# BOUNDEDNESS IN FUNCTIONAL PERTURBED DIFFERENTIAL SYSTEMS 

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Abstract. This paper shows that the solutions to the perturbed differential system

$$
y^{\prime}=f(t, y)+\int_{t_{0}}^{t} g(s, y(s)) d s+h(t, y(t), T y(t))
$$

have bounded property. To show this property, we impose conditions on the perturbed part $\int_{t_{0}}^{t} g(s, y(s)) d s, h(t, y(t), T y(t))$, and on the fundamental matrix of the unperturbed system $y^{\prime}=f(t, y)$.

## 1. Introduction and preliminaries

We consider the nonlinear differential system

$$
\begin{equation*}
x^{\prime}(t)=f(t, x(t)), \quad x\left(t_{0}\right)=x_{0}, \tag{1.1}
\end{equation*}
$$

where $f \in C\left(\mathbb{R}^{+} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right), \mathbb{R}^{+}=[0, \infty)$ and $\mathbb{R}^{n}$ is the Euclidean $n$ space. We assume that the Jacobian matrix $f_{x}=\partial f / \partial x$ exists and is continuous on $\mathbb{R}^{+} \times \mathbb{R}^{n}$ and $f(t, 0)=0$. Also, we consider functional perturbed differential system of (1.1)

$$
\begin{equation*}
y^{\prime}=f(t, y)+\int_{t_{0}}^{t} g(s, y(s)) d s+h(t, y(t), T y(t)), y\left(t_{0}\right)=y_{0} \tag{1.2}
\end{equation*}
$$

where $g \in C\left(\mathbb{R}^{+} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right), h \in C\left(\mathbb{R}^{+} \times \mathbb{R}^{n} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right), g(t, 0)=0$, $h(t, 0,0)=0$, and $T: C\left(\mathbb{R}^{+}, \mathbb{R}^{n}\right) \rightarrow C\left(\mathbb{R}^{+}, \mathbb{R}^{n}\right)$ is a continuous operator.

For $x \in \mathbb{R}^{n}$, let $|x|=\left(\sum_{j=1}^{n} x_{j}^{2}\right)^{1 / 2}$. For an $n \times n$ matrix $A$, define the norm $|A|$ of $A$ by $|A|=\sup _{|x| \leq 1}|A x|$.

[^0]Let $x\left(t, t_{0}, x_{0}\right)$ denote the unique solution of (1.1) with $x\left(t_{0}, t_{0}, x_{0}\right)=$ $x_{0}$, existing on $\left[t_{0}, \infty\right)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$
\begin{equation*}
v^{\prime}(t)=f_{x}(t, 0) v(t), v\left(t_{0}\right)=v_{0} \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
z^{\prime}(t)=f_{x}\left(t, x\left(t, t_{0}, x_{0}\right)\right) z(t), z\left(t_{0}\right)=z_{0} \tag{1.4}
\end{equation*}
$$

The fundamental matrix $\Phi\left(t, t_{0}, x_{0}\right)$ of (1.4) is given by

$$
\Phi\left(t, t_{0}, x_{0}\right)=\frac{\partial}{\partial x_{0}} x\left(t, t_{0}, x_{0}\right)
$$

and $\Phi\left(t, t_{0}, 0\right)$ is the fundamental matrix of (1.3).
We recall some notions of $h$-stability [15].
Definition 1.1. The system (1.1) (the zero solution $x=0$ of (1.1)) is called an $h$-system if there exist a constant $c \geq 1$, and a positive continuous function $h$ on $\mathbb{R}^{+}$such that

$$
|x(t)| \leq c\left|x_{0}\right| h(t) h\left(t_{0}\right)^{-1}
$$

for $t \geq t_{0} \geq 0$ and $\left|x_{0}\right|$ small enough (here $h(t)^{-1}=\frac{1}{h(t)}$ ).
Definition 1.2. The system (1.1) (the zero solution $x=0$ of (1.1)) is called
(hS) $h$-stable if there exists $\delta>0$ such that (1.1) is an $h$-system for $\left|x_{0}\right| \leq \delta$ and $h$ is bounded.

Pinto[14,15] introduced the notion of $h$-stability (hS) which is the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called $h$-systems. Choi, Ryu [5] and Choi, Koo, and Ryu [6] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [8,9,10] and Goo et al. [3] investigated boundedness of solutions for nonlinear perturbed systems.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^{+}$and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class $C^{1}$ with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of $t_{\infty}$-similarity in $\mathcal{M}$ was introduced by Conti [7].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is $t_{\infty}$-similar to a matrix $B(t) \in$ $\mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^{+}$, i.e.,

$$
\int_{0}^{\infty}|F(t)| d t<\infty
$$

such that

$$
\begin{equation*}
\dot{S}(t)+S(t) B(t)-A(t) S(t)=F(t) \tag{1.5}
\end{equation*}
$$

for some $S(t) \in \mathcal{N}$.
The notion of $t_{\infty}$-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^{+}$, and it preserves some stability concepts [7, 12].

The aim of this paper is to obtain some results on boundedness of the nonlinear functional differential systems under suitable conditions on perturbed term using the notion of $t_{\infty}$-similarity.

We give some related properties that we need in the sequal.
Lemma 1.4. [15] The linear system

$$
\begin{equation*}
x^{\prime}=A(t) x, x\left(t_{0}\right)=x_{0}, \tag{1.6}
\end{equation*}
$$

where $A(t)$ is an $n \times n$ continuous matrix, is an $h$-system (respectively $h$-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function $h$ defined on $\mathbb{R}^{+}$such that

$$
\begin{equation*}
\left|\phi\left(t, t_{0}\right)\right| \leq c h(t) h\left(t_{0}\right)^{-1} \tag{1.7}
\end{equation*}
$$

for $t \geq t_{0} \geq 0$, where $\phi\left(t, t_{0}\right)$ is a fundamental matrix of (1.6).
We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$
\begin{equation*}
y^{\prime}=f(t, y)+g(t, y), y\left(t_{0}\right)=y_{0} \tag{1.8}
\end{equation*}
$$

where $g \in C\left(\mathbb{R}^{+} \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$ and $g(t, 0)=0$. Let $y(t)=y\left(t, t_{0}, y_{0}\right)$ denote the solution of (1.8) passing through the point $\left(t_{0}, y_{0}\right)$ in $\mathbb{R}^{+} \times \mathbb{R}^{n}$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.5. [2] Let $x$ and $y$ be a solution of (1.1) and (1.8), respectively. If $y_{0} \in \mathbb{R}^{n}$, then for all $t \geq t_{0}$ such that $x\left(t, t_{0}, y_{0}\right) \in \mathbb{R}^{n}$, $y\left(t, t_{0}, y_{0}\right) \in \mathbb{R}^{n}$,

$$
y\left(t, t_{0}, y_{0}\right)=x\left(t, t_{0}, y_{0}\right)+\int_{t_{0}}^{t} \Phi(t, s, y(s)) g(s, y(s)) d s
$$

THEOREM 1.6. [5] If the zero solution of (1.1) is $h S$, then the zero solution of (1.3) is hS.

Theorem 1.7. [6] Suppose that $f_{x}(t, 0)$ is $t_{\infty}$-similar to $f_{x}\left(t, x\left(t, t_{0}, x_{0}\right)\right)$ for $t \geq t_{0} \geq 0$ and $\left|x_{0}\right| \leq \delta$ for some constant $\delta>0$. If the solution $v=0$ of (1.3) is $h S$, then the solution $z=0$ of (1.4) is $h S$.

Lemma 1.8. (Bihari - type inequality) Let $u, \lambda \in C\left(\mathbb{R}^{+}\right), w \in$ $C((0, \infty))$ and $w(u)$ be nondecreasing in $u$. Suppose that, for some $c>0$,

$$
u(t) \leq c+\int_{t_{0}}^{t} \lambda(s) w(u(s)) d s, t \geq t_{0} \geq 0
$$

Then

$$
u(t) \leq W^{-1}\left[W(c)+\int_{t_{0}}^{t} \lambda(s) d s\right]
$$

where $t_{0} \leq t<b_{1}, W(u)=\int_{u_{0}}^{u} \frac{d s}{w(s)}, W^{-1}(u)$ is the inverse of $W(u)$, and

$$
b_{1}=\sup \left\{t \geq t_{0}: W(c)+\int_{t_{0}}^{t} \lambda(s) d s \in \operatorname{domW}^{-1}\right\}
$$

Lemma 1.9. [3] Let $u, \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}, \lambda_{5}, \lambda_{6} \in C\left(\mathbb{R}^{+}\right), w \in C((0, \infty))$ and $w(u)$ be nondecreasing in $u$, $u \leq w(u)$. Suppose that for some $c>0$ and $0 \leq t_{0} \leq t$

$$
\begin{aligned}
u(t) \leq & c+\int_{t_{0}}^{t} \lambda_{1}(s) u(s) d s+\int_{t_{0}}^{t} \lambda_{2}(s) w(u(s)) d s \\
& +\int_{t_{0}}^{t} \lambda_{3}(s) \int_{t_{0}}^{s} \lambda_{4}(\tau) u(\tau) d \tau d s+\int_{t_{0}}^{t} \lambda_{5}(s) \int_{t_{0}}^{s} \lambda_{6}(\tau) w(u(\tau)) d \tau d s
\end{aligned}
$$

Then

$$
\begin{aligned}
& u(t) \leq W^{-1}\left[W(c)+\int_{t_{0}}^{t}\left(\lambda_{1}(s)+\right.\right. \lambda_{2}(s)+ \\
& \lambda_{3}(s) \int_{t_{0}}^{s} \lambda_{4}(\tau) d \tau \\
&\left.\left.+\lambda_{5}(s) \int_{t_{0}}^{s} \lambda_{6}(\tau) d \tau\right) d s\right]
\end{aligned}
$$

where $t_{0} \leq t<b_{1}, W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$
\begin{aligned}
b_{1}=\sup \left\{t \geq t_{0}: W(c)+\right. & \int_{t_{0}}^{t}\left(\lambda_{1}(s)+\lambda_{2}(s)+\lambda_{3}(s) \int_{t_{0}}^{s} \lambda_{4}(\tau) d \tau\right. \\
& \left.\left.+\lambda_{5}(s) \int_{t_{0}}^{s} \lambda_{6}(\tau) d \tau\right) d s \in \operatorname{domW}^{-1}\right\}
\end{aligned}
$$

For the proof we need the following corollary from Lemma 1.9.
Corollary 1.10. Let $u, \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4} \in C\left(\mathbb{R}^{+}\right), w \in C((0, \infty))$ and $w(u)$ be nondecreasing in $u, u \leq w(u)$. Suppose that for some $c>0$ and $0 \leq t_{0} \leq t$,

$$
\begin{aligned}
u(t) \leq & c+\int_{t_{0}}^{t} \lambda_{1}(s) u(s) d s+\int_{t_{0}}^{t} \lambda_{2}(s) w(u(s)) d s \\
& +\int_{t_{0}}^{t} \lambda_{3}(s) \int_{t_{0}}^{s} \lambda_{4}(\tau) u(\tau) d \tau d s
\end{aligned}
$$

Then

$$
u(t) \leq W^{-1}\left[W(c)+\int_{t_{0}}^{t}\left(\lambda_{1}(s)+\lambda_{2}(s)+\lambda_{3}(s) \int_{t_{0}}^{s} \lambda_{4}(\tau) d \tau\right) d s\right]
$$

where $t_{0} \leq t<b_{1}, W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$
\begin{aligned}
b_{1}=\sup \{t & \geq t_{0}: W(c)+\int_{t_{0}}^{t}\left(\lambda_{1}(s)+\lambda_{2}(s)\right. \\
& \left.\left.+\lambda_{3}(s) \int_{t_{0}}^{s} \lambda_{4}(\tau) d \tau\right) d s \in \operatorname{domW}^{-1}\right\}
\end{aligned}
$$

Lemma 1.11. [4] Let $u, \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}, \lambda_{5}, \lambda_{6}, \lambda_{7} \in C\left(\mathbb{R}^{+}\right), w \in C((0, \infty))$, and $w(u)$ be nondecreasing in $u$, $u \leq w(u)$. Suppose that for some $c>0$ and $0 \leq t_{0} \leq t$,

$$
\begin{aligned}
u(t) \leq & c+\int_{t_{0}}^{t} \lambda_{1}(s) u(s) d s+\int_{t_{0}}^{t} \lambda_{2}(s) \int_{t_{0}}^{s}\left(\lambda_{3}(\tau) w(u(\tau))\right. \\
& \left.+\lambda_{4}(\tau) \int_{t_{0}}^{\tau} \lambda_{5}(r) w(u(r)) d r\right) d \tau d s+\int_{t_{0}}^{t} \lambda_{6}(s) \int_{t_{0}}^{s} \lambda_{7}(\tau) u(\tau) d \tau d s
\end{aligned}
$$

Then

$$
\begin{aligned}
u(t) \leq W^{-1}\left[W(c)+\int_{t_{0}}^{t}\right. & \left(\lambda_{1}(s)+\lambda_{2}(s) \int_{t_{0}}^{s}\left(\lambda_{3}(\tau)+\lambda_{4}(\tau) \int_{t_{0}}^{\tau} \lambda_{5}(r) d r\right) d \tau\right. \\
& \left.\left.+\lambda_{6}(s) \int_{t_{0}}^{s} \lambda_{7}(\tau) d \tau\right) d s\right]
\end{aligned}
$$

where $t_{0} \leq t<b_{1}, W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$
\begin{gathered}
b_{1}=\sup \left\{t \geq t_{0}: W(c)+\int_{t_{0}}^{t}\left(\lambda_{1}(s)+\lambda_{2}(s) \int_{t_{0}}^{s}\left(\lambda_{3}(\tau)+\lambda_{4}(\tau) \int_{t_{0}}^{\tau} \lambda_{5}(r) d r\right) d \tau\right.\right. \\
\left.\left.+\lambda_{6}(s) \int_{t_{0}}^{s} \lambda_{7}(\tau) d \tau\right) d s \in \operatorname{domW}^{-1}\right\}
\end{gathered}
$$

## 2. Main results

In this section, we investigate boundedness for solutions of the functional perturbed differential systems via $t_{\infty}$-similarity.

We need the following lemma to prove Theorem 2.2.
Lemma 2.1. Let $u, \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}, \lambda_{5}, \lambda_{6}, \lambda_{7} \in C\left(\mathbb{R}^{+}\right), w \in C((0, \infty))$ and $w(u)$ be nondecreasing in $u, u \leq w(u)$. Suppose that, for some $c \geq 0$ and $t \geq t_{0}$, we have

$$
\begin{align*}
u(t) \leq & c+\int_{t_{0}}^{t} \lambda_{1}(s) w(u(s)) d s+\int_{t_{0}}^{t} \lambda_{2}(s)\left(\int _ { t _ { 0 } } ^ { s } \left(\lambda_{3}(\tau) u(\tau)\right.\right.  \tag{2.1}\\
& \left.\left.+\lambda_{4}(\tau) \int_{t_{0}}^{\tau} \lambda_{5}(s) w(u(r)) d r\right) d \tau+\lambda_{6}(s) \int_{t_{0}}^{s} \lambda_{7}(\tau) u(\tau) d \tau\right) d s
\end{align*}
$$

Then

$$
\begin{align*}
u(t) \leq W^{-1}\left[W(c)+\int_{t_{0}}^{t}\left[\lambda_{1}(s)+\lambda_{2}(s)( \right.\right. & \int_{t_{0}}^{s}\left(\lambda_{3}(\tau)+\lambda_{4}(\tau) \int_{t_{0}}^{\tau} \lambda_{5}(r) d r\right) d \tau  \tag{2.2}\\
& \left.\left.\left.+\lambda_{6}(s) \int_{t_{0}}^{s} \lambda_{7}(\tau) d \tau\right)\right] d s\right], t \geq t_{0}
\end{align*}
$$

Proof. Define a function $v(t)$ by the right member of (2.1). Then, we have $v\left(t_{0}\right)=c$ and

$$
\begin{aligned}
v^{\prime}(t)= & \lambda_{1}(t) w(u(t))+\lambda_{2}(t)\left(\int _ { t _ { 0 } } ^ { t } \left(\lambda_{3}(s) u(s)\right.\right. \\
& \left.\left.+\lambda_{4}(s) \int_{t_{0}}^{s} \lambda_{5}(\tau) w(u(\tau)) d \tau\right) d s+\lambda_{6}(t) \int_{t_{0}}^{t} \lambda_{7}(s) u(s) d s\right) \\
\leq & {\left[\lambda_{1}(t)+\lambda_{2}(t)\left(\int_{t_{0}}^{t}\left(\lambda_{3}(s)+\lambda_{4}(s) \int_{t_{0}}^{s} \lambda_{5}(\tau) d \tau\right) d s\right.\right.} \\
& \left.\left.+\lambda_{6}(t) \int_{t_{0}}^{t} \lambda_{7}(s) d s\right)\right] w(v(t))
\end{aligned}
$$

$t \geq t_{0}$, since $v(t)$ is nondecreasing, $u \leq w(u)$, and $u(t) \leq v(t)$. Now, by integrating the above inequality on $\left[t_{0}, t\right]$ and $v\left(t_{0}\right)=c$, we have

$$
\begin{array}{r}
v(t) \leq c+\int_{t_{0}}^{t}\left(\lambda_{1}(s)+\lambda_{2}(s) \int_{t_{0}}^{s}\left(\lambda_{3}(\tau)+\lambda_{4}(\tau) \int_{t_{0}}^{\tau} \lambda_{5}(r) d r\right) d \tau\right.  \tag{2.3}\\
\left.+\lambda_{6}(s) \int_{t_{0}}^{s} \lambda_{7}(\tau) d \tau\right) w(z(s)) d s
\end{array}
$$

Thus, (2.3) yields the estimate (2.2).
To obtain the bounded result, the following assumptions are needed:
(H1) $f_{x}(t, 0)$ is $t_{\infty}$-similar to $f_{x}\left(t, x\left(t, t_{0}, x_{0}\right)\right)$ for $t \geq t_{0} \geq 0$ and $\left|x_{0}\right| \leq \delta$ for some constant $\delta>0$.
(H2) The solution $x=0$ of (1.1) is hS with the increasing function $h$.
(H3) $w(u)$ be nondecreasing in $u$ such that $u \leq w(u)$ and $\frac{1}{v} w(u) \leq w\left(\frac{u}{v}\right)$ for some $v>0$.

Theorem 2.2. Let $a, b, c, k, q, u, w \in C\left(\mathbb{R}^{+}\right)$. Suppose that (H1), (H2), (H3), and $g$ in (1.2) satisfies

$$
\begin{equation*}
|g(t, y(t))| \leq a(t)|y(t)|+b(t) \int_{t_{0}}^{t} k(s) w(|y(s)|) d s \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
|h(t, y(t), T y(t))| \leq c(t)(w(|y(t)|)+|T y(t)|),|T y(t)| \leq \int_{t_{0}}^{t} q(s)|y(s)| d s \tag{2.5}
\end{equation*}
$$

where $t \geq t_{0} \geq 0, \int_{t_{0}}^{\infty} a(s) d s<\infty, \int_{t_{0}}^{\infty} b(s) d s<\infty, \int_{t_{0}}^{\infty} c(s) d s<\infty$, $\int_{t_{0}}^{\infty} k(s) d s<\infty$, and $\int_{t_{0}}^{\infty} q(s) d s<\infty$. Then, any solution $y(t)=$ $y\left(t, t_{0}, y_{0}\right)$ of (1.2) is bounded on on $\left[t_{0}, \infty\right)$ and it satisfies

$$
\begin{aligned}
|y(t)| & \leq h(t) W^{-1}\left[W(c)+c_{2} \int_{t_{0}}^{t}\left[c(s)+\int_{t_{0}}^{s}(a(\tau)\right.\right. \\
& \left.\left.\left.+b(\tau) \int_{t_{0}}^{\tau} k(r) d r\right) d \tau+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right] d s\right]
\end{aligned}
$$

where $W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$
\begin{array}{r}
b_{1}=\sup \left\{t \geq t_{0}: W(c)+c_{2} \int_{t_{0}}^{t}\left[c(s)+\int_{t_{0}}^{s}\left(a(\tau)+b(\tau) \int_{t_{0}}^{\tau} k(r) d r\right) d \tau\right.\right. \\
\left.\left.+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right] d s \in \operatorname{domW}^{-1}\right\}
\end{array}
$$

Proof. Let $x(t)=x\left(t, t_{0}, y_{0}\right)$ and $y(t)=y\left(t, t_{0}, y_{0}\right)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x=0$ of (1.1) is hS, the solution $v=0$ of (1.3) is hS. Therefore, from (H1), by Theorem 1.7, the solution $z=0$ of (1.4) is hS. Applying the nonlinear variation of constants formula , the hS condition of $x=0$ of (1.1), together with (2.4) and (2.5), we have
$|y(t)|$

$$
\begin{aligned}
\leq & |x(t)|+\int_{t_{0}}^{t}|\Phi(t, s, y(s))|\left(\int_{t_{0}}^{s}|g(\tau, y(\tau))| d \tau+|h(s, y(s), T y(s))|\right) d s \\
\leq & c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}+\int_{t_{0}}^{t} c_{2} h(t) h(s)^{-1}\left(\int_{t_{0}}^{s}(a(\tau)|y(\tau)|\right. \\
& \left.\left.+b(\tau) \int_{t_{0}}^{\tau} k(r) w(|y(r)|) d r\right) d \tau+c(s)\left(w(|y(s)|)+\int_{t_{0}}^{s} q(\tau)|y(\tau)| d \tau\right)\right) d s
\end{aligned}
$$

By the assumptions (H2) and (H3), we obtain

$$
\begin{aligned}
|y(t)| \leq & c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}+\int_{t_{0}}^{t} c_{2} h(t)\left(c(s) w\left(\frac{|y(s)|}{h(s)}\right)\right. \\
& +\int_{t_{0}}^{s}\left(a(\tau) \frac{|y(\tau)|}{h(\tau)}+b(\tau) \int_{t_{0}}^{\tau} k(r) w\left(\frac{|y(r)|}{h(r)}\right) d r\right) d \tau \\
& \left.+c(s) \int_{t_{0}}^{s} q(\tau) \frac{|y(\tau)|}{h(\tau)} d \tau\right) d s
\end{aligned}
$$

Set $u(t)=|y(t) \| h(t)|^{-1}$. Then, by Lemma 2.1, we have

$$
\begin{aligned}
|y(t)| \leq & h(t) W^{-1}\left[W(c)+c_{2} \int_{t_{0}}^{t}\left[c(s)+\int_{t_{0}}^{s}\left(a(\tau)+b(\tau) \int_{t_{0}}^{\tau} k(r) d r\right) d \tau\right.\right. \\
& \left.\left.+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right] d s\right]
\end{aligned}
$$

where $c=c_{1}\left|y_{0}\right| h\left(t_{0}\right)^{-1}$. The above estimation yields the desired result since the function $h$ is bounded, and so the proof is complete.

REmARK 2.3. Letting $c(t)=0$ in Theorem 2.2, we obtain the similar result as that of Theorem 3.4 in [8].

Theorem 2.4. Let $a, b, k, q, u, w \in C\left(\mathbb{R}^{+}\right)$. Suppose that (H1), (H2), (H3), and $g$ in (1.2) satisfies

$$
\begin{equation*}
\int_{t_{0}}^{s}|g(\tau, y(\tau))| d \tau \leq a(s)|y(s)|+b(s) \int_{t_{0}}^{s} k(\tau)|y(\tau)| d \tau \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
|h(t, y(t), T y(t))| \leq b(t)(w(|y(t)|)+|T y(t)|),|T y(t)| \leq \int_{t_{0}}^{t} q(s)|y(s)| d s \tag{2.7}
\end{equation*}
$$

where $\int_{t_{0}}^{\infty} a(s) d s<\infty, \int_{t_{0}}^{\infty} b(s) d s<\infty, \int_{t_{0}}^{\infty} k(s) d s<\infty, \int_{t_{0}}^{\infty} q(s) d s<$ $\infty$. Then, any solution $y(t)=y\left(t, t_{0}, y_{0}\right)$ of (1.2) is bounded on $\left[t_{0}, \infty\right)$ and it satisfies
$|y(t)| \leq h(t) W^{-1}\left[W(c)+c_{2} \int_{t_{0}}^{t}\left(a(s)+b(s)+b(s) \int_{t_{0}}^{s}(k(\tau)+q(\tau)) d \tau\right] d s\right]$,
where $t_{0} \leq t<b_{1}, W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$
\begin{aligned}
b_{1}= & \sup \left\{t \geq t_{0}: W(c)+c_{2} \int_{t_{0}}^{t}(a(s)+b(s)\right. \\
& \left.\left.+b(s) \int_{t_{0}}^{s}(k(\tau)+q(\tau)) d \tau\right] d s \in \operatorname{domW}^{-1}\right\}
\end{aligned}
$$

Proof. Let $x(t)=x\left(t, t_{0}, y_{0}\right)$ and $y(t)=y\left(t, t_{0}, y_{0}\right)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z=0$ of (1.4) is hS. Using the nonlinear variation of constants formula, the hS condition of $x=0$ of (1.1), together with (2.6) and (2.7), we have

$$
\begin{aligned}
|y(t)| & \leq c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}+\int_{t_{0}}^{t} c_{2} h(t) h(s)^{-1}(a(s)|y(s)|+b(s) w(|y(s)|) \\
& \left.+b(s) \int_{t_{0}}^{s} k(\tau)|y(\tau)| d \tau+b(s) \int_{t_{0}}^{s} q(\tau)|y(\tau)| d \tau\right) d s
\end{aligned}
$$

It follows from (H2) and (H3) that

$$
\begin{aligned}
|y(t)| & \leq c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}+\int_{t_{0}}^{t} c_{2} h(t)\left(a(s) \frac{|y(s)|}{h(s)}+b(s) w\left(\frac{|y(s)|}{h(s)}\right)\right. \\
& \left.+b(s) \int_{t_{0}}^{s}(k(\tau)+q(\tau)) \frac{|y(\tau)|}{h(\tau)} d \tau\right) d s
\end{aligned}
$$

Set $u(t)=|y(t) \| h(t)|^{-1}$. Then, by Corollary 1.10, we have
$|y(t)| \leq h(t) W^{-1}\left[W(c)+c_{2} \int_{t_{0}}^{t}\left[a(s)+b(s)+b(s) \int_{t_{0}}^{s}(k(\tau)+q(\tau)) d \tau\right] d s\right]$,
where $c=c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}$. Thus, any solution $y(t)=y\left(t, t_{0}, y_{0}\right)$ of (1.2) is bounded on $\left[t_{0}, \infty\right)$, and so the proof is complete.

Remark 2.5. Letting $b(s)=0$ in Theorem 2.4, we obtain the same result as that of Theorem 3.3 in [11].

REMARK 2.6. Letting $w(u)=u$ and $h(t, y(t), T y(t))=0$ in Theorem 2.4, we obtain the same result as that of Theorem 3.1 in [10].

Theorem 2.7. Let $a, b, c, k, q, u, w \in C\left(\mathbb{R}^{+}\right)$. Suppose that (H1), (H2), (H3), and $g$ in (1.2) satisfies

$$
\begin{equation*}
|g(t, y(t))| \leq a(t) w(|y(t)|)+b(t) \int_{t_{0}}^{t} k(s) w(|y(s)|) d s \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
|h(t, y(t), T y(t))| \leq c(t)(|y(t)|+|T y(t)|),|T y(t)| \leq \int_{t_{0}}^{t} q(s)|y(s)| d s \tag{2.9}
\end{equation*}
$$

where $t \geq t_{0} \geq 0, \int_{t_{0}}^{\infty} a(s) d s<\infty$, and $\int_{t_{0}}^{\infty} b(s) d s<\infty, \int_{t_{0}}^{\infty} c(s) d s<$ $\infty, \int_{t_{0}}^{\infty} k(s) d s<\infty$, and $\int_{t_{0}}^{\infty} q(s) d s<\infty$. Then, any solution $y(t)=$ $y\left(t, t_{0}, y_{0}\right)$ of (1.2) is bounded on $\left[t_{0}, \infty\right)$ and

$$
\begin{aligned}
|y(t)| \leq & h(t) W^{-1}\left[W(c)+c_{2} \int_{t_{0}}^{t}\left(c(s)+\int_{t_{0}}^{s}(a(\tau)\right.\right. \\
& \left.\left.\left.+b(\tau) \int_{t_{0}}^{\tau} k(r) d r\right) d \tau+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right) d s\right]
\end{aligned}
$$

where $W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$
\begin{aligned}
& b_{1}=\sup \left\{t \geq t_{0}: W(c)+\right. c_{2} \int_{t_{0}}^{t}\left(c(s)+\int_{t_{0}}^{s}\left(a(\tau)+b(\tau) \int_{t_{0}}^{\tau} k(r) d r\right) d \tau\right. \\
&\left.\left.+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right) d s \in \operatorname{domW}^{-1}\right\}
\end{aligned}
$$

Proof. Let $x(t)=x\left(t, t_{0}, y_{0}\right)$ and $y(t)=y\left(t, t_{0}, y_{0}\right)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z=0$ of (1.4) is hS. By Lemma 1.4, Lemma 1.5 , the hS condition of $x=0$ of (1.1), together with (2.8) and (2.9), we have

$$
\begin{aligned}
|y(t)| \leq & c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}+\int_{t_{0}}^{t} c_{2} h(t) h(s)^{-1}\left(\int_{t_{0}}^{s}(a(\tau) w(|y(\tau)|)\right. \\
& \left.\left.+b(\tau) \int_{t_{0}}^{\tau} k(r) w(|y(r)|) d r\right) d \tau+c(s)\left(|y(s)|+\int_{t_{0}}^{s} q(\tau)|y(\tau)| d \tau\right)\right) d s .
\end{aligned}
$$

Using the assumptions (H2) and (H3), we obtain

$$
\begin{aligned}
|y(t)| \leq & c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}+\int_{t_{0}}^{t} c_{2} h(t)\left(c(s) \frac{|y(s)|}{h(s)}\right. \\
& +\int_{t_{0}}^{s}\left(a(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right)+b(\tau) \int_{t_{0}}^{\tau} k(r) w\left(\frac{|y(r)|}{h(r)}\right) d r\right) d \tau
\end{aligned}
$$

$$
\left.+c(s) \int_{t_{0}}^{s} q(\tau) \frac{|y(\tau)|}{h(\tau)} d \tau\right) d s
$$

Set $u(t)=|y(t)||h(t)|^{-1}$. Then, it follows from Lemma 1.11 that we have

$$
\begin{aligned}
|y(t)| \leq & h(t) W^{-1}\left[W(c)+c_{2} \int_{t_{0}}^{t}\left(c(s)+\int_{t_{0}}^{s}(a(\tau)\right.\right. \\
& \left.\left.\left.+b(\tau) \int_{t_{0}}^{\tau} k(r) d r\right) d \tau+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right) d s\right]
\end{aligned}
$$

where $c=c_{1}\left|y_{0}\right| h\left(t_{0}\right)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved.

REmARK 2.8. Letting $c(t)=0$ in Theorem 2.7, we obtain the similar result as that of Theorem 3.6 in [9].

Theorem 2.9. Let $a, b, c, k, q, u, w \in C\left(\mathbb{R}^{+}\right)$. Suppose that (H1), (H2), (H3), and $g$ in (1.2) satisfies

$$
\begin{equation*}
\int_{t_{0}}^{s}|g(\tau, y(\tau))| d \tau \leq a(s) w(|y(s)|)+b(s) \int_{t_{0}}^{s} k(\tau) w(|y(\tau)|) d \tau \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
|h(t, y(t), T y(t))| \leq c(t)(|y(t)|+|T y(t)|),|T y(t)| \leq \int_{t_{0}}^{t} q(s)|y(s)| d s \tag{2.11}
\end{equation*}
$$

where $\int_{t_{0}}^{\infty} a(s) d s<\infty, \int_{t_{0}}^{\infty} b(s) d s<\infty, \int_{t_{0}}^{\infty} c(s) d s<\infty, \int_{t_{0}}^{\infty} k(s) d s<$ $\infty$, and $\int_{t_{0}}^{\infty} q(s) d s<\infty$. Then, any solution $y(t)=y\left(t, t_{0}, y_{0}\right)$ of (1.2) is bounded on $\left[t_{0}, \infty\right)$ and it satisfies

$$
\begin{aligned}
|y(t)| \leq & h(t) W^{-1}\left[W(c)+c_{2} \int_{t_{0}}^{t}\left(a(s)+c(s)+b(s) \int_{t_{0}}^{s} k(\tau) d \tau\right.\right. \\
& \left.\left.+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right) d s\right]
\end{aligned}
$$

where $t_{0} \leq t<b_{1}, W, W^{-1}$ are the same functions as in Lemma 1.8, and

$$
\begin{aligned}
b_{1}= & \sup \left\{t \geq t_{0}: W(c)+c_{2} \int_{t_{0}}^{t}(a(s)+c(s)\right. \\
& \left.\left.+b(s) \int_{t_{0}}^{s} k(\tau) d \tau+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right) d s \in \operatorname{domW}^{-1}\right\}
\end{aligned}
$$

Proof. Let $x(t)=x\left(t, t_{0}, y_{0}\right)$ and $y(t)=y\left(t, t_{0}, y_{0}\right)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z=0$ of (1.4) is hS. Using the nonlinear variation of constants formula, the hS condition of $x=0$ of (1.1), together with (2.10) and (2.11), we have

$$
\begin{aligned}
|y(t)| \leq & c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}+\int_{t_{0}}^{t} c_{2} h(t) h(s)^{-1}((a(s) w(|y(s)|) \\
& \left.+c(s)|y(s)|+b(s) \int_{t_{0}}^{s} k(\tau) w(|y(\tau)|) d \tau+c(s) \int_{t_{0}}^{s} q(\tau)|y(\tau)| d \tau\right) d s
\end{aligned}
$$

Using (H2) and (H3), we obtain

$$
\begin{aligned}
|y(t)| \leq & c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}+\int_{t_{0}}^{t} c_{2} h(t)\left(a(s) w\left(\frac{|y(s)|}{h(s)}\right)+c(s) \frac{|y(s)|}{h(s)}\right. \\
& \left.+b(s) \int_{t_{0}}^{s} k(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) d \tau+c(s) \int_{t_{0}}^{s} q(\tau) \frac{|y(\tau)|}{h(\tau)} d \tau\right) d s
\end{aligned}
$$

Set $u(t)=|y(t)||h(t)|^{-1}$.Then, an application of Lemma 1.9 yields

$$
\begin{aligned}
|y(t)| \leq & h(t) W^{-1}\left[W(c)+c_{2} \int_{t_{0}}^{t}\left(a(s)+c(s)+b(s) \int_{t_{0}}^{s} k(\tau) d \tau\right.\right. \\
& \left.\left.+c(s) \int_{t_{0}}^{s} q(\tau) d \tau\right) d s\right]
\end{aligned}
$$

where $c=c_{1}\left|y_{0}\right| h(t) h\left(t_{0}\right)^{-1}$. Then, any solution $y(t)=y\left(t, t_{0}, y_{0}\right)$ of (1.2) is bounded on $\left[t_{0}, \infty\right)$, and so the proof is complete.

Remark 2.10. Letting $c(t)=0$ in Theorem 2.9, we obtain the same result as that of Theorem 3.2 in [8].

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