cf.[15]).

In a study [11, 12], M.C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and the nonexistence of Riemannian warped metric with some prescribed scalar curvature function.

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In this paper, we also study the existence and the nonexistence of a warped product metric with prescribed scalar curvature functions on some Riemannian warped product manifolds. So, using upper solution and lower solution methods, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M = B \times_f N$ in terms of its warping function f and the scalar curvatures of B and N.

By the results of Kazdan and Warner ([8, 9, 10]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three categories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.
(B) A Smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.
(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [8, 9, 10], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on an open manifold. Results of Gromov and Lawson ([5]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds.

Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded ([5], [13] p.322).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature ([2]). It follows from the results of Aviles and McOwen ([1]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct Riemannian metrics on $M = [a,\infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. It is shown that if the fiber manifold N belongs to class (A), then M admits a Riemannian metric with some prescribed scalar curvature outside a compact set.

Although we will assume throughout this paper that all data (M, metric g, and curvature, etc.) are smooth, this is merely for convenience. Our arguments go through with little or no change if one makes minimal smoothness hypotheses, such as assuming that the give data is Hölder continuous.

2. Main results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a,\infty) \to R^+$ be a smooth function, where a is a positive number. A Riemannian warped product of N and $[a,\infty)$ with warping function f is defined to be the product manifold $([a,\infty) \times_f N, g')$ with

$$g' = dt^2 + f^2(t)g. \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t,x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}[R(g)(x) - 2nf(t)f''(t) - n(n-1)|f'(t)|^2] \tag{2.2}$$

for $t \in [a,\infty)$ and $x \in N$ (For details, [4] or [5]). Here we also know that if $R(g)(x)$ is constant, then $R(t,x)$ is a function of only $t-$ variable.

Now we consider the following problem:

Problem 2.1. *Given a fiber N with constant scalar curvature c, can we find a warping function $f > 0$ on $B = [a,\infty)$ such that for any smooth function $R(t,x) = R(t)$, the warped metric g admits $R(t)$ as the scalar curvature on $M = [a,\infty) \times_f N$?*

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u''(t) + R(t,x)u(t) - R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

If N belongs to (A), then a negative constant function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric g_1 on N with scalar curvature $R(g_1) = -\frac{4n}{n+1}k$, where k is a positive constant. Then equation (2.3) becomes

$$\frac{4n}{n+1}u^{''}(t)+\frac{4n}{n+1}ku(t)^{1-\frac{4}{n+1}}+R(t,x)u(t)=0. \tag{2.4}$$

In order to prove the nonexistence of some Riemannian warped product metric with fiber manifolds of class (A), we have the following theorem whose proof is similar to that of Lemma 3.3 in [6].

THEOREM 2.2. *Let $u(t)$ be a positive smooth function on $[a,\infty)$. If $u(t)$ satisfies*

$$\frac{u''(t)}{u(t)}\leq\frac{C}{t^2}$$

for some constant $2>C>0$, then there exists $t_0>a$ such that for all $t>t_0$

$$u(t)\leq c_0t^\epsilon$$

for some positive constants c_0 and $2>\epsilon>1$.

Proof. Since $2>C>0$, we can choose $2>\epsilon>1$ such that $\epsilon(\epsilon-1)=C$. Then from the hypothesis, we have

$$t^\epsilon u''(t)\leq\epsilon(\epsilon-1)t^{\epsilon-2}u(t).$$

Upon integration from $t_1(\geq a)$ to $t(>t_1\geq a)$, and using integration by parts, we obtain

$$t^\epsilon u'(t)-\epsilon t^{\epsilon-1}u(t)-t_1^\epsilon u'(t_1)+\epsilon t_1^{\epsilon-1}u(t_1)+\epsilon(\epsilon-1)\int_{t_1}^t s^{\epsilon-2}u(s)ds$$
$$\leq C\int_{t_1}^t s^{\epsilon-2}u(s)ds.$$

Therefore we have

$$t^\epsilon u'(t)-\epsilon t^{\epsilon-1}u(t)\leq t_1^\epsilon u'(t_1)-\epsilon t_1^{\epsilon-1}u(t_1). \tag{2.5}$$

We consider two following cases:

[Case 1] There exists $t_1\geq a$ such that $u'(t_1)\leq 0$.

If there is a number $t_1\geq a$ such that $u'(t_1)\leq 0$, then we have

$$t^\epsilon u'(t)-\epsilon t^{\epsilon-1}u(t)\leq 0.$$

This gives

$$(\ln u(t))'\leq\epsilon(\ln t)'.$$

Hence

$$u(t)\leq c_1t^\epsilon$$

for all $t>t_1$, where c_1 is a positive constant.

[Case 2] There does not exist $t_1 \geq a$ such that $u'(t_1) \leq 0$.

In other words, if $u'(t) > 0$ for all $t \geq a$, then $u(t) \geq c'$ for some positive constant c'. Let c_2 be a positive constant such that

$$t_1^{\epsilon} u'(t_1) - \epsilon t_1^{\epsilon-1} u(t_1) \leq c_2,$$

then equation (2.5) gives

$$t^{\epsilon} u'(t) - \epsilon t^{\epsilon-1} u(t) \leq c_2$$

for all $t > t_1$. Thus

$$\frac{u'(t)}{u(t)} \leq \frac{\epsilon}{t} + \frac{c_2}{u(t)t^{\epsilon}} \leq \frac{\epsilon}{t} + \frac{c_2}{c' t^{\epsilon}}.$$

Integrating from t_1 to t we have

$$\ln \frac{u(t)}{u(t_1)} \leq \epsilon \ln \left(\frac{t}{t_1}\right) + \frac{c_2}{(\epsilon-1)c' t_1^{\epsilon-1}} \leq \epsilon \ln \left(\frac{c_3 t}{t_1}\right),$$

as $\epsilon > 1$. Here c_3 is a positive constant such that $\ln c_3 \geq \frac{c_2}{\epsilon(\epsilon-1)c' t_1^{\epsilon-1}}$. Hence we again obtain the inequality

$$u(t) \leq b t^{\epsilon}$$

for some positive constant b and for all $t \geq t_1$.

Thus from two cases we always find $t_0 > a$ and a constant $c_0 > 0$ such that

$$u(t) \leq c_0 t^{\epsilon}$$

for all $t \geq t_0$. □

Using the above theorem, we can prove the following theorem about the nonexistence of warping function, whose proof is similar to that of Lemma 3.3 in [12].

THEOREM 2.3. *Suppose that N belongs to class (A). Let g be a Riemannian metric on N of dimension $n(\geq 3)$. We may assume that $R(g) = -\frac{4n}{n+1}k$, where k is a positive constant. On $M = [a, \infty) \times_f N$, there does not exist a Riemannian warped product metric*

$$g' = dt^2 + f^2(t) g$$

with scalar curvature

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$, where t_0 and a are positive constants.

Proof. Assume that we can find a warped product metric on $M = [a, \infty) \times_f N$ with

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$. In equation (2.4), we have

$$\frac{4n}{n+1}\left[\frac{u^{''}(t)}{u(t)} + \frac{k}{u(t)^{\frac{4}{n+1}}}\right] = -R(t) \leq \frac{n(n-1)}{t^2} \tag{2.6}$$

and

$$\frac{u^{''}(t)}{u(t)} \leq \frac{\frac{(n-1)(n+1)}{4}}{t^2}. \tag{2.7}$$

In equation (2.7), we can apply Theorem 2.2 and take $\epsilon = \frac{n+1}{2}$. Hence we have $t_0 > a$ such that

$$u(t) \leq c_0 t^{\frac{n+1}{2}}$$

for some positive constants c_0 and all $t > t_0$.

Then

$$\frac{k}{u(t)^{\frac{4}{n+1}}} \geq \frac{c^{'}}{t^2}$$

where $0 < c^{'} \leq \frac{k}{c_0^{\frac{4}{n+1}}}$ is a positive constant. Hence equation (2.6) gives

$$\frac{u^{''}(t)}{u(t)} \leq \frac{(n+1)(n-1)-\delta}{4t^2},$$

where $4c^{'} \geq \delta \geq 0$ is a constant. We can choose $\delta^{'} > 0$ such that

$$\frac{(n+1)(n-1)-\delta}{4} = \left(\frac{n+1}{2} - \delta^{'}\right)\left(\frac{n-1}{2} - \delta^{'}\right)$$

for small positive δ. Applying Theorem 2.2 again, we have $t_1 > a$ such that

$$u(t) \leq c_1 t^{\frac{n+1}{2}-\delta^{'}}$$

for some $c_1 > 0$ and all $t > t_1$. And

$$\frac{k}{u(t)^{\frac{4}{n+1}}} \geq \frac{c^{''}}{t^{2-\epsilon}}, \tag{2.8}$$

where $\epsilon = \dfrac{4}{n+1}\delta'$ and $0 < c'' \leq \dfrac{k}{c_1^{\frac{4}{n+1}}}$. Thus equation (2.7) and (2.8) give

$$\frac{u''(t)}{u(t)} \leq \frac{(n-1)(n+1)}{4t^2} - \frac{c''}{t^{2-\epsilon}},$$

which implies that

$$u''(t) \leq 0$$

for t large. Hence $u(t) \leq c_2 t$ for some constant $c_2 > 0$ and large t. From equation (2.5) we have

$$\frac{u''(t)}{u(t)} \leq \frac{-c_3}{t^{\frac{4}{n+1}}} + \frac{(n+1)(n-1)}{4t^2} \leq -\frac{c_3}{t}$$

for t large enough, as $n \geq 3$. Here c_3 is a positive constant. Multiplying $u(t)$ and integrating from t' to t, we have

$$u'(t) - u'(t') \leq -c_3 \int_{t'}^{t} \frac{u(s)}{s} ds, \quad t > t'.$$

We consider two following cases :

[Case 1] There exists $t' \geq max\{t_0, t_1\}$ such that $u'(t') \leq 0$. If $u'(t') \leq 0$ for some t', then $u'(t) \leq -c_4$ for some positive constant c_4. Hence $u(t) \leq 0$ for t large enough, contradicting the fact that u is positive.

[Case 2] There does not exist $t' \geq max\{t_0, t_1\}$ such that $u'(t') \leq 0$. In order words, if $u'(t) > 0$ for all t large, then $u(t)$ is increasing, hence

$$\int_{t'}^{t} \frac{u(s)}{s} ds \geq u(t') \int_{t'}^{t} \frac{1}{s} ds \to \infty.$$

Thus $u'(t)$ has to be negative for some t large, which is a contradiction to the hypothesis. Therefore there does not exist such warped product metric. $\square$

In particular, Theorem 2.3 implies that if $R(g) = -\frac{4n}{n+1}k$, then using Lorentzian warped product it is impossible to obtain a Riemannian metric of positive or zero scalar curvature outside a compact subset.

PROPOSITION 2.4. *Suppose that* $R(g) = -\dfrac{4n}{n+1}k$ *and* $R(t,x) = R(t) \in C^\infty([a,\infty))$. *Assume that for* $t > t_0$, *there exist an upper solution* $u_+(t)$ *and a lower solution* $u_-(t)$ *such that* $0 < u_-(t) \leq u_+(t)$. *Then there exists a solution* $u(t)$ *of equation (2.4) such that for* $t > t_0, 0 < u_-(t) \leq u(t) \leq u_+(t)$.

Proof. We have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a, \infty)$, $\tilde{u}_-(t) \le \tilde{u}_+(t)$. Since $R(t) \in C^\infty([a, \infty))$, there exists a positive constant b such that $|R(t)| \le \dfrac{4n}{n+1} b^2$ for $t \in [a, t_0]$. Since

$$\begin{aligned}
&\frac{4n}{n+1} u_+''(t) + R(t) u_+(t) + \frac{4n}{n+1} k u_+(t)^{1-\frac{4}{n+1}} \\
&\le \frac{4n}{n+1} (u_+''(t) + b^2 u_+(t) + k u_+(t)^{1-\frac{4}{n+1}}),
\end{aligned}$$

if we divide the given interval $[a, t_0]$ into small intervals $\{I_i\}_{i=1}^n$, then for each interval I_i we have an upper solution $u_{i+}(t)$ by parallel transporting $\cos Bt$ such that $0 < \dfrac{1}{\sqrt{2}} \le u_{i+}(t) \le 1$. That is to say, for each interval I_i,

$$\begin{aligned}
&\frac{4n}{n+1} u_{i+}''(t) + R(t) u_{i+}(t) + \frac{4n}{n+1} k u_{i+}(t)^{1-\frac{4}{n+1}} \\
&\le \frac{4n}{n+1} (u_{i+}''(t) + b^2 u_{i+}(t) + k u_{i+}(t)^{1-\frac{4}{n+1}}) \\
&= \frac{4n}{n+1} (-B^2 \cos Bt + b^2 \cos Bt + k(\cos Bt)^{1-\frac{4}{n+1}}) \\
&= \frac{4n}{n+1} \cos Bt(-B^2 + b^2 + k(\cos Bt)^{-\frac{4}{n+1}}) \\
&\le \frac{4n}{n+1} \cos Bt(-B^2 + b^2 + k 2^{\frac{2}{n+1}}) \\
&\le 0
\end{aligned}$$

for large B, which means that $u_{i+}(t)$ is an (weak) upper solution for each interval I_i. Then put $\tilde{u}_+(t) = u_{i+}(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution such that $\dfrac{1}{\sqrt{2}} \le \tilde{u}_+(t) \le 1$ for all $t \in [a, t_0]$. Put $\tilde{u}_-(t) = \dfrac{1}{\sqrt{2}} e^{-\alpha t}$ for $t \in [a, t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a, t_0]$,

$$\begin{aligned}
&\frac{4n}{n+1} u_{i-}''(t) + R(t) u_{i-}(t) + \frac{4n}{n+1} k u_{i-}(t)^{1-\frac{4}{n+1}} \\
&\ge \frac{4n}{n+1} (u_{i-}''(t) - b^2 u_{i-}(t)) \\
&= \frac{4n}{n+1} \frac{1}{\sqrt{2}} e^{-\alpha t} (\alpha^2 - b^2) \ge 0
\end{aligned}$$

for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a, \infty)$, $0 < \tilde{u}_-(t) \leq \tilde{u}_+(t)$. □

However, in this paper, when N is a compact Riemannian manifold of class (A), we consider the existence of some warping functions on Riemannian warped product manifolds $M = [a, \infty) \times_f N$ with prescribed scalar curvatures. If $R(t, x)$ is also the function of only t-variable, then we have the following theorems.

THEOREM 2.5. *Suppose that* $R(g) = -\frac{4n}{n+1}k$. *Assume that* $R(t, x) = R(t) \in C^\infty([a, \infty))$ *is a negative function such that*

$$-\frac{4n}{n+1}be^{t^s} \leq R(t) \leq -\frac{4n}{n+1}\frac{C}{t^\alpha}, \quad for \quad t \geq t_0,$$

where $t_0 > a, 0 < \alpha < 2, C$ *and* $b, s > 1$ *are positive constants. Then equation (2.4) has a positive solution on* $[a, \infty)$.

Proof. We let $u_+(t) = t^m$, where m is some positive number. Then we have

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} + R(t)u_+(t) \\
&\leq \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C}{t^\alpha}u_+(t) \\
&= \frac{4n}{n+1}t^m[\frac{m(m-1)}{t^2} + \frac{k}{t^{\frac{4}{n+1}m}} - \frac{C}{t^\alpha}] \\
&\leq 0, \quad t \geq t_0,
\end{aligned}$$

for some large t_0, which is possible for large fixed m since $0 < \alpha < 2$. Hence, $u_+(t)$ is an upper solution. Now put $u_-(t) = e^{-t^\beta}$, where β is a positive constant, which will be determined later. Then

$$\begin{aligned}
&\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}ku_-(t)^{1-\frac{4}{n+1}} + R(t)u_-(t) \\
&\geq \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}ku_-(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}be^{t^s}u_-(t) \\
&= \frac{4n}{n+1}e^{-t^\beta}[\beta^2 t^{2\beta-2} - \beta(\beta-1)t^{\beta-2} + ke^{t^\beta\frac{4}{n+1}} - be^{t^s}] \\
&\geq 0, \quad t \geq t_0
\end{aligned}$$

for some large t_0 and large β such that $\beta > s$, which means that $u_-(t)$ is a lower solution. And we can take β so large that $0 < u_-(t) < u_+(t)$. So by Theroem 2.4, we obtain a positive solution. □

The above theorem implies that if $R(t)$ is not rapidly decreasing and less than some negative function, then equation (2.4) has a positive solution.

THEOREM 2.6. *Suppose that* $R(g) = -\frac{4n}{n+1}k$. *Assume that* $R(t,x) = R(t) \in C^\infty([a,\infty))$ *is a negative function such that*

$$-\frac{4n}{n+1}be^{t^s} \leq R(t) \leq -\frac{C}{t^2}, \quad for \quad t \geq t_0$$

where $t_0 > a, b$ *and* $C, s > 1$ *are positive constants. If* $C > n(n-1)$, *then equation (2.3) has a positive solution on* $[a,\infty)$.

Proof. In case that $C > n(n-1)$, we may take $u_+(t) = C_+ t^{\frac{n+1}{2}}$, where C_+ is a positive constant. Then

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}ku_+(t)^{1-\frac{4}{n+1}} + R(t)u_+(t) \\
&\leq C_+\frac{4n}{n+1}t^{\frac{n-3}{2}}[\frac{n^2-1}{4} + kC_+^{-\frac{4}{n+1}} - \frac{n+1}{4n}C] \\
&\leq 0,
\end{aligned}$$

which is possible if we take C_+ to be large enough since $\frac{(n+1)(n-1)}{4} - \frac{n+1}{4n}C < 0$. Thus $u_+(t)$ is an upper solution. And we take $u_-(t)$ as in Theorem 2.4. In this case, we also obtain a positive solution. □

REMARK 2.7. *The results in Theorem 2.5, and Theorem 2.6 are almost sharp as we can get as close to* $-\frac{n(n-1)}{t^2}$ *as possible. For example, let* $R(g) = -\frac{4n}{n+1}k$ *and* $f(t) = t\ln t$ *for* $t > a$. *Then we have*

$$R = -\frac{1}{t^2}[\frac{4n}{n+1}\frac{k}{(\ln t)^2} + \frac{2n}{\ln t} + n(n-1)(1+\frac{1}{\ln t})^2],$$

which converges to $-\frac{n(n-1)}{t^2}$ *as* t *goes to* ∞.

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