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ON THE FAILURE OF GORENSTEINESS FOR THE SEQUENCE $(1, 125, 95, 77, 70, 77, 95, 125, 1)$

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Abstract. In [9], the authors determine an infinite class of non-unimodal Gorenstein sequence, which includes the example

$$\bar{h}_1 = (1, 125, 95, 77, 71, 77, 95, 125, 1).$$

They raise a question whether there is a Gorenstein algebra with Hilbert function

$$\bar{h}_2 = (1, 125, 95, 77, 70, 77, 95, 125, 1),$$

which has remained an open question. In this paper, we prove that there is no Gorenstein algebra with Hilbert function $\bar{h}_2$.

1. Introduction

Throughout this paper, k will denote an infinite field, and $R = k[x_1, ..., x_r]$ a graded polynomial ring in r variables. Let $A = R/I$ be an Artinian graded Gorenstein algebra of socle degree e. The Hilbert function of A is a finite symmetric sequence $\bar{h} = (1, r, h_2, h_3, \cdots, h_e)$ and we call it h-vector of A. A Gorenstein h-vector is called unimodal up to the degree $i+1$ if $1 \leq h_1 \leq \cdots \leq h_i \leq h_{i+1}$. A Gorenstein h-vector is called unimodal if it is unimodal up to the degree $[\frac{e}{2}]$.

One of important questions is to find the possible Hilbert function of Gorenstein algebra. The complete answer to this question is known for the case $r \leq 3$ (see [8, 11, 12]). In this case, the h-vector of Gorenstein sequence is unimodal.

Another important related question is to investigate the unimodality of the Gorenstein h-vector. In codimension ≥ 5, not all Gorenstein Hilbert functions are unimodal ([1, 3, 5, 11]). It is not known whether

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all Gorenstein algebras of codimension 4 are unimodal. In this context, one may ask if a given non-unimodal sequence is Gorenstein or if there is a good lower bounds for the growth of an Artinian Gorenstein Hilbert function in the first half. In [9], using the classical results of Macaulay, Green and Stanley about binomial expansions and h-vectors, the authors prove a very important lower bound for h_{i+1}, for any integer $1 \leq i \leq \frac{e}{2} - 1$.

THEOREM 1.1. [9, Theorem 2.4] *Suppose that* $\bar{h} = (1, h_1 = r, h_2, \ldots, h_{e-2}, h_{e-1}, h_e)$ *is the h-vector of an Artinian Gorenstein algebra over* $R = k[x_1, \ldots, x_r]$. *Assume that i is an integer satisfying* $1 \leq i \leq \frac{e}{2} - 1$. *Then,*

$$h_{i+1} \geq \left((h_i)_{(e-i)}\right)_{-1}^{-1} + \left((h_i)_{(e-i)}\right)_{-(e-2i-1)}^{-(e-2i)}.$$

Using this theorem the authors give a very beautiful short proof to the theorem of Stanley that all Gorenstein h-vectors of codimension $r \leq 3$ are unimodal. Also this result is used to show that all Gorenstein h-vectors of codimension 4 and socle degree $e > \frac{1}{6}(i^2 + 12i + 2)$ are unimodal up to degree $i + 1$.

The bound in Theorem 1.1 is often sharp, but one would not expect it to be sharp "too often", since a sharp bound would probably make it easy to decide if non-unimodal Gorenstein Hilbert functions exist. In [9], the authors construct a particular family of Gorenstein algebras with trivial extensions, including the Gorenstein sequence

$$(1, 125, 95, 77, 71, 77, 95, 125, 1),$$

and raise a question whether there is a Gorenstein algebra with Hilbert function

$$\bar{h}_2 = (1, 125, 95, 77, 70, 77, 95, 125, 1),$$

which has remained an open question. In this paper, we prove that there is no Gorenstein algebra with Hilbert function $\bar{h}_2$.

2. Gorenstein Hilbert functions

Recall that if n and i are positive integers, then n can be written uniquely in the form

$$n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$ (see Lemma 4.2.6, [4]).

Following [2], we define, for any integers a and b,

$$\left(n_{(i)}\right)_b^a = \binom{n_i + a}{i + b} + \binom{n_{i-1} + a}{i - 1 + b} + \cdots + \binom{n_j + a}{j + b}$$

where $\binom{m}{n} = 0$ for either $m < n$ or $n < 0$.

THEOREM 2.1 ([6], Chapter 5 in [7]). *Let L be a general linear form in R and we denote by h_d the degree d entry of the Hilbert function of R/I and ℓ_d the degree d entry of the Hilbert function of $R/(I, L)$. Then, we have the following inequalities.*

(a) Macaulay's Theorem: $h_{d+1} \leq \left((h_d)_{(d)}\right)_1^1$.

(b) Green's Hyperplane Restriction Theorem: $\ell_d \leq \left((h_d)_{(d)}\right)_0^{-1}$.

LEMMA 2.2 ([11]). *Let $A = R/I$ be an Artinian Gorenstein algebra, and let $L \notin I$ be a linear form of R. Then the h-vector of A can be written as*

$$h := (h_0, h_1, \ldots, h_s) = (1, b_1 + \ell_1, \ldots, b_{s-1} + \ell_{s-1}, b_s = 1)$$

where $b = (b_1 = 1, b_2, \ldots, b_{s-1}, b_s = 1)$ is the h-vector of $R/(I : L)$ (with the indices shifted by 1), which is a Gorenstein algebra, and

$$\ell = (\ell_0, \ell_1, \ldots, \ell_{s-1}) \quad \textit{with} \quad \ell_0 = 1$$

is the h-vector of $R/(I, L)$.

Let S be an Artinian graded k-algebra with socle degree e. Then $E = \mathrm{Hom}(S.k)$ is the injective envelope of k, regarded as the residue class field of S. E has the structure of S-module in the usual way, $(x\phi)(y) = \phi(xy)$, where $x \in S$ and $\phi \in E$. Let $R = S \times E$, endowed with component wise addition and the multiplication $(x, \phi) \cdot (y, \psi) = (xy, x\psi + y\phi)$. With these notations, we have

LEMMA 2.3 (Trivial Extension). *A graded algebra R is a 0-dimensional Gorenstein k-algebra. Moreover, its Hilbert function satisfies*

$$H(R, d) = H(S, d) + H(S, e + 1 - d), \quad (0 \leq d \leq e + 1).$$

This is a way to construct Gorenstein algebra and we call R trivial extension algebra induced by S.

The set of all Artinian Gorenstein algebras of codimension $\leq r$ and socle degree e arises, via Macaulay's inverse systems (or divided powers in characteristic p), by looking at the annihilators of forms $F \in R_e$. So the set of all such algebras is parametrized by the projective space $\mathbb{P}(R_e)$. Macaulay's inverse system is another way of constructing Gorenstein algebras. Using it, we have the following result. [13, Proposition 8] Suppose that $(1, h_1, h_2, \ldots, h_{e-1}, 1)$ is a Gorenstein h-vector. Then also

$$(1, h_1 + 1, h_2 + 1, \ldots, h_{e-1} + 1, 1)$$

is always a Gorenstein h-vector.

3. The sequence $(1, 125, 95, 77, 70, 77, 95, 125, 1)$ is not a Gorenstein sequence.

In [9], Migliore-Nagel-Zanello constructed a Gorenstein sequence

$$(1, 125, 95, 77, 71, 77, 95, 125, 1),$$

Indeed, let $S = k[x_1, x_2, x_3, x_4]/(x_1, x_2, x_3, x_4)^8$ and $E = \mathrm{Hom}_k(S, k)$. Note that S has the Hilbert function

d	0	1	2	3	4	5	6	7	8
$H(S,d)$	1	4	10	20	35	56	84	120	0

Then E has the structure of an S-module in the usual way, i.e.,

$$(f\varphi)g = \varphi(fg),$$

where $\varphi \in E$ and $f, g \in R$. Let $T = R \times E$, endowed with componentwise addition and the multiplication

$$(x, \varphi) \cdot (y, \psi) = (xy, x\psi + y\varphi).$$

Then T is a 0-dimensional Gorenstein k-algebra (see [11, Example 4.3]) with Hilbert function $H(T, d) = H(S, d) + H(S, 8 - d)$ and thus, we have

(3.1)

d	1	2	3	4	5	6	7	8	9
$H(T,d)$	1	124	94	76	70	76	94	124	1

Since there is a Gorenstein algebra with Hilbert function (3.1), it follows from [13, Proposition 8], the following h-vector

$$(1, 125, 95, 77, 71, 77, 95, 125, 1)$$

is a Gorenstein sequence. Note that this sequence satisfies the equality in Theorem 1.1 in each degree $d \leq 3$. For the case $d = 4$, we have

$$h_4 = 71 > \big((h_3)_{(5)}\big)^{-1}_{-1} + \big((h_3)_{(5)}\big)^{-(2)}_{-1} = 70.$$

In [9], the authors raise a question whether there is a Gorenstein algebra with Hilbert function

$$\bar{h} = (1, 125, 95, 77, 70, 77, 95, 125, 1),$$

which has remained an open question. Our main result shows that there is no Gorenstein algebra with Hilbert function $\bar{h}$.

THEOREM 3.1. $\bar{h} = (1, 125, 95, 77, 70, 77, 95, 125, 1)$ *is not a Gorenstein sequence.*

Proof. Suppose that $\bar{h} = (1, 125, 95, 77, 70, 77, 95, 125, 1)$ is a Gorenstein sequence. Then there is an Artinian Gorenstein algebra $A = R/I$ with the Hilbert function $\bar{h}$. and let $L \notin I$ be a linear form of R. Then the h-vector of A can be written as

$$\bar{h} = (h_0, h_1, \ldots, h_s) = (1, b_1 + \ell_1, \ldots, b_{s-1} + \ell_{s-1}, b_s = 1)$$

where $\bar{b} = (b_1 = 1, b_2, \ldots, b_{s-1}, b_s = 1)$ is the h-vector of $R/(I : L)$ (with the indices shifted by 1), which is a Gorenstein algebra, and

$$\bar{\ell} = (\ell_0, \ell_1, \ell_2 \cdots, \ell_{s-1}) \quad \text{with } \ell_0 = 1, \ell_1 = 124$$

is the h-vector of $R/(I, L)$. So we have the following table:

d	0	1	2	3	4	5	6	7	8
h_d	1	125	95	77	70	77	95	125	1
ℓ_d	1	124	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	0
b_d	0	1	b_2	b_3	b_4	b_5	b_6	b_7	1

Note that Green's Restriction Theorem implies $\ell_i \leq (h_i)_0^{-1}$ for each $i \geq 0$. From the following table

d	0	1	2	3	4	5	6	7	8
h_d	1	125	95	77	70	77	95	125	1
$(h_d)_0^{-1}$	1	124	81	50	35	27	30	36	0

we have $(1, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8) <_{\text{Lex}} (1, 124, 81, 50, 35, 27, 30, 36)$.

Now let $\ell_5 = 27 - i$ for some $i \geq 0$ and $b_5 = h_5 - \ell_5 = 50 + i$. Since $\bar{b}$ is a Gorenstein sequence, $\bar{b}$ should be symmetric. So we have $b_4 = b_5 = 50 + i$ and thus $\ell_4 = h_4 - b_4 = 20 - i$.

For such i with $0 \leq i \leq 19$, consider the following table:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
$\ell_4 = 20 - i$	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	$\cdots$
$(\ell_4)_{+1}^{+1}$	27	26	24	23	22	21	18	16	15	13	12	11	9	8	7	6	4	$\cdots$
$\ell_5 = 27 - i$	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	$\cdots$

By Macaulay's Theorem, we see that the following inequality has to be satisfied,

$$\ell_5 \leq (\ell_4)_{+1}^{+1}$$

and it happens if and only if $i = 0, 1$. So we have $\ell_4 = 19$ or 20.

On the other hand, if $\ell_6 = 30 - j$ then $b_3 = b_6 = h_6 - \ell_6 = 65 + j$ and thus $\ell_3 = h_3 - b_3 = 12 - j \leq 12$. Since the numerical function $(-)^{+1}_{+1}$ is a strictly increasing function, we have

$$19 \leq \ell_4 \leq (\ell_3)^{+1}_{+1} \leq (12_{(3)})^{+1}_{+1} = 17,$$

which is impossible. Therefore, we conclude that the sequence

$$(1, 125, 95, 77, 70, 77, 95, 125, 1)$$

is not a Gorenstein sequence. □

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