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EXISTENCE OF SOLUTIONS OF A CLASS OF IMPULSIVE PERIODIC TYPE BVPS FOR SINGULAR FRACTIONAL DIFFERENTIAL SYSTEMS

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Abstract. A class of periodic type boundary value problems of coupled impulsive fractional differential equations are proposed. Sufficient conditions are given for the existence of solutions of these problems. We allow the nonlinearities $p(t)f(t,x,y)$ and $q(t)g(t,x,y)$ in fractional differential equations to be singular at $t=0,1$ and be involved a sup-multiplicative-like function. So both f and g may be super-linear and sub-linear. The analysis relies on a well known fixed point theorem. An example is given to illustrate the efficiency of the theorems.

1. Introduction

Fractional calculus has many applications (see Chapter 10 in [36]). Boundary value problems for nonlinear fractional differential equations have been addressed by several researchers during last decades. That is why, the fractional derivatives serve an excellent tool for the description of hereditary properties of various materials and processes. Actually, fractional differential equations arise in many engineering and scientific

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disciplines such as, physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and image processing, aerodynamics, and porous media. There have been many results obtained on the existence of solutions of boundary value problems for nonlinear fractional differential equations (see [6, 7, 29, 31, 32, 43, 51, 54]).

In recent years, many authors [1,14,19,20,22,23,25,26,30,37,42,43,50, 55] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems (see [2–4, 39]), impulsive periodic boundary value problems (see [40]), impulsive initial value problems (see [9, 13, 28, 46]), two-point, three-point or multi-point impulsive boundary value problems (see [5, 41, 53]), impulsive boundary value problems on infinite intervals (see [52]).

In [40], the following periodic boundary value problem of impulse type fractional differential equation

$$\begin{cases} D^{\alpha}x(t)-\lambda x(t)=f(t,x(t)), \quad t\in(0,1], t\neq t_1,\\ x(1)-\lim\limits_{t\to 0}t^{1-\alpha}x(t)=0,\\ \lim\limits_{t\to t_1^+}(t-t_1)^{1-\alpha}[x(t)-x(t_1)]=I(x(t_1)) \end{cases}$$

where $0<\alpha<1$, D^{α} is the standard Riemann-Liouville fractional derivative, $\lambda\in R$ with $\lambda\neq 0$, $0=t_0<t_1<t_2=1$, $I\in C(R,R)$, f is continuous at every point $(t,u)\in[0,1]\times R$.

In [8], authors studied the following periodic boundary value problem of impulse type fractional differential equation

$$\begin{cases} D^{\alpha}_{t_k^+}x(t)-\lambda x(t)=f(t,x(t)), \quad t\in(t_k,t_{k+1}), k=0,1,\cdots,p,\\ x(1)-\lim\limits_{t\to 0}t^{1-\alpha}x(t)=0,\\ \lim\limits_{t\to t_k^+}(t-t_k)^{1-\alpha}[x(t)-x(t_k)]=I(x(t_k)), k=1,2,\cdots,p, \end{cases}$$

where $0<\alpha<1$, D^{α} is the standard Riemann-Liouville fractional derivative, $\lambda\in R$ with $\lambda\neq 0$, $0=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=1$, $I\in C(R,R)$, f is continuous at every point $(t,u)\in(t_k,t_{k+1}]\times R(k=0,1,2,\cdots,p)$.

Applications of fractional order differential systems are in many fields, as for example, rheology, mechanics, chemistry, physics, bioengineering, robotics and many others, see [10]. Diethehm [11] proposed the model of the type (which is called a multi-order fractional differential system):

$${}^{c}D^{n_i}_{0^+}y_i(t)=f_i(t,y_1(t),\cdots,y_n(t)), i=1,2,\cdots,n$$

subjected to the initial conditions

$$y_j(0) = y_{j,0}(j = 1, 2, \cdots, n).$$

In [15, 33, 45], the fractional order nonlinear dynamical model of interpersonal relationships

$$\begin{cases} D^{\alpha}x(t) + \alpha_1 x(t) = A_1 + \beta_1 y(t)(1 - \epsilon y^2(t)), \\ \\ D^{\alpha}y(t) + \alpha_2 y(t) = A_2 + \beta_2 x(t)(1 - \epsilon x^2(t)), \end{cases}$$

was proposed, where $0 < \alpha \leq 1$, $\alpha_i, \beta_i, A_i, \epsilon$ are real constants. These systems contain many models as special cases, see Chen's fractional order system [47,48] with a double scroll attractor, Genesio-Tesi fractional-order system [18], Lu's fractional order system [12], Volta's fractional-order system [34, 35], Rossler's fractional-order system [24] and so on. To the authors knowledge, there has been no paper discussing the existence of solutions of impulsive periodic type boundary value problems of singular fractional differential systems.

Motivated by mentioned applications and reason, in this paper, we discuss the following impulsive periodic type boundary value problem of singular fractional differential system

(1)

$$\begin{cases} D_{t_i^+}^{\alpha}x(t) - \lambda x(t) = p(t)f(t, x(t), y(t)), t \in (t_i, t_{i+1}), i \in N[0, m], \\ D_{t_i^+}^{\beta}y(t) - \mu y(t) = q(t)g(t, x(t), y(t)), t \in (t_i, t_{i+1}), i \in N[0, m], \\ x(1) - a \lim\limits_{t \to 0} t^{1-\alpha}x(t) = \int_0^1 \phi(s)G(s, x(s), y(s))ds, \\ y(1) - b \lim\limits_{t \to 0} t^{1-\beta}y(t) = \int_0^1 \psi(s)H(s, x(s), y(s))ds, \\ \lim\limits_{t \to t_i^+} (t - t_i)^{1-\alpha}x(t) = I(t_i, x(t_i), y(t_i)), i \in N[1, m], \\ \lim\limits_{t \to t_i^+} (t - t_i)^{1-\beta}y(t) = J(t_i, x(t_i), y(t_i)), i \in N[1, m], \end{cases}$$

where

(a) $0 < \alpha, \beta < 1$, $D_{t_i^+}^{\alpha}$ (or $D_{t_i^+}^{\beta}$) is the Riemann-Liouville fractional derivative of order α (or β),

(b) $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$ with $m \geq 1$, $a, b \in R$ with $ab \neq 0$, $\lambda, \mu \in R$, $N[c, d] = \{c, c + 1, \cdots, d\}$ for integers c and d,

(c) $\phi, \psi : (0, 1) \to R$ satisfy $\phi, \psi \in L^1(0, 1)$,

(d) $p, q : \bigcup\limits_{i=0}^{m}(t_i, t_{i+1}) \to R$ satisfy the growth conditions: there exist constants $k_i, l_i (i = 1, 2)$ with $k_1 > -1, k_2 > -1$ and $\max\{-\alpha, -k_1 - 1\} \leq$

$l_1 \leq 0$ and $\max\{-\beta, -k_2-1\} \leq l_2 \leq 0$ such that $|p(t)| \leq (t-t_i)^{k_1}(t_{i+1}-t)^{l_1}$, $|q(t)| \leq (t-t_i)^{k_2}(t_{i+1}-t)^{l_2}$, $t \in (t_i, t_{i+1}), i = 0, 1, \cdots, m$,

(e) f, g, G, H defined on $(0,1) \times R \times R$ are *impulsive Caratheodory functions*(see Definition 2.3), I, J are *Caratheodory functions*(see Definition 2.4).

A pair of functions $x, y : (0,1] \to R$ is called a solution of BVP(1) if

$$x|_{(t_k,t_{k+1}]} \in C^0(t_k, t_{k+1}],\ y|_{(t_k,t_{k+1}]} \in C^0(t_k, t_{k+1}],\quad k = 0, 1, 2, \cdots, m$$

and x, y satisfy all equations in (1). As in [40], for clarity and brevity, we restrict our attention to BVPs with one impulse, the difference between the theory of one or an arbitrary number of impulses is quite minimal.

To the best of the authors knowledge, no one has studied the existence of solutions of BVP (1) in which the nonlinearities are singular functions. We fill this gap by establishing existence results on solutions of BVP(1). The assumptions (D) in Theorem 3.1 in this paper are more general that the assumptions (H1) and (H2) in Theorem 3.18 in [8,40]. Two examples are given to illustrate the efficiency of the main theorems.

The remainder of this paper is as follows: in Section 2, we present preliminary results. The main theorems and their proofs are given in Section 3. In Section 4, an example is given to illustrate the main results.

2. Preliminary results

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [21,36].

Let the Gamma function, Beta function and the classical Mittag-Leffler special function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx,\ \ \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx,$$

$$E_{\delta,\delta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\delta k+\delta)}$$

respectively for , $\alpha > 0, p > 0, q > 0, \delta > 0$. We note that $E_{\delta,\delta}(x) > 0$ for all $x \in R$ and $E_{\delta,\delta}(x)$ is strictly increasing in x. Then for $x > 0$ we have $E_{\delta,\delta}(-x) < E_{\delta,\delta}(0) = \frac{1}{\Gamma(\delta)} < E_{\delta,\delta}(x)$.

Definition 2.1. ([21]) Let $c \in R$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (c, \infty) \to R$ is given by

$$I_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} g(s) ds,$$

provided that the right-hand side exists.

Definition 2.2. ([21]) Let $c \in R$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (c, \infty) \to R$ is given by

$$D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $\alpha < n \le \alpha + 1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

For readers convenience, choose

$$\delta_{\alpha,\lambda}(t, t_i) = (t-t_i)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-t_i)^{\alpha}), t \in (t_i, t_{i+1}], i \in N[0, m],$$

$$\delta_{\beta,\mu}(t, t_i) = (t-t_i)^{\beta-1} E_{\beta,\beta}(\mu(t-t_i)^{\beta}), t \in (t_i, t_{i+1}], i \in N[0, m].$$

Definition 2.3. We call $F : \bigcup_{i=0}^{m} (t_i, t_{i+1}) \times R^2 \to R$ an *impulsive Caratheodory function* if it satisfies

(i) $t \to F(t, \delta_{\alpha,\lambda}(t, t_i)u, \delta_{\beta,\mu}(t, t_i)v)$ is measurable on $(t_i, t_{i+1})(i \in N[0, m])$ for any $(u, v) \in R^2$,

(ii) $(u, v) \to F(t, \delta_{\alpha,\lambda}(t, t_i)u, \delta_{\beta,\mu}(t, t_i)v)$ is continuous on R^2 for almost all $t \in (t_i, t_{i+1})(i = 0, 1, 2, \cdots, m)$,

(iii) for each $r > 0$ there exists $M_r > 0$ such that

$$|F(t, \delta_{\alpha,\lambda}(t, t_i)u, \delta_{\beta,\mu}(t, t_i)v)| \le M_r, t \in (t_i, t_{i+1}), |u|, |v| \le r, i \in N[0, m].$$

Definition 2.4. We call $I : \{t_i : i \in N[1, m]\} \times R^2 \to R$ an *Caratheodory function* if it satisfies

(i) $(u, v) \to I(t_i, \delta_{\alpha,\lambda}(t_i, t_{i-1})u, \delta_{\beta,\mu}(t_i, t_{i-1})v)$ is continuous on R^2 for almost all $i = 1, 2, \cdots, m$,

(ii) for each $r > 0$ there exists $M_r > 0$ such that

$$|I(t_i, \delta_{\alpha,\lambda}(t_i, t_{i-1})u, \delta_{\beta,\mu}(t_i, t_{i-1})v)| \le M_r, i \in N[1, m].$$

Definition 2.5. ([19]) An odd homeomorphism Φ of the real line R onto itself is called a sup-multiplicative-like function if there exists a homeomorphism ω of $[0, +\infty)$ onto itself which *supports* Φ in the sense that for all $v_1, v_2 \ge 0$ it holds

$$\Phi(v_1 v_2) \ge \omega(v_1) \Phi(v_2). \tag{2}$$

ω is called the supporting function of Φ.

REMARK 2.1. From [19], any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in R$$

is a sup-multiplicative-like function, provided that $c_j \geq 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, \quad u\}$, $u \geq 0$.

REMARK 2.2. ([19]) It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero and moreover their inverses Φ^{-1} and ν respectively are increasing and such that

$$\Phi^{-1}(w_1 w_2) \leq \nu(w_1)\Phi^{-1}(w_2), \tag{3}$$

for all $w_1, w_2 \geq 0$ and ν is called the supporting function of Φ^{-1}.

In this paper we suppose that $\Phi : R \to R$ is a sup-multiplicative-like function with supporting function ω, its inverse function is denoted by $\Phi^{-1} : R \to R$ with supporting function ν.

Suppose that $\lambda > 0, \mu > 0$. We use the Banach spaces (similarly to [8], we can give the proofs)

$$X = \left\{ x : (0,1] \to R : \begin{array}{c} x|_{(t_i,t_{i+1}]} \in C^0(t_i, t_{i+1}], i \in N[0,m], \\ \text{there exist the limits} \\ \lim\limits_{t \to t_i^+} \frac{x(t)}{\delta_{\alpha,\lambda}(t,t_i)},\ i \in N[0,m] \end{array} \right\}$$

with the norm

$$||x|| = ||x||_X = \max\left\{ \sup_{t \in (t_i,t_{i+1}]} \frac{|x(t)|}{\delta_{\alpha,\lambda}(t,t_i)} : i \in N[0,m] \right\}$$

$$Y = \left\{ y : (0,1] \to R : \begin{array}{c} y|_{(t_i,t_{i+1}]} \in C^0(t_i, t_{i+1}], i \in N[0,m], \\ \text{there exist the limits} \\ \lim\limits_{t \to t_i^+} \frac{y(t)}{\delta_{\beta,\mu}(t,t_i)},\ i \in N[0,m] \end{array} \right\}$$

with the norm

$$||y|| = ||y||_Y = \max\left\{\sup_{t\in(t_i,t_{i+1}]} \frac{|y(t)|}{\delta_{\beta,\mu}(t,t_i)} : i \in N[0,m]\right\}.$$

Choose $E = X \times Y$ with the norm $||(x,y)|| = \max\{||x||_X, ||y||_Y\}$. Then E is a Banach space.

Lemma 2.1. *Suppose that $\sigma : (0,1) \to R$ satisfies that there exist numbers $k > -1$ and $\max\{-\alpha, -k-1\} < l \leq 0$ such that $|\sigma(t)| \leq (t-t_i)^k(t_{i+1}-t)^l$ for all $t \in (t_i, t_{i+1})$, $i = 0, 1, \cdots, m$. The x is a solutions of*

$$(4) \qquad \begin{cases} D^\alpha_{t_i^+}x(t) - \lambda x(t) = \sigma(t), t \in (t_i,t_{i+1}), i \in N[0,m], \\ x(1) - a\lim\limits_{t\to 0} t^{1-\alpha}x(t) = a_0, \\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}x(t) = I_i, i \in N[1,m] \end{cases}$$

if and only if $x \in X$ and

$$(5) \qquad x(t) = \begin{cases} \Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)\frac{I_m\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_m)+\int_{t_m}^1\delta_{\alpha,\lambda}(1,s)\sigma(s)ds-a_0}{a} \\ \quad +\int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (0,t_1], \\ \Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_i)I_i + \int_{t_i}^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (t_i,t_{i+1}], i \in N[1,m]. \end{cases}$$

Proof. Let x be a solution of (4). One sees from $l \leq 0$, for $t \in (t_i, t_{i+1}]$, that

$$\begin{aligned} &(t-t_i)^{1-\alpha}\left|\int_{t_i}^t \delta_{\alpha,\alpha}(t,s)\sigma(s)ds\right| \\ &\leq (t-t_i)^{1-\alpha}\int_{t_i}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)(s-t_i)^k(t_{i+1}-s)^l ds \\ &= (t-t_i)^{1-\alpha}\int_{t_i}^t (t-s)^{\alpha-1}\sum_{i=0}^\infty \frac{\lambda^i(t-s)^{\alpha i}}{\Gamma(\alpha i+\alpha)}(s-t_i)^k(t_{i+1}-s)^l ds \\ &\leq (t-t_i)^{1-\alpha}\int_{t_i}^t (t-s)^{\alpha+l-1}\sum_{i=0}^\infty \frac{\lambda^i(t-s)^{\alpha i}}{\Gamma(\alpha i+\alpha)}(s-t_i)^k ds \end{aligned}$$

$$= (t-t_i)^{1-\alpha}\sum_{i=0}^{\infty}\frac{\lambda^i}{\Gamma(\alpha i+\alpha)}\int_{t_i}^{t}(t-s)^{\alpha+\alpha i+l-1}(s-t_i)^k ds$$

$$= (t-t_i)^{1-\alpha}\sum_{i=0}^{\infty}\frac{\lambda^i}{\Gamma(\alpha i+\alpha)}(t-t_i)^{\alpha+\alpha i+l+k}\int_0^1(1-w)^{\alpha+\alpha i+l-1}w^k dw$$

$$\leq (t-t_i)^{1-\alpha}\sum_{i=0}^{\infty}\frac{\lambda^i}{\Gamma(\alpha i+\alpha)}(t-t_i)^{\alpha+\alpha i+l+k}\int_0^1(1-w)^{\alpha+l-1}w^k dw$$

$$= (t-t_i)^{1+l+k}\mathbf{B}(\alpha+l,k+1)\sum_{i=0}^{\infty}\frac{\lambda^i(t-t_i)^{\alpha i}}{\Gamma(\alpha i+\alpha)}$$

$$= (t-t_i)^{1+l+k}\mathbf{B}(\alpha+l,k+1)E_{\alpha,\alpha}(\lambda(t-t_i)^{\alpha}).$$

From $k+l+1>0$, we get

$$\lim_{t\to t_i^+}(t-t_i)^{1-\alpha}\left|\int_{t_i}^{t}\delta_{\alpha,\lambda}(t,s)\sigma(s)ds\right| = 0.$$

By (3.26) in [7], we know that there exist numbers A_i such that
(6)

$$x(t) = A_i\Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_i)+\int_{t_i}^{t}\delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t\in(t_i,t_{i+1}], i\in N[0,m].$$

Note $E_{\alpha,\alpha}(0)=\frac{1}{\Gamma(\alpha)}$. It follows from the boundary conditions and the impulse assumption in (4) that

$$A_m\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_m)+\int_{t_m}^{1}\delta_{\alpha,\lambda}(1,s)\sigma(s)ds-aA_0=a_0,$$

$$A_i=I_i,\ i\in N[1,m].$$

Then

$$A_0=\frac{I_m\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_m)+\int_{t_m}^{1}\delta_{\alpha,\lambda}(1,s)\sigma(s)ds-a_0}{a}.$$

Substituting $A_i(i=0,1,2,\cdots,m)$ into (6), we get (5) obviously.

It is easy to see that both $x|_{(0,t_1]}$ and $x|_{(t_1,1]}$ are continuous and the limits $\lim\limits_{t\to 0}t^{1-\alpha}x(t)$ and $\lim\limits_{t\to t_1}x(t)$. So $x\in X$.

On the other hand, if x satisfies (5), we can prove that $x\in X$ and x satisfies (4). The proof is completed. $\square$

LEMMA 2.2. *Suppose that* $\sigma : (0,1) \to R$ *satisfies that there exist numbers* $k > -1$ *and* $\max\{-\beta, -k-1\} < l \le 0$ *such that* $|\sigma(t)| \le (t-t_i)^k (t_{i+1}-t)^l$ *for all* $t \in (t_i, t_{i+1}), i \in N[0,m]$. *The* y *is a solutions of*

$$(7) \qquad \begin{cases} D_{t_i^+}^{\beta} y(t) - \mu y(t) = \sigma(t), t \in (t_i, t_{i+1}), i \in N[0,m], \\ y(1) - b\lim_{t\to 0} t^{1-\beta} y(t) = b_0, \\ \lim\limits_{t\to t_i^+} (t-t_i)^{1-\beta} y(t) = J_i, i \in N[1,m] \end{cases}$$

if and only if $y \in Y$ *and*

(8)

$$y(t) = \begin{cases} \Gamma(\beta)\delta_{\beta,\mu}(t,0)\frac{J_m\Gamma(\beta)\delta_{\beta,\mu}(1,t_m)+\int_{t_m}^1 \delta_{\beta,\mu}(1,s)\sigma(s)ds - b_0}{b} \\ +\int_0^t \delta_{\beta,\mu}(t,s)\sigma(s)ds, t \in (0,t_1], \\ \Gamma(\beta)\delta_{\beta,\mu}(t,t_i)J_i + \int_{t_i}^t \delta_{\beta,\mu}(t,s)\sigma(s)ds, t \in (t_i, t_{i+1}], i \in N[1,m]. \end{cases}$$

Proof. The proof is similar to that of the proof of Lemma 2.1 and is omitted.

□

Define the nonlinear operator T on E by

$$T(x,y)(t) = ((T_1(x,y))(t), (T_2(x,y))(t)) \text{ with}$$

$$(T_1(x,y))(t) = \begin{cases} \frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(t,0)\delta_{\alpha,\lambda}(1,t_m)}{a} I(t_m, x(t_m), y(t_m)) \\ +\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{a}\int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)p(s)f(s,x(s),y(s))ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)\frac{\int_0^1 \phi(s)G(s,x(s),y(s))ds}{a} \\ +\int_0^t \delta_{\alpha,\lambda}(t,s)p(s)f(s,x(s),y(s))ds, t \in (0,t_1], \\ \Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_i)I(t_i, x(t_i), y(t_i)) \\ +\int_{t_i}^t \delta_{\alpha,\lambda}(t,s)p(s)f(s,x(s),y(s))ds, t \in (t_i, t_{i+1}], i \in N[1,m]. \end{cases}$$

$$(T_2(x,y))(t)=$$
$$\begin{cases} \frac{\Gamma(\beta)^2\delta_{\beta,\mu}(t,0)\delta_{\beta,\mu}(1,t_m)}{b}J(t_m,x(t_m),y(t_m)) \\ \frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b}\int_{t_m}^1\delta_{\beta,\mu}(1,s)q(s)g(s,x(s),y(s))ds \\ -\frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b}\int_0^1\psi(s)H(s,x(s),y(s))ds \\ +\int_0^t\delta_{\beta,\mu}(t,s)q(s)g(s,x(s),y(s))ds, t\in(0,t_1], \\ \Gamma(\beta)\delta_{\beta,\mu}(t,t_i)J(t_i,x(t_i),y(t_i)) \\ +\int_{t_i}^t\delta_{\beta,\mu}(t,s)q(s)g(s,x(s),y(s))ds, t\in(t_i,t_{i+1}], i\in N[1,m] \end{cases}$$

for $(x,y)\in E$.

Lemma 2.3. *Suppose that (a)-(e) hold and $\lambda>0,\mu>0$. Then $T:E\to E$ is well defined and is completely continuous.*

Proof. **Step (i)** We prove that $T:E\to E$ is well defined. It comes from that $T_j(x,y)|_{(t_i,t_{i+1}]}(i=0,1,\cdots,m,j=1,2)$ are continuous and the limits

$$\lim_{t\to t_i^+}\delta_{\alpha,\lambda}(t,t_i)(T_1(x,y))(t)(i=0,1,\cdots,m),$$

$$\lim_{t\to t_i}\delta_{\beta,\mu}(t,t_i)(T_2(x,y))(t)(i=0,1,\cdots,m)\text{ exist.}$$

Step (ii) We prove that T is continuous.

Let $(x_n,y_n)\in E$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We can show that $T(x_n,y_n)\to T(x_0,y_0)$ as $n\to\infty$ by using the dominant convergence theorem. We refer the readers to the papers [38, 44, 49].

Step (iii) Prove that T is compact, i.e., prove that $T(\overline{\Omega})$ is relatively compact for every bounded closed subset $\overline{\Omega}\subset E$.

Let $\overline{\Omega}$ be a bounded closed nonempty subset of E. We have $||(x,y)||\le r<+\infty$ for all $(x,y)\in\overline{\Omega}$. Since f,g,G,H are *impulsive Caratheodory functions*, I,J are *Caratheodory functions*, then there exists a constant

$M_I, M_J, M_f, M_g, M_G, M_H \geq 0$ such that

(9)

$$|f\left(t, x(t), y(t)\right)| = \left|f\left(t, \delta_{\alpha,\lambda}(t,t_i)\frac{x(t)}{\delta_{\alpha,\lambda}(t,t_i)}, \delta_{\beta,\mu}(t,t_i)\frac{y(t)}{\delta_{\beta,\mu}(t,t_i)}\right)\right|$$
$$\leq M_f, t \in (t_i, t_{i+1}], i \in N[0,m],$$

$$|g\left(t, x(t), y(t)\right)| \leq M_g, t \in (t_i, t_{i+1}], i \in N[0,m],$$

$$|G\left(t, x(t), y(t)\right)| \leq M_G, t \in (t_i, t_{i+1}], i \in N[0,m],$$

$$|H\left(t, x(t), y(t)\right)| \leq M_H, t \in (t_i, t_{i+1}], i \in N[0,m],$$

$$|I\left(t_i, x(t_i), y(t_i)\right)| = \left|I\left(t_i, \delta_{\alpha,\lambda}(t_i,t_{i-1})\frac{x(t_i)}{\delta_{\alpha,\lambda}(t_i,t_{i-1})}, \delta_{\beta,\mu}(t_i,t_{i-1})\frac{y(t_i)}{\delta_{\beta,\mu}(t_i,t_{i-1})}\right)\right|$$
$$\leq M_I, i \in N[1,m],$$

$$|J\left(t_i, x(t_i), y(t_i)\right)| \leq M_J, i \in N[1,m].$$

This step is done by the following two sub-steps:

Sub-step (iii1) Prove that $T(\overline{\Omega})$ is uniformly bounded.

Using (d), (10), $\lambda > 0, \mu > 0$ and the definition of T_1, we have for $t \in (0, t_1]$ that

$$\begin{aligned}
\frac{|(T_1(x,y))(t)|}{\delta_{\alpha,\lambda}(t,0)} \leq\ & \frac{1}{\delta_{\alpha,\lambda}(t,0)}\frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(t,0)\delta_{\alpha,\lambda}(1,t_m)}{|a|}M_I \\
& +\frac{1}{\delta_{\alpha,\lambda}(t,0)}\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{|a|}\int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)(s-t_m)^{k_1}(1-s)^{l_1}M_f ds \\
& +\frac{1}{\delta_{\alpha,\lambda}(t,0)}\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)\frac{||\phi||_1 M_G}{|a|} \\
& +\frac{1}{\delta_{\alpha,\lambda}(t,0)}\int_0^t \delta_{\alpha,\lambda}(t,s)s^{k_1}(t_1-s)^{l_1}M_f ds \\
\leq\ & \frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(1,t_m)}{|a|}M_I + \frac{\Gamma(\alpha)}{|a|}\int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)(s-t_m)^{k_1}(1-s)^{l_1}M_f ds \\
& +\Gamma(\alpha)\frac{||\phi||_1 M_G}{|a|} + \frac{1}{\delta_{\alpha,\lambda}(t,0)}\int_0^t \delta_{\alpha,\lambda}(t,s)s^{k_1}(t_1-s)^{l_1}M_f ds
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \tfrac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(1,t_m)}{|a|}M_I+\tfrac{\Gamma(\alpha)||\phi||_1}{|a|}M_G\\
&+\tfrac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}\int_{t_m}^1(1-s)^{\alpha-1}(s-t_m)^{k_1}(1-s)^{l_1}dsM_f\\
&+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}s^{k_1}(t-s)^{l_1}dsM_f\\
=\ & \tfrac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(1,t_m)}{|a|}M_I+\tfrac{\Gamma(\alpha)||\phi||_1}{|a|}M_G\\
&+\left(\tfrac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}(1-t_m)^{1+k_1+l_1}+t^{1+k_1+l_1}\right)\mathbf{B}(\alpha+l_1,k_1+1)M_f\\
\leq\ & \tfrac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}M_I+\tfrac{\Gamma(\alpha)||\phi||_1}{|a|}M_G\\
&+\left(\tfrac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)M_f.
\end{aligned}
$$

For $t\in(t_i,t_{i+1}](i\in N[1,m])$, similarly we have

$$
\begin{aligned}
&\tfrac{|(T_1(x,y))(t)|}{\delta_{\alpha,\lambda}(t,t_i)}\leq\Gamma(\alpha)M_I+\tfrac{1}{\delta_{\alpha,\lambda}(t,t_i)}\int_{t_i}^t\delta_{\alpha,\lambda}(t,s)(s-t_i)^{k_1}(t_{i+1}-s)^{l_1}dsM_f\\
&\leq\Gamma(\alpha)M_I+(t-t_i)^{1-\alpha}\int_{t_i}^t(t-s)^{\alpha-1}(s-t_i)^{k_1}(t-s)^{l_1}dsM_f\\
&\leq\Gamma(\alpha)M_I+\mathbf{B}(\alpha+l_1,k_1+1)M_f.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
||T_1(x,y)||\leq&\left(\tfrac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)M_I+\tfrac{\Gamma(\alpha)||\phi||_1}{|a|}M_G\\
&+\left(\tfrac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)M_f.
\end{aligned}\tag{10}
$$

Similarly we have

$$
\begin{aligned}
||T_2(x,y)||\leq&\left(\tfrac{\Gamma(\beta)^2(1-t_m)^{\beta-1}\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)M_J+\tfrac{\Gamma(\beta)||\psi||_1}{|b|}M_H\\
&+\left(\tfrac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)M_g.
\end{aligned}\tag{11}
$$

Then $T(\overline{\Omega})$ is uniformly bounded.

From above discussion, $T(\overline{\Omega})$ is uniformly bounded.

Sub-step (iii2) Prove that both $\left\{t \to \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)} : (x,y) \int \overline{\Omega}\right\}$ and $\left\{t \to \frac{(T_2(x,y))(t)}{\delta_{\beta,\mu}(t,t_i)} : (x,y) \int \overline{\Omega}\right\}$ are equi-continuous on $(t_i, t_{i+1}](i \in N[0,m])$, respectively.

Let

$$\frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)} = \begin{cases} \lim\limits_{t \to t_i^+} \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)}, t = t_i, \\ \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)}, t \in (t_i, t_{i+1}] \end{cases}$$

Since $t \to \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)}$ is continuous on $[t_i, t_{i+1}]$, $\left\{t \to \frac{(T_1(x,y))(t)}{\delta_{\alpha,\lambda}(t,t_i)} : (x,y) \int \overline{\Omega}\right\}$ is equi-continuous on $(t_i, t_{i+1}](i \in N[0,m])$. We can prove similarly that $\left\{t \to \frac{(T_2(x,y))(t)}{\delta_{\beta,\mu}(t,t_i)} : (x,y) \int \overline{\Omega}\right\}$ is equi-continuous on $(t_i, t_{i+1}](i \in N[0,m])$.

So $T(\overline{\Omega})$ is relatively compact. Then T is completely continuous. The proofs are completed. □

3. Main results

Now, we prove that main theorem in this paper by using the Schauder's fixed point theorem [27]. We need the following assumptions:

(C) Φ is a sup-multiplicative-like function with its supporting function w, the inverse function of Φ is Φ^{-1} with supporting function ν.

(D) f, g, H, G are *impulsive caratheodory functions*, I, J are continuous functions and satisfy that there exist nonnegative constants I_0, J_0, $b_i, a_i (i = 1,2)$, $B_i, A_i (i = 1,2)$ and $\overline{B}_i, \overline{A}_i (i = 1,2)$, bounded measurable functions $\phi_i, \psi_i : (0,1) \to R (i = 1,2)$ such that

$$\left|f\left(t, \frac{x}{\delta_{\alpha,\lambda}(t,t_i)}, \frac{y}{\delta_{\beta,\mu}(t,t_i)}\right) - \phi_1(t)\right| \le b_1|x| + a_1\Phi^{-1}(|y|), t \in (t_i, t_{i+1}],$$

$$\left|g\left(t, \frac{x}{\delta_{\alpha,\lambda}(t,t_i)}, \frac{y}{\delta_{\beta,\mu}(t,t_i)}\right) - \phi_2(t)\right| \le b_2\Phi(|x|) + a_2|y|, t \in (t_i, t_{i+1}],$$

$$\left|G\left(t, \frac{x}{\delta_{\alpha,\lambda}(t,t_i)}, \frac{y}{\delta_{\beta,\mu}(t,t_i)}\right) - \psi_1(t)\right| \le B_1|x| + A_1\Phi^{-1}(|y|), t \in (t_i, t_{i+1}],$$

$$\left|H\left(t, \frac{x}{\delta_{\alpha,\lambda}(t,t_i)}, \frac{y}{\delta_{\beta,\mu}(t,t_i)}\right) - \psi_2(t)\right| \le B_2\Phi(|x|) + A_2|y|, t \in (t_i, t_{i+1}]$$

hold for $x, y \in R, i \in N[0,m]$ and

$$\left|I\left(t_i, \frac{x}{\delta_{\alpha,\lambda}(t_i,t_{i-1})}, \frac{y}{\delta_{\beta,\mu}(t_i-t_{i-1})}\right) - I_0\right| \le \overline{B}_1|x| + \overline{A}_1\Phi^{-1}(|y|),$$

$$\left|J\left(t_i, \frac{x}{\delta_{\alpha,\lambda}(t_i,t_{i-1})}, \frac{y}{\delta_{\beta,\mu}(t_i-t_{i-1})}\right) - J_0\right| \le \overline{B}_2\Phi(|x|) + \overline{A}_2|y|$$

hold for $i \in N[1,m], x, y \in R$.

Denote

$$\Phi_1(t) = \begin{cases} \frac{\Gamma(\alpha)^2\delta_{\alpha,\lambda}(t,0)\delta_{\alpha,\lambda}(1,t_m)}{a}I_0 + \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{a}\int_{t_m}^1 \delta_{\alpha,\lambda}(1,s)p(s)\phi_1(s)ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)\frac{\int_0^1 \phi(s)\psi_1(s)}{a} + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)\phi_1(s)ds, t \in (0,t_1], \\ \Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_i)I_0 + \int_{t_i}^t \delta_{\alpha,\lambda}(t,s)p(s)\phi_1(s)ds, t \in (t_i,t_{i+1}], i \in N[1,m], \end{cases}$$

$$\Phi_2(t) = \begin{cases} \frac{\Gamma(\beta)^2\delta_{\beta,\mu}(t,0)\delta_{\beta,\mu}(1,t_m)}{b}J_0 + \frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b}\int_{t_m}^1 \delta_{\beta,\mu}(1,s)q(s)\phi_2(s)ds \\ -\frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b}\int_0^1 \psi(s)\psi_2(s)ds + \int_0^t \delta_{\beta,\mu}(t,s)q(s)\phi_2(s)ds, t \in (0,t_1], \\ \Gamma(\beta)\delta_{\beta,\mu}(t,t_i)J_0 + \int_{t_i}^t \delta_{\beta,\mu}(t,s)q(s)\phi_2(s)ds, t \in (t_i,t_{i+1}], i \in N[1,m] \end{cases}$$

and

$$\begin{aligned} M_2 = \ & \left(\frac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|} + 1\right)\overline{B}_1 + \frac{\Gamma(\alpha)||\phi||_1}{|a|}B_1 \\ & + \left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|} + 1\right)\mathbf{B}(\alpha+l_1, k_1+1)b_1, \\ M_3 = \ & \left(\frac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|} + 1\right)\overline{A}_1 \\ & + \frac{\Gamma(\alpha)||\phi||_1}{|a|}A_1 + \left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|} + 1\right)\mathbf{B}(\alpha+l_1, k_1+1)a_1, \end{aligned}$$

$$N_2 = \left(\frac{\Gamma(\beta)^2(1-t_m)^{\beta-1}\mathbf{E}_{\beta,\beta}(\mu)}{|b|} + 1\right)\overline{B}_2 + \frac{\Gamma(\beta)||\psi||_1}{|b|}B_2$$

$$+ \left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|} + 1\right)\mathbf{B}(\beta + l_2, k_2 + 1)b_2,$$

$$N_3 = \left(\frac{\Gamma(\beta)^2(1-t_m)^{\beta-1}\mathbf{E}_{\beta,\beta}(\mu)}{|b|} + 1\right)\overline{A}_2 + \frac{\Gamma(\beta)||\psi||_1}{|b|}A_2$$

$$+ \left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|} + 1\right)\mathbf{B}(\beta + l_2, k_2 + 1)a_2.$$

THEOREM 3.1. *Suppose that $\lambda > 0, \mu > 0$ and (a)-(e), (C), (D) hold. Then BVP(1) has at least one solution if*

$$M_2 < 1, \quad N_3 < 1, \quad \lim_{r\to+\infty} \frac{\nu(\Phi(r))}{r} < \frac{1-M_2}{M_3}\left[\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\right]^{-1} \tag{12}$$

or

$$M_2 < 1, \quad N_3 < 1, \quad \lim_{r\to+\infty} \omega(1/\Phi^{-1}(r))r > \frac{N_2}{1-N_3}\Phi\left(\frac{M_3}{1-M_2}\right). \tag{13}$$

Proof. To apply the Schauder's fixed point theorem, we should define an closed convex bounded subset Ω of E such that $T(\Omega) \subseteq \Omega$.

For $r_1 > 0, r_2 > 0$, denote $\Omega = \{(x,y) \in E : ||x-\Phi_1|| \le r_1, ||y-\Phi_2|| \le r_2\}$. For $(x,y) \in \Omega$, we get

$$\begin{aligned} ||x|| &\le ||x - \Phi_1|| + ||\Phi_1|| \le r_1 + ||\Phi_1||, \\ ||y|| &\le ||y - \Phi_2|| + ||\Phi_2|| \le r_2 + ||\Phi_2||. \end{aligned} \tag{14}$$

Then

$$\begin{aligned} &|f(t,x(t),y(t)) - \phi_1(t)| \\ &= \left|f\left(t, \delta_{\alpha,\lambda}(t,t_i)\frac{x(t)}{\delta_{\alpha,\lambda}(t,t_i)}, \delta_{\beta,\mu}(t,t_i)\frac{y(t)}{\delta_{\beta,\mu}(t,t_i)}\right) - \phi_1(t)\right| \\ &\le b_1\delta_{\alpha,\lambda}(t,t_i)|x(t)| + a_1\Phi^{-1}(\delta_{\beta,\mu}(t,t_i)|y(t)|) \\ &\le b_1||x|| + a_1\Phi^{-1}(||y||) \le b_1[r_1 + ||\Phi_1||] + a_1\Phi^{-1}(r_2 + ||\Phi_2||), \end{aligned}$$

$$|g(t,x(t),y(t))-\phi_2(t)|\le b_2\Phi(r_1+||\Phi_1||)+a_2[r_2+||\Phi_2||],$$

$$|G(t,x(t),y(t))-\psi_1(t)|\le B_1[r_1+||\Phi_1||]+A_1\Phi^{-1}(r_2+||\Phi_2||),$$

$$|H(t,x(t),y(t))-\psi_2(t)|\le B_2\Phi(r_1+||\Phi_1||)+A_2[r_2+||\Phi_2||]$$

hold for $t\in(t_i,t_{i+1}],i\in N[0,m]$ and

$$|I(t_i,x(t_i),y(t_i))-I_0|\le \overline{B}_1[r_1+||\Phi_1||]+\overline{A}_1\Phi^{-1}(r_2+||\Phi_2||),$$

$$|J(t_i,x(t_i),y(t_i))-J_0|\le \overline{B}_2\Phi(r_1+||\Phi_1||)+\overline{A}_2[r_2+||\Phi_2||]$$

hold for $i\in N[1,m]$.

By the definition of T, using the methods proving (10) and (11), in Step (iii1) of the proof of Lemma 2.3, we have that

$$\begin{aligned}&||T_1(x,y)-\Phi_1||\\
&\le\left(\tfrac{\Gamma(\alpha)^2(1-t_m)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)[\overline{B}_1[r_1+||\Phi_1||]+\overline{A}_1\Phi^{-1}(r_2+||\Phi_2||)]\\
&+\tfrac{\Gamma(\alpha)||\phi||_1}{|a|}[B_1[r_1+||\Phi_1||]+A_1\Phi^{-1}(r_2+||\Phi_2||)]\\
&+\left(\tfrac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)[b_1[r_1+||\Phi_1||]+a_1\Phi^{-1}(r_2+||\Phi_2||)],\end{aligned}$$

and

$$\begin{aligned}&||T_2(x,y)-\Phi_2||\\
&\le\left(\tfrac{\Gamma(\beta)^2(1-t_m)^{\beta-1}\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)[\overline{B}_2\Phi(r_1+||\Phi_1||)+\overline{A}_2[r_2+||\Phi_2||]]\\
&+\tfrac{\Gamma(\beta)||\psi||_1}{|b|}[B_2\Phi(r_1+||\Phi_1||)+A_2[r_2+||\Phi_2||]]\\
&+\left(\tfrac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)[b_2\Phi(r_1+||\Phi_1||)+a_2[r_2+||\Phi_2||]].\end{aligned}$$

It follows that

$$\begin{aligned}&||T_1(x,y)-\Phi_1||\le M_2(r_1+||\Phi_1||)+M_3\Phi^{-1}(r_2+||\Phi_2||),\\
&||T_2(x,y)-\Phi_2||\le N_2\Phi(r_1+||\Phi_1||)+N_3(r_2+||\Phi_2||).\end{aligned}\tag{15}$$

We claim that there exists $r_1, r_2 > 0$ such that

$$\begin{aligned} & M_2(r_1+||\Phi_1||)+M_3\Phi^{-1}(r_2+||\Phi_2||)\le r_1, \\ & N_2\Phi(r_1+||\Phi_1||)+N_3(r_2+||\Phi_2||)\le r_2. \end{aligned} \tag{16}$$

We consider two cases:

Case (i) $M_2<1,\ \ N_3<1,\ \ \lim\limits_{r\to+\infty}\frac{\nu(\Phi(r))}{r}<\frac{1-M_2}{M_3}\left[\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\right]^{-1}$.

First we prove that that exists $r_1>0$ such that

$$r_1\ge\frac{M_2||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}\left(\frac{N_2}{1-N_3}\Phi(r_1+||\Phi_1||)+\frac{||\Phi_2||}{1-N_3}\right). \tag{17}$$

In fact, if

$$r<\frac{M_2||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}\left(\frac{N_2}{1-N_3}\Phi(r+||\Phi_1||)+\frac{||\Phi_2||}{1-N_3}\right)$$

for every $r>0$, using (3), we get

$$1<\frac{M_2||\Phi_1||}{1-M_2}\frac{1}{r}+\frac{M_3}{1-M_2}\frac{1}{r}\Phi^{-1}\left(\frac{N_2}{1-N_3}\Phi(r+||\Phi_1||)+\frac{||\Phi_2||}{1-N_3}\right)$$

$$\le\frac{M_2||\Phi_1||}{1-M_2}\frac{1}{r}+\frac{M_3}{1-M_2}\frac{\nu(\Phi(r))}{r}\Phi^{-1}\left(\frac{\frac{N_2}{1-N_3}\Phi(r+||\Phi_1||)+\frac{||\Phi_2||}{1-N_3}}{\Phi(r)}\right).$$

Let $r\to+\infty$, we get

$$1\le\frac{M_3}{1-M_2}\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\lim_{r\to+\infty}\frac{\nu(\Phi(r))}{r},$$

which contradicts

$$\lim_{r\to+\infty}\frac{\nu(\Phi(r))}{r}<\frac{1-M_2}{M_3}\left[\Phi^{-1}\left(\frac{N_2}{1-N_3}\right)\right]^{-1}.$$

Then there exists $r_1>0$ such that (17) holds. Choose $r_2>0$ satisfying $r_2\ge\frac{N_2}{1-N_3}\Phi(r_1+||\Phi_1||)+\frac{N_3||\Phi_2||}{1-N_3}$. Then $r_1>0$ and $r_2>0$ satisfy (16).

Case (ii) $M_2<1,\ \ N_3<1,\ \ \lim\limits_{r\to+\infty}\omega(1/\Phi^{-1}(r))r>\frac{N_2}{1-N_3}\Phi\left(\frac{M_3}{1-M_2}\right)$.

First we prove that that exists $r_2>0$ such that

$$r_2\ge\frac{N_2}{1-N_3}\Phi\left(\frac{||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}(r_2+||\Phi_2||)\right)+\frac{N_3||\Phi_2||}{1-N_3}. \tag{18}$$

In fact, if

$$r<\frac{N_2}{1-N_3}\Phi\left(\frac{||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}(r+||\Phi_2||)\right)+\frac{N_3||\Phi_2||}{1-N_3}$$

holds for all $r > 0$. using (2), we get $\Phi(xy) \le \frac{1}{\omega(1/x)}\Phi(y)$. Then

$$
\begin{aligned}
1 &< \frac{\frac{N_2}{1-N_3}\Phi\Big(\frac{||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}(r+||\Phi_2||)\Big)}{r} + \frac{N_3||\Phi_2||}{1-N_3}\frac{1}{r} \\
&= \frac{N_2}{1-N_3}\Phi\left(\frac{\Phi^{-1}(r)\frac{||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}(r+||\Phi_2||)}{\Phi^{-1}(r)}\right)\frac{1}{r} + \frac{N_3||\Phi_2||}{1-N_3}\frac{1}{r} \\
&\le \frac{N_2}{1-N_3}\Phi\left(\frac{\frac{||\Phi_1||}{1-M_2}+\frac{M_3}{1-M_2}\Phi^{-1}(r+||\Phi_2||)}{\Phi^{-1}(r)}\right)\frac{1}{\omega(1/\Phi^{-1}(r))r} + \frac{N_3||\Phi_2||}{1-N_3}\frac{1}{r}.
\end{aligned}
$$

Let $r \to \infty$. We get

$$
1 \le \frac{N_2}{1-N_3}\Phi\left(\frac{M_3}{1-M_2}\right)\frac{1}{\lim\limits_{r\to+\infty}\omega(1/\Phi^{-1}(r))r}.
$$

Hence there is $r_2 > 0$ such that (18) holds. Now choose $r_1 > 0$ such that

$$
r_1 \ge \frac{M_2||\Phi_1||}{1-M_2} + \frac{M_3}{1-M_2}\Phi^{-1}(r_2 + ||\Phi_2||).
$$

Then $r_1 > 0$ and $r_2 > 0$ satisfy (16).

We choose $\Omega = \{(x,y) \in E : ||x-\Phi_1|| \le r_1, ||y-\Phi_2|| \le r_2\}$. Then we get $T(\Omega) \subset \Omega$. Hence the Schauder's fixed point theorem implies that T has a fixed point $(x,y) \in \Omega$. So (x,y) is a solution of BVP(1). The proof of Theorem 3.1 is complete. ☐

REMARK 3.1. When the limits $\lim\limits_{r\to+\infty}\frac{\nu(\Phi(r))}{r}$ and $\lim\limits_{r\to+\infty}\omega(1/\Phi^{-1}(r))r$ exist, we note, from Theorem 3.1, that (12) and (13) hold for sufficiently small nonnegative constants I_0, J_0, $b_i, a_i(i = 1,2)$, $B_i, A_i(i = 1,2)$ and $\overline{B}_i, \overline{A}_i(i = 1,2)$. So it is easy to see that BVP(1) has at least one solution if the nonnegative constants I_0, J_0, $b_i, a_i(i = 1,2)$, $B_i, A_i(i = 1,2)$ and $\overline{B}_i, \overline{A}_i(i = 1,2)$ are very small.

REMARK 3.2. In BVP(1) when $\lambda < 0, \mu < 0$, or $\lambda < 0, \mu > 0$, or $\lambda > 0, \mu < 0$, similar result to Theorem 3.1 can be obtained. The details are omitted.

Remark 3.3. Consider the following periodic boundary value problem

$$
\left\{\begin{array}{l}
D_{t_i^+}^{\alpha}x(t) - \lambda x(t) = p(t)f(t,x(t),y(t)), \quad t \in (t_i,t_{i+1}], i = 0,1,\\
D_{t_i^+}^{\beta}y(t) - \mu y(t) = q(t)g(t,x(t),y(t)), \quad t \in (t_i,t_{i+1}], i = 0,1,\\
x(1) - \lim\limits_{t\to 0} t^{1-\alpha}x(t) = 0, \;\; y(1) - \lim\limits_{t\to 0} t^{1-\beta}y(t) = 0,\\
\lim\limits_{t\to t_1^+}(t-t_t)^{1-\alpha}x(t) - x(t_1) = \lim\limits_{t\to t_1^+}(t-t_1)^{1-\beta}y(t) - y(t_1) = 0,
\end{array}\right. \tag{19}
$$

where

(i) $0 < \alpha, \beta < 1$, $\lambda, \mu \in R$ with $\lambda \neq 0, \mu \neq 0$, $D_{t_i^+}^{\alpha}$ (or $D_{t_i^+}^{\beta}$) is the Riemann-Liouville fractional derivative of order α (or β),

(ii) $0 = t_0 < t_1 < t_2 = 1$,

(iii) $p, q : (0,1) \to R$ satisfy the growth conditions: there exist constants $k_i, l_i (i = 1,2)$ with $k_1 > -1, k_2 > -1$ and $\max\{-\alpha, -k_1 - 1\} \leq l_1 \leq 0$ and $\max\{-\beta, -k_2 - 1\} \leq l_2 \leq 0$ such that

$$|p(t)| \leq (t-t_i)^{k_1}(t_{i+1}-t)^{l_1}, \; |q(t)| \leq (t-t_i)^{k_2}(t_{i+1}-t)^{l_2}, \; t \in (t_i,t_{i+1}), i = 0,1,$$

(iv) f, g defined on $(0,1] \times R \times R$ are *impulsive Caratheodory functions.*

Theorem 3.2. *Suppose that $\lambda > 0, \mu > 0$ and (i)-(iv), (C) hold and (D1) f, g are impulsive caratheodory functions, and satisfy that there exist nonnegative constants $b_i, a_i (i = 1,2)$ and bounded measurable functions $\phi_i : (0,1) \to R (i = 1,2)$ such that*

$$|f(t, \delta_{\alpha,\lambda}(t,t_i)x, \delta_{\beta,\mu}(t,t_i)y) - \phi_1(t)| \leq b_1|x| + a_1\Phi^{-1}(|y|), t \in (t_i,t_{i+1}],$$

$$|g(t, \delta_{\alpha,\lambda}(t,t_i)x, \delta_{\beta,\mu}(t,t_i)y) - \phi_2(t)| \leq b_2\Phi(|x|) + a_2|y|, t \in (t_i,t_{i+1}]$$

hold for $x, y \in R, i \in N[0,m]$.

Then BVP(19) has at least one solution if

$$M_2 < 1, \;\; N_3 < 1, \;\; \lim_{r\to+\infty} \tfrac{\nu(\Phi(r))}{r} < \tfrac{1-M_2}{M_3}\left[\Phi^{-1}\left(\tfrac{N_2}{1-N_3}\right)\right]^{-1} \tag{20}$$

or

$$M_2 < 1, \;\; N_3 < 1, \;\; \lim_{r\to+\infty} \omega(1/\Phi^{-1}(r))r > \tfrac{N_2}{1-N_3}\Phi\left(\tfrac{M_3}{1-M_2}\right), \tag{21}$$

where

$$\Phi_1(t)=\begin{cases}\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{a}\int_{t_m}^1\delta_{\alpha,\lambda}(1,s)p(s)\phi_1(s)ds\\ +\int_0^t\delta_{\alpha,\lambda}(t,s)p(s)\phi_1(s)ds, t\in(0,t_1],\\ \int_{t_i}^t\delta_{\alpha,\lambda}(t,s)p(s)\phi_1(s)ds, t\in(t_i,t_{i+1}], i\in N[1,m],\end{cases}$$

$$\Phi_2(t)=\begin{cases}\frac{\Gamma(\beta)\delta_{\beta,\mu}(t,0)}{b}\int_{t_m}^1\delta_{\beta,\mu}(1,s)q(s)\phi_2(s)ds\\ +\int_0^t\delta_{\beta,\mu}(t,s)q(s)\phi_2(s)ds, t\in(0,t_1],\\ \int_{t_i}^t\delta_{\beta,\mu}(t,s)q(s)\phi_2(s)ds, t\in(t_i,t_{i+1}], i\in N[1,m],\end{cases}$$

and

$$M_2=\left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)b_1,$$

$$M_3=\left(\frac{\Gamma(\alpha)\mathbf{E}_{\alpha,\alpha}(\lambda)}{|a|}+1\right)\mathbf{B}(\alpha+l_1,k_1+1)a_1,$$
$$N_2=\left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)b_2,$$

$$N_3=\left(\frac{\Gamma(\beta)\mathbf{E}_{\beta,\beta}(\mu)}{|b|}+1\right)\mathbf{B}(\beta+l_2,k_2+1)a_2.$$

Proof. In Theorem 3.1, choose $G(t,x,y)\equiv H(t,x,y)\equiv 0$, $I(t,x,y)\equiv J(t,x,y)\equiv 0$. The theorem follows Theorem 3.1. The details of proof is omitted. □

REMARK 3.4. Similar results can be obtained for BVP(19) when $\lambda<0,\mu<0$, $\lambda<0,\mu>0$ and $\lambda>0,\mu<0$ respectively. The details are omitted. When the limits $\lim\limits_{r\to+\infty}\frac{\nu(\Phi(r))}{r}$ and $\lim\limits_{r\to+\infty}\omega(1/\Phi^{-1}(r))r$ exist, we note, from Theorem 3.2, that (18) and (19) hold for sufficiently small nonnegative constants $b_i,a_i(i=1,2)$. So it is easy to see that BVP(19) has at least one solution if the nonnegative constants $b_i,a_i(i=1,2)$ are very small.

4. Applications

Now, we present an example, which can not be covered by known results, to illustrate Theorem 3.1.

Example 4.1. *Consider the following periodic type boundary value problem for fractional differential equation*
(22)

$$\begin{cases} D_{t_i^+}^{\frac{2}{3}}x(t)-x(t)=(t-t_i)^{-\frac{1}{4}}(t_{i+1}-t)^{-\frac{1}{4}}f(t,x(t),y(t)),t\in(t_i,t_{i+1}], \\ \\ D_{t_i^+}^{\frac{1}{2}}y(t)-y(t)=(t-t_i)^{-\frac{1}{4}}(t_{i+1}-t)^{-\frac{1}{4}}g(t,x(t),y(t)),t\in(t_i,t_{i+1}], \\ \\ x(1)-\lim\limits_{t\to 0}t^{\frac{1}{3}}x(t)=\frac{1}{2}\int_0^1 s^{-\frac{1}{2}}G(s,x(s),y(s))ds, \\ \\ y(1)-\lim\limits_{t\to 0}t^{\frac{1}{2}}y(t)=\frac{1}{2}\int_0^1 s^{-\frac{1}{2}}H(s,x(s),y(s))ds, \\ \\ \lim\limits_{t\to\frac{1}{2}^+}\left(t-\frac{1}{2}\right)^{\frac{1}{3}}x(t)-x(1/2)=1,\ \lim\limits_{t\to\frac{1}{2}^+}\left(t-\frac{1}{2}\right)^{\frac{1}{2}}y(t)-y(1/2)=1. \end{cases}$$

where $0=t_0<t_1=\frac{1}{2}<t_2=1$ *and*

$$f(t,x,y)=c_1+b_1\delta_{2/3,1}(t,t_i)x+a_1[\delta_{1/2,1}(t,t_i)]^{\frac{1}{3}}y^{\frac{1}{3}},t\in(t_i,t_{i+1}],$$

$$g(t,x,y)=c_2+b_2[\delta_{2/3,1}(t,t_i)]^3x^3+a_2\delta_{1/2,1}(t,t_i)y,t\in(t_i,t_{i+1}],$$

$$G(t,x,y)=C_1+B_1\delta_{2/3,1}(t,t_i)x+A_1[\delta_{1/2,1}(t,t_i)]^{\frac{1}{3}}y^{\frac{1}{3}},t\in(t_i,t_{i+1}],$$

$$H(t,x,y)=C_2+B_2[\delta_{2/3,1}(t,t_i)]^3x^3+A_2\delta_{1/2,1}(t,t_i)y,t\in(t_i,t_{i+1}],$$

with $c_i,b_i,a_i,C_i,B_i,A_i(i=1,2)$ *being nonnegative numbers. Then, BVP(22) has at least one solution for sufficiently small* $b_i,a_i,B_i,A_i(i=1,2)$..

Proof. Corresponding to BVP(1), $\alpha=\frac{2}{3}$, $\beta=\frac{1}{2}$, $\lambda=\mu=1$, $a=b=1$, $t_1=\frac{1}{2}$, $p(t)=q(t)=(t-t_i)^{-\frac{1}{4}}(t_{i+1}-t)^{-\frac{1}{4}}$ for $t\in(t_i,t_{i+1})(i=0,1)$, $\phi(t)=\psi(t)=\frac{1}{2}t^{-\frac{1}{2}}$, $\Phi(x)=x^3$ with $\Phi^{-1}(x)=x^{\frac{1}{3}}$, the supporting function of Φ is $\omega(x)=x^3$ and the supporting function of Φ^{-1} is $\nu(x)=x^{\frac{1}{3}}$, $I(t,x,y)=J(t,x,y)=1$.

It is easy to see that $k_1 = l_1 = k_2 = l_2 = -\frac{1}{4}$, $||\phi||_1 = ||\psi||_1 = 1$ and

$$f\left(t, \frac{x}{\delta_{2/3,1}(t,t_i)}, \frac{y}{\delta_{1/2,1}(t,t_i)}\right) = c_1 + b_1x + a_1\Phi^{-1}(y), t \in (t_i, t_{i+1}], i = 0, 1,$$

$$g\left(t, \frac{x}{\delta_{2/3,1}(t,t_i)}, \frac{y}{\delta_{1/2,1}(t,t_i)}\right) = c_2 + b_2\Phi(x) + a_1y, t \in (t_i, t_{i+1}], i = 0, 1,$$

$$G\left(t, \frac{x}{\delta_{2/3,1}(t,t_i)}, \frac{y}{\delta_{1/2,1}(t,t_i)}\right) = C_1 + B_1x + A_1\Phi^{-1}(y), t \in (t_i, t_{i+1}], i = 0, 1,$$

$$H\left(t, \frac{x}{\delta_{2/3,1}(t,t_i)}, \frac{y}{\delta_{1/2,1}(t,t_i)}\right) = C_2 + B_2\Phi(x) + A_2y, t \in (t_i, t_{i+1}], i = 0, 1.$$

It is easy to see that $I_0 = J_0 = 1$ and $\overline{B}_1 = \overline{B}_2 = \overline{A}_1 = \overline{A}_2 = 0$.

One sees that (C) and (D) hold. By computation, we get by direct computation that

$$M_2 = \Gamma(2/3)B_1 + \big(\Gamma(2/3)\mathbf{E}_{2/3,2/3}(1) + 1\big)\,\mathbf{B}(5/12, 3/4)b_1,$$

$$M_3 = \Gamma(2/3)A_1 + \big(\Gamma(2/3)\mathbf{E}_{2/3,2/3}(1) + 1\big)\,\mathbf{B}(5/12, 3/4)a_1,$$

$$N_2 = \Gamma(1/2)B_1 + \big(\Gamma(1/2)\mathbf{E}_{1/2,1/2}(1) + 1\big)\,\mathbf{B}(1/4, 3/4)b_2,$$

$$N_3 = \Gamma(1/2)A_2 + \big(\Gamma(1/2)\mathbf{E}_{1/2,1/2}(1) + 1\big)\,\mathbf{B}(1/4, 3/4)a_2.$$

From Theorem 3.1, we know that BVP(22) has at least one solution if

$$M_2 < 1, \quad N_3 < 1, \quad \lim_{r\to+\infty} \frac{\nu(\Phi(r))}{r} = 1 < \frac{1-M_2}{M_3}\sqrt[3]{\frac{1-N_3}{N_2}}$$

or

$$M_2 < 1, \quad N_3 < 1, \quad \lim_{r\to+\infty} \omega(1/\Phi^{-1}(r))r = 1 > \frac{N_2}{1-N_3}\sqrt[3]{\frac{M_3}{1-M_2}}.$$

So BVP(22) has at least one solution for sufficiently small $b_i, a_i, B_i, A_i (i = 1, 2)$. □

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