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SYMMETRIC BI-DERIVATIONS OF BCH-ALGEBRAS

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Abstract. The aim of this paper is to introduce the notion of left-right (resp. right-left) symmetric bi-derivation of BCH-algebras and some related properties are investigated.

1. Introduction

In 1966, Imai and Iseki introduced two classes of abstract algebras, BCK-algebra and BCI-algebras [6]. It is known that the class of BCI-algebras is a generalization of the class of BCK-algebras In 1983, Hu and Li [3] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK-algebras and BCI-algebras. They have studied a few properties of these algebras. In this paper, we introduce the notion of left-right (resp. right-left) symmetric bi-derivations of BCH algebras and investigate some properties of symmetric bi-derivations in a BCH-algebra. Moreover, we prove that the set of all symmetric bi-derivations on a medial BCH-algebra forms a semigroup under a suitably defined binary composition.

2. Preliminary

By a *BCH-algebra,* we mean an algebra $(X, *, 0)$ with a single binary operation "$*$" that satisfies the following identities for any $x, y, z \in X$:

(BCH1) $x * x = 0$,
(BCH2) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(BCH3) $(x * y) * z = (x * z) * y$, where $x \leq y$ if and only if x*y=0.

In a BCH-algebra, the following identities are true for all $x, y \in X$:

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(BCH4) $(x * (x * y)) * y = 0$,
(BCH5) $x * 0 = 0$ implies $x = 0$,
(BCH6) $0 * (x * y) = (0 * x) * (0 * y)$,
(BCH7) $x * 0 = x$,
(BCH8) $(x * y) * x = 0 * y$,
(BCH9) $x * y = 0$ implies $0 * x = 0 * y$.

DEFINITION 2.1. Let I be a nonempty subset of a BCH-algebra X. Then I is called an *ideal* of X if it satisfies:
(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

DEFINITION 2.2. A BCH-algebra is said to be *medial* if it satisfies

$$(x * y) * (z * w) = (x * z) * (y * w)$$

for all x, y, z, w.

In a medial BCH-algebra, the following identity hold:
(BCH10) $x * (x * y) = y$ for all $x, y \in X$.

DEFINITION 2.3. A BCH-algebra X is said to be *commutative* if $y * (y * x) = x * (x * y)$ for all $x, y \in X$. For a BCH-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

DEFINITION 2.4. Let X be a BCH-algebra. A map $d : X \to X$ is a *left-right derivation* (briefly, (l, r)-*derivation*) of X if it satisfies the identity

$$d(x * y) = (d(x) * y) \wedge (x * d(y))$$

for all $x, y \in X$. If d satisfies the identity

$$d(x * y) = (x * d(y)) \wedge (d(x) * y)$$

for all $x, y \in X$, then d is a *right-left derivation* (briefly, (r, l)-*derivation*) of X. Moreover, if d is both an (l, r) and (r, l)-derivation of X, then d is a *derivation* of X.

DEFINITION 2.5. A BCH-algebra is said to be it associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.

DEFINITION 2.6. For any BCH-algebra, we define the set $G(X)$ by as follows

$$G(X) = \{x \in X | 0 * x = x\}.$$

DEFINITION 2.7. Let X be a BCH-algebra. Then the set $X_+ = \{x \in X | 0 * x = 0\}$ is called a *BCA-part* of X.

3. Symmetric bi-derivations of BCH-algebras

In what follows, let X denote a BCH-algebra unless otherwise specified.

Definition 3.1. Let $X,*,0)$ be a BCH-algebra. Define a binary composition "+" on X as follows:

$$x + y = x * (0 * y)$$

for any $x, y \in X$.

Theorem 3.2. *In any medial BCH-algebra $(X,*,0)$, if we define "+" as $x+y = x*(0*y)$ for any $x, y \in X$, Then the following properties hold:*

(1) $x + 0 = x = 0 + x$,

(2) Addition is associative,

(3) Addition is commutative,

(4) Additive inverse of x *is* $0 * x$.

Proof. (1) Let X be a medial BCH-algebra and $x \in X$. Then

$$x + 0 = x * (0 * 0) = x * 0 = x = 0 * (0 * x) = 0 + x.$$

(2) Applying the definition of "+" repeatedly and simplifying, we have the result.

(3) For any $x, y \in X$,

$$\begin{aligned} x + y &= 0 + (x + y) = (y * y) + (x * (0 * y)) \\ &= (y * y) * (0 * (x * (0 * y)) \\ &= (y * y) * ((0 * x) * (0 * (0 * y))) && ((x * y) * z = (x * z) * y) \\ &= (y * y) * ((0 * x) * y) && (y * (y * x) = x) \\ &= (y * (0 * x)) * (y * y) = y * (0 * x) \\ &= y * (0 * x) = y + x \end{aligned}$$

(4) For any $x \in X$,

$$x + (0 * x) = x * (0 * (0 * x)) = x * x = 0.$$

Hence the additive inverse of x is written as as $-x = 0 * x$. □

Definition 3.3. Let X be a medial BCH-algebra. If we define an addition "+" as $x + y = x * (0 * y)$ for all $x, y \in X$, then $(X, +)$ is an abelian group with identity 0 and the additive inverse denoted by $-x = 0 * x$ for any $x \in X$.

If we have a medial BCH-algebra $(X, *, 0)$, it follows from the above definition that $(X, +)$ is an abelian group with $-y = 0*y$ for any $y \in X$. Then we have $x - y = x * y$ for any $x, y \in X$. On the other hand, if we choose an abelian group $(X, +)$ with an identity 0 and define $x*y = x-y$, we obtain a medial BCH-algebra $(X, *, 0)$ where $x + y = x * (0 * y)$ for any $x, y \in X$.

Since $x + (0 * y) = x * (0 * (0 * y)) = x * y$, for all $x, y \in X$, we have $x * y = x + (0 * y) = x - y$.

Definition 3.4. Let X, Y be BCH-algebras. An operation $*$ on the Cartesian product $X \times X$ of X, Y as follows: For $x_1, x_2 \in X, y_1, y_2 \in Y$,

1. $(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$,
2. $(0, 0) = 0$.

Lemma 3.5. *A cartesian product of two BCH-algebras is again a BCH-algebras.*

Proof. (1) For all $(x, y) \in X \times Y$, we have $(x, y)*(x, y) = (x*x, y*y) = (0, 0)$.

(2) For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, let $(x_1, y_1) * (x_2, y_2) = (0, 0)$ and $(x_2, y_2) * (x_1, y_1) = (0, 0)$. Then we have $x_1 * x_2 = 0$ and $x_2 * x_1 = 0$, which means that $x_1 = x_2$. Also, $y_1 * y_2 = 0$ and $y_2 * y_1 = 0$. Thus we get $y_1 = y_2$. Hence $(x_1, y_1) = (x_2, y_2)$.

(3) For any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, we get $((x_1, y_1) * (x_2, y_2)) * (x_3, y_3) = ((x_1 * x_2) * x_3, (y_1 * y_2*) * y_3) = ((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) = ((x_1, y_1) * (x_3, y_3)) * (x_2, y_2)$. □

Definition 3.6. Let X be a BCH-algebra. A map $D : X \times X \to X$ is a *symmetric map* if $D(x, y) = D(y, x)$ holds for all pairs of elements $x, y \in X$.

Example 3.7. Let $X = \{0, 1, 2, 3\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The map $D : X \times X \to X$ defined by $D(x, y) = x * (0 * y)$ is a symmetric map.

Definition 3.8. Let X be a BCH-algebra and let $D : X \times X \to X$ be a symmetric mapping. A mapping $d : X \to X$ defined by $d(x) = D(x, x)$ is called a *trace* of D.

EXAMPLE 3.9. In Example 3.4, $d(0) = D(0,0) = 0+0 = 0, d(1) = D(1,1) = 1+1 = 0, d(2) = D(2,2) = 2+2 = 0, d(3) = D(3,3) = 3+3 = 0$.

DEFINITION 3.10. Let X be a BCH-algebra and let $D : X \times X \to X$ be a symmetric mapping. If D satisfies the identity, $D(x * y, z) = (D(x,z) * y) \wedge (x * D(y,z))$ for all $x, y, z \in X$, then D is called a *left-right symmetric bi-derivation* (briefly, *(l,r)-symmetric bi-derivation*) of X.

If D satisfies the identity, $D(x * y, z) = (x * D(y,z)) \wedge (D(x,z) * y)$ for all $x, y, z \in X$, then D is called a *right-left symmetric bi-derivation* (briefly, *(r,l)-symmetric bi-derivation*) of X.

If D is both an (l,r)-symmetric bi-derivation and an (r,l)-symmetric bi-derivation, then D is called a *symmetric bi-derivation* of X.

EXAMPLE 3.11. In Example 3.4, define a mapping $D : X \times X \to X$ by $D(x,y) = x*(0*y)$ for all $x, y \in X$. Then D is a symmetric bi-derivation of X.

EXAMPLE 3.12. Let $X = \{0,1,2\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a map $D : X \times X \to X$ by

$$D(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) = (0,1) \\ 0 & \text{if } (x,y) = (1,0) \\ 2 & \text{if } (x,y) = (0,2) \\ 2 & \text{if } (x,y) = (2,0) \\ 1 & \text{if } (x,y) = (1,1) \\ 0 & \text{if } (x,y) = (2,2) \\ 2 & \text{if } (x,y) = (2,1) \\ 2 & \text{if } (x,y) = (1,2) \end{cases}$$

Then it is easily checked that D is a symmetric bi-derivation of X.

PROPOSITION 3.13. *Let X be a medial BCH-algebra. Define a symmetric map $D : X \times X \to X$ by $D(x,y) = x + y$ for all $x, y \in X$. Then D is a (l,r)-symmetric bi-derivation of X.*

Proof. For all $x, y, z \in X$, we have

$$\begin{aligned} D(x*y,z) &= (x*y)+z = (x*y)*(0*z) \\ &= (x*(0*z))*y = (x+z)*y \quad (\because (x*y)*z = (x*z)*y) \\ &= (x*(y+z))*((x*(y+z))*((x+z)*y)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\because y*(y*x) = x) \\ &= ((x+z)*y) \wedge (x*(y+z)) \\ &= (D(x,z)*y) \wedge (x*(D(y,z)). \end{aligned}$$

This proves that D is a (l, r)-symmetric bi-derivation of X. □

THEOREM 3.14. *Let X be an associative medial BCH-algebra. Then the symmetric map $D : X \times X \to X$ defined by $D(x, y) = x + y$ for all $x, y \in X$ is a symmetric bi-derivation of X.*

Proof. By the above proposition, D is a (l, r)-symmetric bi-derivation of X. For all $x, y, z \in X$, we have

$$\begin{aligned} D(x*y,z) &= (x*y)+z = (x*y)*(0*z) \\ &= (x*(0*z))*y = ((x*0)*z)*y \quad (\because X \ \ is \ \ associative) \\ &= (x*z)*y = (x*y)*z. \qquad\qquad\qquad (1) \end{aligned}$$

Also, we have for any $x, y, z \in X$,

$$\begin{aligned} (x*D(y,z)) \wedge (D(x,z)*y) &= x*D(y,z) \qquad (\because x \wedge y = y*(y*x) = x) \\ &= x*(y+z) = x*(y*(0*z)) \\ &= x*((y*0)*z) \qquad (\because X \ \ is \ \ associative) \\ &= x*(y*z) \\ &= (x*y)*z. \qquad (2) \ \ (\because X \ \ is \ \ associative) \end{aligned}$$

From (1) and (2), $D(x * y), z) = (x * D(y, z)) \wedge (D(x, z) * y)$ for all $x, y, z \in X$. This proves that D is a (r, l)-symmetric bi-derivation, and so a symmetric bi-derivation of X. □

PROPOSITION 3.15. *Let X be a medial BCH-algebra and let D be a symmetric map. Then we have for any $x \in X$,*

(1) *if D is a (l, r)-symmetric bi-derivation of X and $(x * z) * (y * z) = x * y$, then $D(x, y) = D(x, y) \wedge x$,*
(2) *if D is a (r, l)-symmetric bi-derivation of X, then $D(x, y) = x \wedge D(x, y)$ for all $x, y \in X$ if and only if $D(0, y) = 0$ for all $x \in X$.*

Proof. (1) Let D be a (l,r)-symmetric bi-derivation of X. Then we have

$$\begin{aligned} D(x,y) &= D(x*0,y) = D(x,y)*0) \wedge (x*D(0,y)) \\ &= D(x,y) \wedge (x*D(0,y)) \\ &= (x*D(0,y))*((x*D(0,y))*D(x,y)) \\ &= (x*D(0,y))*((x*D(x,y))*D(0,y)) \\ & \qquad\qquad (\because (x*y)*z=(x*z)*y) \\ &= x*(x*D(x,y)) \qquad (\because (x*z)*(y*z)=x*y) \\ &= D(x,y) \wedge x. \end{aligned}$$

(2) Let D be a (r,l)-symmetric bi-derivation of X and $D(0,y)=0$ for all $y \in X$. Then we have

$$\begin{aligned} D(x,y) &= D(x*0,y) \\ &= (x*D(0,y)) \wedge (D(x,y)*0) \\ &= (x*0) \wedge D(x,y) \\ &= x \wedge D(x,y). \end{aligned}$$

Conversely, if $D(x,y) = x \wedge D(x,y)$ for all $x,y \in X$, then

$$\begin{aligned} D(0,y) &= 0 \wedge D(0,y) \\ &= D(0,y)*(D(0,y)*0) \\ &= D(0,y)*D(0,y) = 0. \end{aligned}$$

□

Proposition 3.16. *Let X be a medial BCH-algebra and let $D : X \times X \to X$ be a (l,r)-symmetric bi-derivation of X. Then*
(1) $D(a,y) = D(0,y)*(0,a) = D(0,y)+a$ for all $a,x,y \in X$,
(2) $D(a+b,y) = D(a,y)+D(b,y)-D(0,y)$ for all $a,b,x,y \in X$,
(3) $D(a,y) = a$ if and only if $D(0,y) = 0$ for all $a,y \in X$.

Proof. (1) Let (l,r)-symmetric bi-derivation of X and let $a = 0*(0*a)$. Then we have

$$\begin{aligned} D(a,y) &= D(0*(0*a),y) \\ &= (D(0,y)*(0*a)) \wedge (0*D(0*a,y)) \\ &= D(0,y)*(0*a) \qquad (\because x \wedge y = x) \\ &= D(0,y)+a \end{aligned}$$

for for any $a,x,y \in X$,

(2) By (1), we get for any $a, b, y \in X$,

$$\begin{aligned} D(a+b, y) &= D(0, y) + a + b \\ &= D(0, y) + a + D(0, y) + b - D(0, y) \\ &= D(a, y) * D(b, y) - D(0, y). \end{aligned}$$

(3) Let $D(a, y) = a$ for any $a, y \in X$. Putting $a = 0$, then we get $D(0, y) = 0$ for any $y \in X$. Conversely, if $D(0, y) = 0$, then $D(a, y) = D(0, y) + a = 0 + a = a$. □

PROPOSITION 3.17. *Let X be a medial BCH-algebra and let $D : X \times X \to X$ be a (r, l)-symmetric bi-derivation of X. Then*
(1) $D(a, y) \in G(X)$ for any $a \in G(X)$,
(2) $D(a, y) = a * D(0, y) = a + D(0, y)$ for any $a, y \in X$,
(3) $D(a + b, y) = D(a, y) + D(b, y) - D(0, y)$ for all $a, b, y \in X$,
(4) $D(a, y) = a$ for any $a, y \in X$ if and only if $D(0, y) = 0$.

Proof. (1) Let $a \in G(X)$. Then $0 * a = a$, and so

$$\begin{aligned} D(a, y) &= D(0 * a, y) \\ &= (0 * (D(a, y)) \wedge (D(0, y) * a) \\ &= (D(0, y) * a) * ((D(0, y) * a) * (0 * D(a, y))) \\ &= 0 * D(a, y). \end{aligned}$$

This implies that $D(a, y) \in G(X)$.

(2) For any $a, y \in X$, we get

$$\begin{aligned} D(a, y) &= D(a * 0, y) \\ &= (a * (D(0, y)) \wedge (D(a, y) * 0) \\ &= (a * (D(0, y)) \wedge D(a, y) \\ &= D(a, y) * (D(a, y) * (a * D(0, y))) \\ &= a * D(0, y). \end{aligned}$$

Again, for any $a, y \in X$, we get

$$\begin{aligned} D(a, y) &= a * D(0, y) \\ &= (a * (D(0, y)) \wedge (D(a, y) * 0) \\ &= a * D(0 * (D(0, y)) \wedge (D(0, y) * 0) \\ &= a * (0 * D(0, y)) \\ &= a + D(0, y). \end{aligned}$$

(3) For any a, b, y, we have

$$\begin{aligned} D(a+b,y) &= a+b+D(0,y) \\ &= a+D(0,y)+b+D(0,y)-D(0,y) \\ &= D(a,y)+D(b,y)-D(0,y). \end{aligned}$$

(4) If $D(0.y)=0$, then $D(a,y)=D(a*0,y)=a*D(0,y)=a*0=a$ by (2). Conversely, if $D(a,y)=a$ for any $a \in X$, we get $D(0,y)=0$. □

Definition 3.18. Let X be a BCH-algebra and let $D: X \times X \to X$ be a symmetric mapping. If $D(0,z)=0$, for all $z \in X$, D is called *componentwise regular*. In particular, if $D(0,0)=d(0)=0$, D is called *d-regular*.

Proposition 3.19. *Let D be a (r,l)-symmetric bi-derivation of X and $0*x=0$ for all $x \in X$. Then D is d-regular.*

Proof. Let D be a system bi-derivation of X and $0*x=0$ for all $x \in X$. Then we have

$$\begin{aligned} D(0,0) = D(0*x,0) &= (0*D(x,0)) \wedge (D(0,0)*x) \\ &= 0 \wedge (D(0,0)*x) \\ &= 0 \end{aligned}$$

Hence D is d-regular. □

Theorem 3.20. *Let D be an (l,r)-symmetric bi-derivation of X. If there exists $a \in X$ such that $D(x,z)*a=0$ for all $x,z \in X$, then D is componentwise regular.*

Proof. Let $D(x,y)*a=0$ for all $x,z \in X$. Then

$$\begin{aligned} 0 = D(x*a,z)*a &= ((D(x*z)*a) \wedge (D(0,0)*x)*a \\ &= (0 \wedge (D(0,0)*x))*a \\ &= 0*a, \end{aligned}$$

that is, $0 \leq a$,and so

$$\begin{aligned} D(0,z) &= D(0*a,z) \\ &= (D(0,z)*a) \wedge (0*D(a,z)) \\ &= 0 \wedge (0*D(a,z)) = 0. \end{aligned}$$

Hence d is componentwise regular. □

Corollary 3.21. *Let D be an (l,r)-symmetric bi-derivation of X. If there exists $a \in X$ such that $D(x,z)*a=0$ for all $x,z \in X$, then D is d-regular.*

Theorem 3.22. *Let D be an (r,l)-symmetric bi-derivation of X. If there exists $a \in X$ such that $a * D(x,z) = 0$ for all $x, z \in X$, then D is componentwise regular.*

Proof. Let $D(x,y) * a = 0$ for all $x, z \in X$. Then

$$\begin{aligned} 0 = a * D(x * a, z) &= a * ((a * D(x * z)) \wedge (D(a,z) * x)) \\ &= a * (0 \wedge (D(a,z) * x)) \\ &= a * 0, \end{aligned}$$

This shows that

$$\begin{aligned} D(0,z) &= D(a * 0, z) \\ &= (a * D(0,z)) \wedge (D(a,z) * 0) \\ &= 0 \wedge D(a,z) = 0. \end{aligned}$$

Hence D is componentwise regular. □

Corollary 3.23. *Let D be an (r,l)-symmetric bi-derivation of X. If there exists $a \in X$ such that $a * D(x,z) = 0$ for all $x, z \in X$, then D is d-regular.*

Let D be a symmetric bi-derivation of X and $a \in X$. Define a set $Fix_a(X)$ by

$$Fix_a(X) := \{x \in X \mid D(x,a) = x\}$$

for all $x \in X$.

Proposition 3.24. *Let D be a symmetric bi-derivation of X. Then $Fix_a(X)$ is a subalgebra of X.*

Proof. Let $x, y \in Fix_a(X)$. Then we have $D(x,a) = x$ and $D(y,a) = y$, and so

$$\begin{aligned} D(x * y, a) &= (D(x,a) * y) \wedge (x * D(y,a)) \\ &= (x * y) \wedge (x * y) \\ &= (x * y) * ((x * y) * (x * y)) \\ &= (x * y) * 0 = x * y. \end{aligned}$$

Hence we get $x * y \in Fix_a(X)$. This completes the proof. □

Proposition 3.25. *Let D be a symmetric bi-derivation of X. If $x, y \in Fix_a(X)$, we obtain $x \wedge y \in Fix_a(X)$.*

Proof. Let $x, y \in Fix_a(X)$. Then we have $D(x,a) = x$ and $D(y,a) = y$, and so

$$\begin{aligned} D(x \wedge y, a) &= D(y * (y * x), a) = (D(y,a) * (y * x)) \wedge (y * D(y * x, a)) \\ &= (y * (y * x)) \wedge (y * ((D(y * a) * x) \wedge (y * D(x,a)))) \\ &= (y * (y * x)) \wedge (y * ((y * x) \wedge (y * x))) \\ &= y * (y * x) \wedge y * (y * x) \\ &= y * (y * x) = x \wedge y. \end{aligned}$$

Hence we get $x \wedge y \in Fix_a(X)$. This completes the proof. □

Proposition 3.26. *Let X be a commutative BCH-algebra and d a trace of D. Then, if $x \leq y$ for all $x, y \in X$, then $d(x \wedge y) = d(x)$.*

Proof. Let $x \leq y$. Then we get $x * y = 0$ and

$$\begin{aligned} d(x \wedge y) &= D(x \wedge y, x \wedge y) \\ &= D(y * (y * x), y * (y * x)) \\ &= D(x * (x * y), x * (x * y)) \\ &= D(x,x) = d(x). \end{aligned}$$

This completes the proof. □

Definition 3.27. Let X be a BCH-algebra. A self-map d on X is said to be *isotone* if $x \leq y$ implies $d(x) \leq d(y)$ for $x, y \in X$.

Let $Der(X)$ denote the set of all (l,r)-symmetric bi-derivation on X. Define the binary operation "$\wedge$" on $Der(X)$ as follows:

$$(D_1 \wedge D_2)(x,y) = D_1(x * y) \wedge D_2(x,y)$$

for any $D_1, D_2 \in Der(X)$ and $x, y \in X$.

Proposition 3.28. *Let D_1 and D_2 are (l,r)-symmetric bi-derivations on X. Then $D_1 \wedge D_2$ is also a (l,r)-symmetric bi-derivation of X.*

Proof. Let D_1 and D_2 are (l,r)-symmetric bi-derivations on X. Then

$$(D_1 \wedge D_2)(x * y, z) = ((D_1 \wedge D_2)(x,z) * y) \wedge (x * ((D_1 \wedge D_2)(y,z))).$$

$$\begin{aligned} (D_1 \wedge D_2)(x * y, z) &= D_1(x * y, z) \wedge D_2(x * y, z) \\ &= D_2(x * y, z) * (D_2(x * y, z) * D_1(x * y, z)) \\ &= D_1(x * y, z) \\ &= (D_1(x,z) * y) \wedge (x * D_1(y,z)) \\ &= (x * D_1(y,z)) * ((x * D_1(y,z)) * (D_1(x,z) * y)) \\ &= D_1(x,z) * y \qquad (1) \end{aligned}$$

$$
\begin{aligned}
&((D_1 \wedge D_2)(x,z) * y) \\
&= (x * (D_1 \wedge D_2)(y,z) * ((x * (D_1 \wedge D_2)(y,z)) * ((D_1 \wedge D_2)(x,z) * y)) \\
&= (D_1(x,z) \wedge D_2(x * y, z) * (D_2(x * y, z) * D_1(x * y, z)) \\
&= D_1(x * y, z) \\
&= (D_1 \wedge D_2)(x,z) * y \\
&= (D_1(x,z) \wedge D_2(x,z)) * y \\
&= (D_2(x,z) * (D_2(x,z) * D_1(x,z))) * y \\
&= D_1(x,z) * y \qquad (2)
\end{aligned}
$$

Combining (1) and (2), we prove that $D_1 \wedge D_2$ is a (l,r)-symmetric bi-derivation of X. □

PROPOSITION 3.29. *The binary composition "$\wedge$" defined on $Der(X)$ is associative.*

Proof. Let D_1, D_2 and D_2 are (l,r)-symmetric bi-derivations on X. Then

$$
\begin{aligned}
&((D_1 \wedge D_2) \wedge D_3)(x * y, z) \\
&= ((D_1 \wedge D_2)(x * y, z)) \wedge D_3)(x * y, z)) \\
&= (D_1(x,z) * y) \wedge D_3(x * y, z)) \\
&= (D_3(x * y, z) * (D_3(x,z) * D_1(x,z) * y) \\
&= D_1(x,z) * y \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
&(D_1 \wedge (D_2 \wedge D_3))(x * y, z) \\
&= (D_1(x * y, z)) \wedge ((D_2 \wedge D_3)(x * y, z)) \\
&= (D_1(x * y, z)) \wedge (D_2(x,z) * y) \\
&= (D_2(x,z) * y) * ((D_2(x,z) * y) * (D_1(x * y, z))) \\
&= D_1(x * y, z) \\
&= (D_1(x,z) * y) \wedge (x * D_1(y,z)) \\
&= (x * D_1(y,z)) * ((x * D_1(y,z)) * (D_1(x,z) * y) \\
&= D_1(x,z) * y. \qquad (1)
\end{aligned}
$$

Combining (1) and (2), we have $(D_1 \wedge D_2) \wedge D_3 = D_1 \wedge (D_2 \wedge D_3)$, which implies that "$\wedge$" is associative. □

Combining the above two propositions, we obtain the following theorem.

THEOREM 3.30. *$Der(X)$ is a a semigroup under the binary composition "$\wedge$".*

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