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# APPROXIMATION OF INTEGRATED SEMIGROUPS

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Abstract. The purpose of this paper is to show an integrated semigroup on a Banach space can be approximated by a sequence of integrated semigroups acting on different Banach spaces.

## 1. Introduction

The initial value problem in a Banach space $X$

$$u'(t) = Au(t), \quad u(0) = x$$

has been extensively studied if $A$ is the generator of a $C_0$ semigroup. Hille-Yosida theorem gives the necessary and sufficient conditions in order that $A$ is the generator of a $C_0$ semigroup [4]. One of these conditions is the density of the domain of $A$. But there are many examples that is formulated in the above problem without the density of the domain of $A$ (see [3]). In this case the concept of integrated semigroup introduced by Arendt [1] is very useful to treat the above problem.

In this paper we study the approximation of an integrated semigroup on a Banach space $X$ by a sequence of the integrated semigroups on Banach spaces $X_n$. In order to prove our result, we use Theorem 2.2 in [5] that the convergence of the sequence of functions $\{f_n : [0, \infty) \to X\}$ is equivalent to the convergence of their Laplace transforms and the equicontinuity of $\{f_n\}$.

Let $X$ and $X_n$ be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_n$, $n = 1, 2, \cdots$, respectively. For each $n$, there exist bounded linear operators $P_n : X \to X_n$ and $E_n : X_n \to X$ satisfying

(i) $\|P_n\| \leq M_1$ and $\|E_n\|_n \leq M_2$, where $M_1$ and $M_2$ are independent of $n$.

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(ii) $\lim_{n\to\infty} \|E_n P_n x - x\| = 0$ for every $x \in X$.
(iii) $P_n E_n = I_n$, where $I_n$ is the identity operator on $X_n$.

In general we do not have $X_n \subset X$. If one has numerical approximation in mind, then the spaces $X_n$ are finite dimensional.

Throughout this paper, $X$ is a Banach space and $B(X)$ is the space of all bounded linear operators from $X$ to $X$. For a linear operator $A$, we denote the domain, the range, the resolvent set and the resolvent by $D(A)$, $Ran(A)$, $\rho(A)$ and $R(\lambda,\ A)$, respectively.

## 2. Approximation

First we recall the definition of integrated semigroups.

Definition 2.1. A linear operator $A$ on a Banach space $X$ is called the generator of an integrated semigroup if there exist constants $M$, $\omega \geq 0$ and a strongly continuous function $S : [0,\ \infty) \to B(X)$ with $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ such that $(\omega,\ \infty) \subset \rho(A)$ and $R(\lambda,\ A)x = \lambda \int_0^\infty e^{-\lambda t} S(t)x dt$ for $\lambda > \omega$ and $x \in X$.

In this case, $\{S(t)\}_{t\geq 0}$ is called the integrated semigroup generated by $A$.

It is known in [2] that a closed linear operator $A$ in $X$ is the generator of a locally Lipschitz continuous integrated semigroup on $X$ if and only if there exist constants $M,\ \omega \geq 0$ such that

$$(\omega,\ \infty) \subset \rho(A) \quad \text{and} \quad \|(\lambda I - A)^{-k}\| \leq \frac{M}{(\lambda - \omega)^k}$$

for $\lambda > \omega$ and $k \geq 1$, and every locally Lipschitz continuous integrated semigroup is exponentially bounded.

Main result of this paper is given by the following theorem.

Theorem 2.2. *Let $A$ be the generator of an integrated semigroup $\{S(t)\}_{t\geq 0}$ on $X$ satisfying $\|S(t)\| \leq Me^{\omega t}$ for some constants $M,\ \omega \geq 0$ and all $t \geq 0$. Let $\{T_n\}$ be a sequence of linear operators with $T_n \in B(X_n)$ and let $\{h_n\}$ be a positive null sequence with the following properties.*

(i) $\|T_n\|_n \leq Me^{\omega k h_n}$ *for* $k \geq 0$ *and* $n \geq 1$.
(ii) *For* $x \in D(A)$ *there exists a sequence* $\{x_n\}$ *with* $x_n \in X_n$ *such that* $\lim_{n\to\infty} E_n x_n = x$ *and* $\lim_{n\to\infty} E_n A_n x_n = Ax$, *where* $A_n = (T_n - I_n)/h_n$.

*Then*

$$\lim_{n\to\infty}\int_0^t E_nT_n^{[s/h_n]}P_nxds = S(t)x \quad \text{for } x\in X$$

*and the convergence is uniform on bounded* $t-$*intervals, where* $[r]$ *is the integer part of* $r\geq 0$.

*Proof.* Since $A_n\in B(X_n)$, $A_n$ is the generator of a uniformly continuous semigroup $\{e^{tA_n}\}_{t\geq 0}$ on $X_n$ and

$$\begin{aligned}\|e^{tA_n}\|_n &\leq e^{-t/h_n}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t}{h_n}\right)^k\|T_n^k\|_n\\ &\leq Me^{t/h_n(e^{\omega h_n}-1)}\leq Me^{te^{\omega}}.\end{aligned}$$

Choose $a>e^{\omega}$. Then $\|e^{tA_n}\|\leq Me^{at}$ for all $t\geq 0$. By Hille-Yosida theorem, $(a,\ \infty)\subset\rho(A_n)$ and $\|R(\lambda,\ A_n)\|\leq M/(\lambda-a)$ for $\lambda>a$.

For $y\in Ran(\lambda I-A)$, there exists $x\in D(A)$ such that $y=(\lambda I-A)x$. By hypothesis there exist $x_n\in X_n$ such

$$\lim_{n\to\infty}E_nx_n=x \quad\text{and}\quad \lim_{n\to\infty}E_nA_nx_n=Ax.$$

Set $(\lambda I_n-A_n)x_n=y_n$. Then we have

$$\lim_{n\to\infty}E_ny_n=\lim_{n\to\infty}E_n(\lambda I_n-A_n)x_n=(\lambda I-A)x=y.$$

So we have for $\lambda>a$

$$\begin{aligned}&\|E_nR(\lambda,\ A_n)P_ny-R(\lambda,\ A)y\|\\ &\leq\|E_nR(\lambda,\ A_n)P_ny-E_nR(\lambda,\ A_n)y_n\|\\ &\quad+\|E_nR(\lambda,\ A_n)y_n-R(\lambda,\ A)y\|\\ &\leq M_2\|R(\lambda,\ A_n)P_ny-R(\lambda,\ A_n)y_n\|_n+\|E_nx_n-x\|\\ &\leq\frac{M_2M}{\lambda-a}\|P_ny-y_n\|_n+\|E_nx_n-x\|\\ &=\frac{M_2M}{\lambda-a}\|P_ny-E_nP_ny_n\|_n+\|E_nx_n-x\|\\ &\leq\frac{M_1M_2M}{\lambda-a}\|y-E_ny_n\|+\|E_nx_n-x\| \ \to\ 0 \ \text{ as }\ n\to\infty.\end{aligned}$$

By the density of $Ran(\lambda I-A)$, we have

$$\lim_{n\to\infty}E_nR(\lambda,\ A_n)P_nx=R(\lambda,\ A)x \quad\text{for } x\in X.$$

Let $x \in X$. Then

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\int_0^t T_n^{[s/h_n]}P_nxdsdt\\
&=\int_0^\infty\int_s^\infty e^{-\lambda t}T_n^{[s/h_n]}P_nxdtds\\
&=\frac{1}{\lambda}\int_0^\infty e^{-\lambda s}T_n^{[s/h_n]}P_nxds\\
&=\frac{1}{\lambda}\sum_{k=0}^\infty\int_{kh_n}^{(k+1)h_n} e^{-\lambda s}T_n^kP_nxds\\
&=\frac{1-e^{-\lambda h_n}}{\lambda^2}\sum_{k=0}^\infty e^{-\lambda kh_n}T_n^kP_nx\\
&=\frac{1-e^{-\lambda h_n}}{\lambda^2}\left(I_n-e^{-\lambda h_n}T_n\right)^{-1}P_nx\\
&=\frac{1-e^{-\lambda h_n}}{\lambda^2}\left(I_n-e^{-\lambda h_n}(I_n+h_nA_n)\right)^{-1}P_nx\\
&=\frac{1-e^{-\lambda h_n}}{\lambda^2}\frac{e^{\lambda h_n}}{h_n}\left(\frac{e^{\lambda h_n}-1}{h_n}I_n-A_n\right)^{-1}P_nx.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^\infty e^{-\lambda t}\int_0^t E_nT_n^{[s/h_n]}P_nxdsdt\\
&=\lim_{n\to\infty}\frac{1-e^{-\lambda h_n}}{\lambda^2}\frac{e^{\lambda h_n}}{h_n}E_n\left(\frac{e^{\lambda h_n}-1}{h_n}I_n-A_n\right)^{-1}P_nx\\
&=\frac{1}{\lambda}(\lambda I-A)^{-1}x=\int_0^\infty e^{-\lambda t}S(t)xdt.
\end{aligned}$$

We have proved that the Laplace transforms of $\int_0^t E_nT_n^{[s/h_n]}P_nxds$ converge to the Laplace transform of the integrated semigroup $\{S(t)\}_{t\geq 0}$.

Next we will show the equicontinuity of $\{\int_0^t E_nT_n^{[s/h_n]}P_nxds\}$. For $0\leq s<t\leq T$,

$$\begin{aligned}
&\|\int_0^t T_n^{[r/h_n]}P_nxdr-\int_0^s T_n^{[r/h_n]}P_nxdr\|_n\\
&=\|\int_s^t T_n^{[r/h_n]}P_nxdr\|_n
\end{aligned}$$

$$\leq \int_s^t Me^{\omega[r/h_n]h_n}\|P_n x\|_n dr$$
$$\leq M\int_s^t e^{\omega r}dr\|P_n x\|_n$$
$$\leq Me^{\omega T}\|P_n\|_n|t-s|.$$

Hence $\{\int_0^t E_n T_n^{[s/h_n]} P_n x ds\}$ is equicontinuous. By Theorem 2.2 in [5] we have the result. □

Example 2.3. *Let $X = C([0,\ 1])$ with the supremum norm and let $A : D(A) \subset X \to X$ be a linear operator defined by $Au = -u'$ with $D(A) = \{u \in X : u(0) = 0,\ u' \in X\}$.*

Then the closure of $D(A)$ is $C_0([0,\ 1])$, which is not dense in X. For $\lambda > 0$ and $v \in X$, define

$$u(t) = \int_0^t e^{-\lambda s} v(t-s)ds, \quad t \in [0,\ 1].$$

Then $u \in D(A)$, $(\lambda I - A)u = v$ and

$$\begin{aligned} |u(t)| &\leq \int_0^t e^{-\lambda s}|v(t-s)|ds \\ &\leq \|v\|\int_0^t e^{-\lambda s}ds \leq \frac{1}{\lambda}\|v\| \end{aligned}$$

So $(0,\ \infty) \subset \rho(A)$ and $\|R(\lambda,\ A)\| \leq 1/\lambda$, that is, $A$ is a Hille-Yosida operator. By Theorem 2.4 in [2], $A$ is the generator of an integrated semigroup $\{S(t)\}_{t\geq 0}$.

Let $X_n = R^n$ with the supremum norm. Define $P_n : X \to R^n$ and $E_n : R^n \to X$ by

$$P_n u = (u(1/n),\ u(2/n),\ \cdots,\ u(n/n)) \text{ and } E_n x^{(n)} = f_n,$$

where $u \in X$, $x^{(n)} = (x_1^{(n)},\ x_2^{(n)},\ \cdots,\ x_n^{(n)}) \in R^n$ and $f_n(0) = x_1^{(n)}$, $f_n(k/n) = x_k^{(n)}$, $k = 1,\ 2,\ \cdots,\ n$ and linear between two consecutive points. Then $\|P_n\| \leq 1$, $\|E_n\| \leq 1$, $P_n E_n = I_n$ and $\lim_{n\to\infty} E_n P_n u = u$ for all $u \in X$.

Define $A_n : R^n \to R^n$ by

$$A_n x^{(n)} = n(-x_1^{(n)},\ x_1^{(n)} - x_2^{(n)},\ \cdots,\ x_{n-1}^{(n)} - x_n^{(n)}).$$

Then $A_n$ is linear and $\|A_n\|_n \leq 2n$. Let $u \in D(A)$. Then

$$\begin{aligned} &A_n P_n u \\ &= n(-u(1/n),\ u(1/n) - u(2/n),\ \cdots,\ u((n-1)/n) - u(n/n)) \\ &= -(u'(c_1),\ u'(c_2),\ \cdots,\ u'(c_n)) \end{aligned}$$

for some $c_i \in ((i-1)/n,\ i/n)$, $i = 1,\ 2,\ \cdots,\ n$ and

$$P_n A u = -(u'(1/n),\ u'(2/n),\ \cdots,\ u'(n/n)).$$

Since $u'$ is continuous, $\lim_{n\to\infty} \|A_n P_n u - P_n A u\|_n = 0$. Hence we have

$$\begin{aligned} &\|E_n A_n P_n u - Au\| \\ &\leq \|E_n A_n P_n u - E_n P_n A u\| + \|E_n P_n A u - Au\| \\ &\leq M_2 \|A_n P_n u - P_n A u\|_n + \|E_n P_n A u - Au\| \to 0 \text{ as } \ n \to \infty. \end{aligned}$$

Choose a sequence $\{h_n\}$ with $0 < h_n < 1/n$. Then

$$\begin{aligned} T_n x^{(n)} &= x^{(n)} + h_n A_n x^{(n)} \\ &= \Big( (1 - nh_n) x_1^{(n)},\ nh_n x_1^{(n)} + (1 - nh_n) x_2^{(n)}, \\ &\qquad \cdots,\ nh_n x_{n-1}^{(n)} + (1 - nh_n) x_n^{(n)} \Big) \end{aligned}$$

Then $\|T_n\|_n \leq 1$ and so we have

$$\lim_{n\to\infty} \int_0^t E_n T_n^{[s/h_n]} P_n u ds = S(t)u \text{ for } \ u \in X.$$

That is, the values computed by the difference equations converge to the integrated semigroup.

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