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QUALITATIVE UNCERTAINTY PRINCIPLES FOR THE INVERSE OF THE HYPERGEOMETRIC FOURIER TRANSFORM

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Abstract. In this paper, we prove an L^p version of Donoho-Stark's uncertainty principle for the inverse of the hypergeometric Fourier transform on $\mathbb{R}^d$. Next, using the ultracontractive properties of the semigroups generated by the Heckman-Opdam Laplacian operator, we obtain an L^p Heisenberg-Pauli-Weyl uncertainty principle for the inverse of the hypergeometric Fourier transform on $\mathbb{R}^d$.

1. Introduction

We consider the differential-difference operators $T_j,\ j = 1, 2, ..., d$, associated with a root system $\mathcal{R}$ and a multiplicity function k, introduced by Cherednik in [5], and called the Cherednik operators in the literature. These operators were helpful for the extension and simplification of the theory of Heckman-Opdam which is a generalization of the harmonic analysis on the symmetric spaces G/K, (cf. [23, 24, 26]).

The Cherednik and Heckman-Opdam theories are based on the Opdam-Cherednik kernel $G_\lambda, \lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$T_j u(x) = -i\lambda_j u(x), \quad j = 1, 2, ..., d,$$

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This paper is dedicated to Professor Khalifa Trimèche on the occasion of his promotion to Professor Emeritus.

satisfying the normalizing condition $u(0) = 1$, and the Heckman-Opdam kernel $F_\lambda, \lambda \in \mathbb{C}^d$, which is defined by

$$\forall\, x \in \mathbb{R}^d, F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx),$$

where W is the Weyl group associated with the root system $\mathcal{R}$, (cf. [23, 24]).

With the kernel G_λ Opdam and Cherednik have defined in [5, 23] the Opdam-Cherednik transform $\mathcal{H}$ and have used the kernel F_λ to define the Opdam-Cherednik transform $\mathcal{H}_k^W$ on spaces of W-invariant functions, and have established some of their properties (see also [24]).

Classical uncertainty principles give us information about a function and its Fourier transform. If we try to limit the behavior of one we lose control of the other. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics they tell us that a particles speed and position cannot both be measured with infinite precision. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. The mathematical equivalent is that a function and its Fourier transform cannot both be arbitrarily localized. There is two categories of uncertainty principles: Quantitative uncertainty principles and Qualitative uncertainty principles.

Quantitative uncertainty principles is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relate. They are called uncertainty principles since they are similar to the classical Heisenberg Uncertainty Principle, which has had a big part to play in the development and understanding of quantum physics. For example: Benedicks [2], Slepian and Pollak [27], Landau and Pollak [15], and Donoho and Stark [8] paid attention to the supports of functions and gave quantitative uncertainty principles for the Fourier transforms.

Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [11], Morgan [21], Cowling and Price [7], Beurling [3], Miyachi [20] theorems enter within the framework of the qualitative uncertainty principles.

The quantitative and qualitative uncertainty principles has been studied by many authors for various Fourier transforms, for examples (cf. [6, 9, 14, 16, 17, 19, 31]).

Our aim here is to consider quantitative uncertainty principles when the transform under consideration is the inverse of the hypergeometric Fourier transform. The hypergeometric Fourier transform have been studied by many authors from many points of view [18, 22, 26, 28].

The remaining part of the paper is organized as follows. In §2, we recall the main results about the harmonic analysis associated with the Cherednik operators and the Heckman-opdam theory. §3 is devoted to study the Donoho-Stark's uncertainty principle and variants of Heisenberg's inequalities for $(\mathcal{H}^W)^{-1}$.

2. Preliminaries

This section gives an introduction to the theory of Cherednik operators, hypergeometric Fourier transform, and hypergeometric convolution. Main references are [5, 23, 24, 26, 28, 30].

2.1. Reflection groups, root systems and multiplicity functions. The basic ingredient in the theory of Cherednik operators are root systems and finite reflection groups, acting on $\mathbb{R}^d$ with the standard Euclidean scalar product $\langle ., . \rangle$ and $||x|| = \sqrt{\langle x, x \rangle}$. On $\mathbb{C}^d$, $||.||$ denotes also the standard Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^{d} z_j \overline{w}_j$.

For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let $\alpha^\vee = \frac{2}{||\alpha||}\alpha$ be the coroot associated to α and let

$$r_\alpha(x) = x - \langle \alpha^\vee, x \rangle \alpha. \tag{2.1}$$

be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α.

A finite set $\mathcal{R} \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $\mathcal{R} \cap \mathbb{R}.\alpha = \{\alpha, -\alpha\}$ and $r_\alpha(\mathcal{R}) = \mathcal{R}$ for all $\alpha \in \mathcal{R}$, where $\mathbb{R}.\alpha := \{\lambda\alpha,\ \lambda \in \mathbb{R}\}$.

For a given root system $\mathcal{R}$ the reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathcal{R}$. All reflections in W correspond to suitable pairs of roots. We fix a positive root system $\mathcal{R}_+ = \Big\{\alpha \in \mathcal{R} :\ \langle \alpha, \beta \rangle > 0\Big\}$ for some $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha \in \mathcal{R}} H_\alpha$.

Let

$$C_+ = \Big\{x \in \mathbb{R}^d : \forall\, \alpha \in \mathcal{R}_+, \quad \langle \alpha, x\rangle > 0\Big\},$$

be the positive chamber. We denote by $\overline{C}_+$ its closure.

A function $k : \mathcal{R} \longrightarrow [0, \infty)$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha). \tag{2.2}$$

Moreover, let A_k denotes the weight function

$$\forall\, x \in \mathbb{R}^d,\ A_k(x) = \prod_{\alpha \in \mathcal{R}_+} |\sinh\langle \frac{\alpha}{2}, x\rangle|^{2k(\alpha)}. \tag{2.3}$$

We note that this function is W invariant and satisfies

$$\forall\, x \in \overline{C}_+,\ A_k(x) \le \exp(2\langle \varrho, x\rangle), \tag{2.4}$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha.$$

2.2. The eigenfunctions of the Cherednik operators.

The Cherednik operators $T_j,\ j\ = 1\ , ...,\ d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha_j}{1 - e^{-\langle \alpha, x\rangle}} \{f(x) - f(r_\alpha(x))\} - \rho_j f(x). \tag{2.5}$$

The operators T_j can also be written in the form

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_j \coth\langle \frac{\alpha}{2}, x\rangle \{f(x) - f(r_\alpha(x))\} - \frac{1}{2} S_j f(x),$$

with

$$\forall\, x \in \mathbb{R}^d, \quad S_j f(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_j f(r_\alpha(x)).$$

In the case $k(\alpha) = 0$, for all $\alpha \in \mathcal{R}_+$, the $T_j, j = 1, 2, ..., d$, reduce to the corresponding partial derivatives.

EXAMPLE 1. For $d = 1$, the root systems are $\mathcal{R} = \{-\alpha, \alpha\}$, $\mathcal{R} = \{-2\alpha, 2\alpha\}$ or $\mathcal{R} = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$ with α the positive root. We take the normalization $\alpha = 2$.

For $\mathcal{R}_+ = \{\alpha\}$, we have the Cherednik operator

$$T_1 f(x) = \frac{d}{dx} f(x) + \frac{2k_\alpha}{1 - e^{-2x}} \{f(x) - f(-x)\} - \rho f(x),$$

with $\rho = k_\alpha$. This operator can also be written in the form

$$T_1 f(x) = \frac{d}{dx} f(x) + k_\alpha \coth(x) \{f(x) - f(-x)\} - k_\alpha f(-x). \tag{2.6}$$

For $\mathcal{R}_+ = \{2\alpha\}$, we have the Cherednik operator

$$T_1 f(x) = \frac{d}{dx} f(x) + \frac{4k_{2\alpha}}{1 - e^{-4x}} \{f(x) - f(-x)\} - \rho f(x).$$

This operator can also be written in the form

$$T_1 f(x) = \frac{d}{dx} f(x) + (k_{2\alpha} \coth(x) + k_{2\alpha} \tanh(x)) \{f(x) - f(-x)\} - \rho f(-x). \tag{2.7}$$

with $\rho = 2k_{2\alpha}$.

For $\mathcal{R}_+ = \{\alpha, 2\alpha\}$, we have the Cherednik operator

$$T_1 f(x) = \frac{d}{dx} f(x) + \left(\frac{2k_\alpha}{1 - e^{-2x}} + \frac{4k_{2\alpha}}{1 - e^{-4x}} \right) \{f(x) - f(-x)\} - \rho f(x),$$

with $\rho = k_\alpha + 2k_{2\alpha}$. It is also equal to

$$T_1 f(x) = \frac{d}{dx} f(x) + ((k_\alpha + k_{2\alpha}) \coth(x) + k_{2\alpha} \tanh(x)) \{f(x) - f(-x)\} - \rho f(-x). \tag{2.8}$$

The operators (2.6), (2.7) and (2.8) are particular cases of the differential-difference operator

$$\Lambda_{k,k'} f(x) = \frac{d}{dx} f(x) + (k \coth(x) + k' \tanh(x)) \{f(x) - f(-x)\} - \rho f(-x), \tag{2.9}$$

which is refereed to as the Jacobi-Cherednik operator (cf. [1, 10]).

The Heckman-Opdam Laplacian $\triangle_k$ is defined by

$$\begin{aligned}\triangle_k f(x) := & \sum_{j=1}^{d} T_j^2 f(x) = \triangle f(x) \\ & + \sum_{\alpha\in\mathbb{R}_+} k(\alpha)(\coth\langle\frac{\alpha}{2}, x\rangle)\langle\nabla f(x),\alpha\rangle + \|\rho\|^2 f(x) \\ & - \sum_{\alpha\in R_+} k(\alpha)\frac{\|\alpha\|^2}{4(\sinh\langle\frac{\alpha}{2}, x\rangle)^2}\{f(x) - f(r_\alpha(x))\},\end{aligned} \tag{2.10}$$

where $\triangle$ and ∇ are respectively the Laplacian and the gradient on $\mathbb{R}^d$.

The Heckman-Opdam Laplacian on W-invariant functions is denoted by $\triangle_k^W$ and has the expression

$$\triangle_k^W f(x) = \triangle f(x) + \sum_{\alpha\in R_+} k(\alpha)(\coth\langle\frac{\alpha}{2}, x\rangle)\langle\nabla f(x),\alpha\rangle + \|\rho\|^2 f(x).$$

Example 2. For $d = 1$, $W = \mathbb{Z}_2$ and $k \geq k' \geq 0$, $k \neq 0$, the Heckman-Opdam Laplacian $\triangle_k^W$ is the Jacobi operator defined for even functions f of class C^2 on $\mathbb{R}$ by

$$\triangle_k^W f(x) = \frac{d^2}{dx^2} f(x) + (2k\coth x + 2k'\tanh x)\frac{d}{dx} f(x) + \varrho^2 f(x),$$

with $\varrho = k + k'$.

We denote by G_λ the eigenfunction of the operators $T_j, j = 1, 2, ..., d$. It is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} T_j u(x) & = -i\lambda_j u(x), j = 1, 2, ..., d, x \in \mathbb{R}^d \\ u(0) & = 1. \end{cases}$$

It is called the Opdam-Cherednik kernel.
We consider the function F_λ defined by

$$\forall\, x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|}\sum_{w\in W} G_\lambda(wx).$$

This function is the unique analytic W-invariant function on $\mathbb{R}^d$, which satisfies the differential equations

$$\begin{cases} p(T)u(x) & = p(-i\lambda)u(x), \quad x \in \mathbb{R}^d, \lambda \in \mathbb{R}^d \\ u(0) & = 1, \end{cases}$$

for all W-invariant polynomial p on $\mathbb{R}^d$ and $p(T) = p(T_1, ..., T_d)$.
In particular for all $\lambda \in \mathbb{R}^d$ we have

$$\triangle_k^W F_\lambda(x) = -\|\lambda\|^2 F_\lambda(x).$$

The function F_λ is called the Heckman-Opdam kernel.

The functions G_λ and F_λ possess the following properties

i) For all $x \in \mathbb{R}^d$, the functions G_λ and F_λ are entire on $\mathbb{C}^d$.

ii) For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have

$$\overline{G_\lambda(x)} = G_{-\overline{\lambda}}(x) \quad \text{and} \quad \overline{F_\lambda(x)} = F_{-\overline{\lambda}}(x). \tag{2.11}$$

iii) There exists a positive constant $M_0 := \sqrt{|W|}$ such that

$$\forall\, x \in \mathbb{R}^d\ ,\forall\, \lambda \in \mathbb{R}^d\ , |F_\lambda(x)| \leq M_0, \tag{2.12}$$

and

$$\forall\, x \in \mathbb{R}^d,\ \forall\, \lambda \in \mathbb{R}^d, |G_\lambda(x)| \leq M_0.$$

iv) We have

$$\forall\, x \in \overline{C}_+, F_0(x) \asymp e^{-\langle \rho, x\rangle} \prod_{\alpha \in R_+^0} (1 + \langle \alpha, x\rangle).$$

v) Let p and q be polynomials of degree n and m. Then there exists a positive constant M' such that for all $\lambda \in \mathbb{C}^d$ and for all $x \in \mathbb{R}^d$, we have

(2.13)
$$|p(\frac{\partial}{\partial \lambda})q(\frac{\partial}{\partial x})F_\lambda(x)| \leq M'(1 + \|x\|)^n(1 + \|\lambda\|)^m F_0(x)e^{\max_{w\in W}(\mathrm{Im}\langle w\lambda, x\rangle)}.$$

vi) The preceding estimate holds true for G_λ too.

Example 3. When $d = 1$ and $W = \mathbb{Z}_2$, and $k \geq k' \geq 0$, $k \neq 0$, the Opdam-Cherednik kernel $G_\lambda(x)$ is given for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ by

$$G_\lambda(x) = \varphi_\lambda^{(k-\frac{1}{2},k'-\frac{1}{2})}(x) - \frac{1}{\rho - i\lambda}\frac{d}{dx}\varphi^{(k-\frac{1}{2},k'-\frac{1}{2})}(x),$$

where $\varphi_\lambda^{(\alpha,\beta)}(x)$ is the Jacobi function of index (α, β) defined by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda); \alpha + 1; -(\sinh x)^2),$$

with $\rho = \alpha + \beta + 1$ and ${}_2F_1$ is the Gauss hypergeometric function.

In this case the Heckman-Opdam kernel $F_\lambda(x)$ is given for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ by

$$F_\lambda(x) = \varphi_\lambda^{(k-\frac{1}{2},k'-\frac{1}{2})}(x).$$

2.3. The Hypergeometric Fourier transform on W-invariant function.

`Notations`. We denote by

$\mathcal{E}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant.

$D(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant and with compact support.

$\mathcal{S}(\mathbb{R}^d)^W$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$, which are W-invariant.

$\mathcal{S}_2(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$ which are W-invariant, and such that for all $\ell, n \in \mathbb{N}$, we have

$$\sup_{\substack{|\mu|\leq n\\ x\in\mathbb{R}^d}} (1+\|x\|)^\ell F_0^{-1}(x)|D^\mu f(x)| < +\infty,$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial_{x_1}^{\mu_1}...\partial_{x_d}^{\mu_d}}, \quad \mu=(\mu_1,...,\mu_d)\in\mathbb{N}^d.$$

$PW(\mathbb{C}^d)^W$ the space of entire functions on $\mathbb{C}^d$, which are W-invariant, rapidly decreasing and of exponential type.

$\mathcal{PW}(\mathbb{C}^d)^W$ the space of entire functions on $\mathbb{C}^d$, which are W-invariant, slowly increasing and of exponential type.

$L^p_{A_k}(\mathbb{R}^d)^W$, $1\leq p\leq\infty$, the space of measurable functions f on $\mathbb{R}^d$ which are W-invariant and satisfying

$$\begin{aligned}\|f\|_{L^p_{A_k}(\mathbb{R}^d)^W} &= \left(\int_{\mathbb{R}^d}|f(x)|^p A_k(x)dx\right)^{1/p} < \infty, \quad \text{if } 1\leq p<\infty\\ \|f\|_{L^\infty_{A_k}(\mathbb{R}^d)^W} &= ess\sup_{x\in\mathbb{R}^d}|f(x)| < +\infty.\end{aligned}$$

$L^p_{\nu_k}(\mathbb{R}^d)^W$, $1\leq p\leq\infty$, the space of measurable functions f on $\mathbb{R}^d$ which are W-invariant and satisfying

$$\begin{aligned}\|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W} &= \left(\int_{\mathbb{R}^d}|f(x)|^p d\nu_k(x)\right)^{1/p} < \infty, \quad \text{if } 1\leq p<\infty\\ \|f\|_{L^\infty_{\nu_k}(\mathbb{R}^d)^W} &= ess\sup_{x\in\mathbb{R}^d}|f(x)| < \infty,\end{aligned}$$

where

$$\begin{aligned}d\nu_k(\lambda) &:= C_k(\lambda)d\lambda\\ &= c\prod_{\alpha\in R_+}\frac{\Gamma(-i\langle\lambda,\alpha^\vee\rangle+k(\alpha)+\frac{1}{2}k(\frac{\alpha}{2}))\,\Gamma(i\langle\lambda,\alpha^\vee\rangle+k(\alpha)+\frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(-i\langle\lambda,\alpha^\vee\rangle+\frac{1}{2}k(\frac{\alpha}{2}))\,\Gamma(i\langle\lambda,\alpha^\vee\rangle+\frac{1}{2}k(\frac{\alpha}{2}))}d\lambda,\end{aligned}$$

with c a normalizing constant and $k(\frac{\alpha}{2}) = 0$ if $\frac{\alpha}{2} \notin R_+$.

The measure $d\nu_k(\lambda)$ is called the symmetric Plancherel measure or Harish-Chandra measure (cf. [23, 26]).

Remark 1. The function C_k is a positive, continuous on $\mathbb{R}^d$ and satisfies the estimate

$$\forall\, \lambda \in \mathbb{R}^d, \quad |C_k(\lambda)| \leq const.||\lambda||^{|\mathcal{R}_0^+|}(1 + ||\lambda||)^{2\gamma - |\mathcal{R}_0^+|},$$

where $\mathcal{R}_0^+ = \{\alpha \in \mathcal{R}^+ : \quad \frac{\alpha}{2} \notin \mathcal{R}^+\}$.

The Hypergeometric Fourier transform of a function f in $D(\mathbb{R}^d)^W$ is given by

$$\mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x) F_\lambda(-x) A_k(x) dx, \quad \text{for all } \lambda \in \mathbb{R}^d. \tag{2.14}$$

Proposition 1. *For all $f \in D(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) we have the following relations*

$$\mathcal{H}^W(\bar{f})(\lambda) = \overline{\mathcal{H}^W(\check{f})(\lambda)}, \quad \text{for all } \lambda \in \mathbb{R}^d \tag{2.15}$$

$$\mathcal{H}^W(f)(\lambda) = \mathcal{H}^W(\check{f})(-\lambda), \quad \text{for all } \lambda \in \mathbb{R}^d, \tag{2.16}$$

where $\check{f}$ is the function defined by $\check{f}(x) = f(-x)$.

Proof. We deduce these relations from (2.11) and (2.14). □

Proposition 2. *The transform $\mathcal{H}^W$ is a topological isomorphism from*

i) *$D(\mathbb{R}^d)^W$ onto $PW(\mathbb{C}^d)^W$.*
ii) *$\mathcal{S}_2(\mathbb{R}^d)^W$ onto $\mathcal{S}(\mathbb{R}^d)^W$.*

The inverse transform is given by

$$\forall\, x \in \mathbb{R}^d,\ (\mathcal{H}^W)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda) F_\lambda(x) d\nu_k(\lambda). \tag{2.17}$$

Proof. See [26]. □

Proposition 3. *For f in $L^1_{\nu_k}(\mathbb{R}^d)^W$ the function $(\mathcal{H}^W)^{-1}(f)$ is continuous on $\mathbb{R}^d$ and we have*

$$\|(\mathcal{H}^W)^{-1}(f)\|_{L^\infty_{A_k}(\mathbb{R}^d)^W} \leq M_0 \|f\|_{L^1_{\nu_k}(\mathbb{R}^d)^W}. \tag{2.18}$$

where M_0 is the constant given by the relation (2.12).

Proof. For all $\lambda \in \mathbb{R}^d$, the function $x \mapsto f(\lambda)F_\lambda(x)$ is continuous on $\mathbb{R}^d$, and from the relation (2.12) we have

$$\forall\, x \in \mathbb{R}^d,\ |f(\lambda)F_\lambda(x)| \leq M_0|f(\lambda)|.$$

As f belongs to $L^1_{\nu_k}(\mathbb{R}^d)^W$, then from the theorem of continuity of integral depending with parameter, we deduce the continuity of $(\mathcal{H}^W)^{-1}(f)$.

Moreover, we have

$$\forall\, x \in \mathbb{R}^d, \quad |(\mathcal{H}^W)^{-1}(f)(x)| \leq \int_{\mathbb{R}^d} |f(\lambda)|\,|F_\lambda(x)| d\nu_k(\lambda).$$

From the relation (2.12), we obtain

$$\forall\, x \in \mathbb{R}^d, \quad |(\mathcal{H}^W)^{-1}(f)(x)| \leq M_0 \int_{\mathbb{R}^d} |f(\lambda)| d\nu_k(\lambda).$$

This completes the proof. □

Definition 1. Let x be in $\mathbb{R}^d$. The hypergeometric translation operator $f \mapsto \tau^W_x f$ is defined on $D(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) by

$$\mathcal{H}^W(\tau^W_x f)(\lambda) = F_\lambda(x)\mathcal{H}^W(f)(\lambda), \quad \text{for all } \lambda \in \mathbb{R}^d. \tag{2.19}$$

Using the hypergeometric translation operator, we define the hypergeometric convolution product, of functions as follows.

Definition 2. The hypergeometric convolution product of two functions f, g in $D(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) is defined by

$$f *_{\mathcal{H}^W} g(x) = \int_{\mathbb{R}} \tau^W_x f(-y) g(y) A_k(y) dy, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.20}$$

Proposition 4. ([30]). i*) For all f, g in $D(\mathbb{R}^d)^W$ (resp . $\mathcal{S}_2(\mathbb{R}^d)^W$), the function $f *_{\mathcal{H}^W} g$ belongs to $D(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$).*

ii*) For all f, g in $D(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$), we have*

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{H}^W(f *_{\mathcal{H}^W} g)(\lambda) = \mathcal{H}^W(f)(\lambda)\mathcal{H}^W(g)(\lambda). \tag{2.21}$$

Proposition 5. i*) Plancherel formula.*

For all f, g in $D(\mathbb{R}^d)^W$ (resp . $\mathcal{S}_2(\mathbb{R}^d)^W$) we have

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)} A_k(x) dx = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(g)(\lambda)} d\nu_k(\lambda). \tag{2.22}$$

ii*) Plancherel theorem.*

The transform $\mathcal{H}^W$ extends uniquely to an isomorphism from $L^2_{A_k}(\mathbb{R}^d)^W$ onto $L^2_{\nu_k}(\mathbb{R}^d)^W$.

Proof. i) By applying the relation (2.17) to the relation (2.21) we obtain

$$\forall\, x \in \mathbb{R}^d, \quad f *_{\mathcal{H}^W} \bar{g}(x) = \int_{\mathbb{R}^d} F_\lambda(x)\mathcal{H}^W(f)(\lambda)\mathcal{H}^W(\bar{g})(\lambda)d\nu_k(\lambda).$$

The relations (2.15), (2.20) permit to write this relation in the following form

$$\forall\, x \in \mathbb{R}^d, \quad \int_{\mathbb{R}} \tau_x^W f(y)\overline{\check{g}(y)}A_k(y)dy = \int_{\mathbb{R}^d} F_\lambda(x)\mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(\check{g})(\lambda)}d\nu_k(\lambda).$$

We obtain (2.22) by changing $\check{g}$ by g in the two members, by taking $x = 0$, and by using the relations

$$\forall\, y \in \mathbb{R}^d, \quad \tau_0^W f(y) = f(y) \quad \text{and} \quad \forall\, \lambda \in \mathbb{R}^d, \quad F_\lambda(0) = 1.$$

ii) We deduce the result from the relation (2.22) and the fact that the space $\mathcal{S}_2(\mathbb{R}^d)^W$ is dense in $L^2_{\nu_k}(\mathbb{R}^d)^W$. □

PROPOSITION 6. *Let f be in $L^p_{\nu_k}(\mathbb{R}^d)^W$, $p \in [1, 2]$. Then $(\mathcal{H}^W)^{-1}(f)$ belongs to $L^{p'}_{A_k}(\mathbb{R}^d)^W$, with $\frac{1}{p} + \frac{1}{p'} = 1$, and we have*

$$\left\|(\mathcal{H}^W)^{-1}(f)\right\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \leqslant M_0^{2-p} \left\|f\right\|_{L^p_{\nu_k}(\mathbb{R}^d)^W}.$$

Proof. From Proposition 3, we have

$$\left\|(\mathcal{H}^W)^{-1}(f)\right\|_{L^\infty_{A_k}(\mathbb{R}^d)^W} \leqslant M_0 \left\|f\right\|_{L^1_{\nu_k}(\mathbb{R}^d)^W}$$

for any $f \in L^1_{\nu_k}(\mathbb{R}^d)^W$. Moreover, by Proposition 5 we have

$$\left\|(\mathcal{H}^W)^{-1}(f)\right\|_{L^2_{A_k}(\mathbb{R}^d)^W} = \left\|f\right\|_{L^2_{\nu_k}(\mathbb{R}^d)^W}$$

for any $f \in L^2_{\nu_k}(\mathbb{R}^d)^W$. The result follows then from the Riesz-Thorin interpolation theorem. □

3. Quantitative Uncertainty Principle For the generalized Fourier transform

We shall investigate the case where f and $(\mathcal{H}^W)^{-1}(f)$ are close to zero outside measurable sets. Here the notion of "close to zero" is formulated as follows. We say that a function $f \in L^p_{\nu_k}(\mathbb{R}^d)^W$, $1 \leq p \leq 2$, is

ε-concentrated on a measurable set $E \subset \mathbb{R}^d$ if there is a measurable function g vanishing outside E such that $||f - g||_{L^p_{\nu_k}(\mathbb{R}^d)^W} \leq \varepsilon \|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W}$. Therefore, if we introduce a projection operator P_E as

$$P_E f(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E, \end{cases}$$

then f is ε-concentrated on E if and only if $||f - P_E f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} \leq \varepsilon_E \|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W}$.

Let T a subset of $\mathbb{R}^d$. We define a projection operator Q_T as

$$Q_T f(\lambda) = \mathcal{H}^W\Big(P_T((\mathcal{H}^W)^{-1}(f))\Big)(\lambda). \tag{3.23}$$

Similarly, we say that $(\mathcal{H}^W)^{-1}(f)$ is ε_T-concentrated to T in $L^{p'}_{A_k}(\mathbb{R}^d)^W$ if and only if

$$\|(\mathcal{H}^W)^{-1}(f) - (\mathcal{H}^W)^{-1}(Q_T f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \leq \varepsilon_T \|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W}. \tag{3.24}$$

If E and T are sets of finite measure, we define $mes_{A_k}(T)$ and $mes_{\nu_k}(E)$, as follow

$$mes_{A_k}(T) := \int_T A_k(x)dx, \quad mes_{\nu_k}(E) := \int_E d\nu_k(y).$$

Lemma 1. *Let T a measurable set of $\mathbb{R}^d$ such that $mes_{A_k}(T) < \infty$. Let $f \in L^p_{\nu_k}(\mathbb{R}^d)^W$ with $p \in [1, 2]$. We have*

$$Q_T f(\lambda) = \int_T F_\lambda(-x)(\mathcal{H}^W)^{-1}(f)(x)A_k(x)dx.$$

Proof. Let $f \in L^p_{\nu_k}(\mathbb{R}^d)^W$ with $p \in [1, 2]$. By (2.12), Hölder's inequality and Proposition 6

$$\begin{aligned} ||P_T((\mathcal{H}^W)^{-1}(f))||_{L^1_{A_k}(\mathbb{R}^d)^W} &= \int_T |(\mathcal{H}^W)^{-1}(f)(x)|A_k(x)dx \\ &\leq \Big(mes_{A_k}(T)\Big)^{\frac{1}{p}} \|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\leq M_0^{2-p}\Big(mes_{A_k}(T)\Big)^{\frac{1}{p}} \|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W}. \end{aligned}$$

and

$$\begin{aligned}||P_T((\mathcal{H}^W)^{-1}(f))||_{L^2_{A_k}(\mathbb{R}^d)^W} &= \int_T |(\mathcal{H}^W)^{-1}(f)(x)|^2 A_k(x)dx \\ &\leq \Big(mes_{A_k}(T)\Big)^{\frac{p'-2}{p'}} \|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\leq M_0^{2-p}\Big(mes_{A_k}(T)\Big)^{\frac{p'-2}{p'}} \|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W}.\end{aligned}$$

Hence $P_T((\mathcal{H}^W)^{-1}(f)) \in L^1_{A_k}(\mathbb{R}^d)^W \bigcap L^2_{A_k}(\mathbb{R}^d)^W$. This combined with (3.23) gives the result. □

We note that, for measurable sets E and T of $\mathbb{R}^d$, where T has finite measure

$$Q_T P_E f(\lambda) = \int_{\mathbb{R}^d} q(y,\lambda) f(y) d\nu_k(y),$$

where

$$q(y,\lambda) = \begin{cases} \displaystyle\int_T F_\lambda(-x)F_y(x)A_k(x)dx & \text{if } y \in E \\ 0 & \text{if } y \notin E. \end{cases} \tag{3.25}$$

Indeed, by Fubini's theorem we see that

$$\begin{aligned}Q_T P_E f(\lambda) &= \int_T (\mathcal{H}^W)^{-1}(P_E f)(x) F_\lambda(-x) A_k(x)dx \\ &= \int_T \Big(\int_E f(y)F_y(x)d\nu_k(y)\Big) F_\lambda(-x)A_k(x)dx \\ &= \int_E f(y)\Big(\int_T F_\lambda(-x)F_y(x)A_k(x)dx\Big)d\nu_k(y).\end{aligned}$$

The Hilbert-Schmidt norm $\|Q_T P_E\|_{HS}$ is given by

$$||Q_T P_E||_{HS} = \Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |q(y,\lambda)|^2 d\nu_k(\lambda)d\nu_k(y)\Big)^{\frac{1}{2}}.$$

We denote by $\|\mathcal{L}\|_2$ the operator norm on $L^2_{\nu_k}(\mathbb{R}^d)$. Since P_E and Q_T are projections, it is clear that $\|P_E\|_2 = \|Q_T\|_2 = 1$. Moreover, it follows that

$$||Q_T P_E||_{HS} \geq ||Q_T P_E||_2. \tag{3.26}$$

Lemma 2. *If E and T are sets of finite measure, then*

$$||Q_T P_E||_{HS} \leq M_0 \sqrt{mes_{A_k}(T) mes_{\nu_k}(E)}.$$

Proof. For $y \in E$, let $g_y(\lambda) = q(y, \lambda)$. (3.25) implies that

$$(\mathcal{H}^W)^{-1}(g_y)(x) = P_T(F_y(x)).$$

Then by Parseval's identity (2.22) and (2.12) it follows that

$$\int_{\mathbb{R}^d} |q(y,\lambda)|^2 d\nu_k(\lambda) = \int_{\mathbb{R}^d} |g_y(\lambda)|^2 d\nu_k(\lambda)$$

$$= \int_{\mathbb{R}^d} |(\mathcal{H}^W)^{-1}(g_y)(x)|^2 A_k(x) dx \leq M_0^2 mes_{A_k}(T).$$

Hence, integrating over $y \in E$, we see that $||Q_T P_E||_{HS}^2 \leq M_0^2 mes_{A_k}(T) mes_{\nu_k}(E)$. □

Proposition 7. *Let E and T be measurable sets and suppose that*

$$\|f\|_{L^2_{\nu_k}(\mathbb{R}^d)^W} = \|(\mathcal{H}^W)^{-1}(f)\|_{L^2_{A_k}(\mathbb{R}^d)^W} = 1.$$

Assume that $\varepsilon_E + \varepsilon_T < 1$, f is ε_T-concentrated on T and $(\mathcal{H}^W)^{-1}(f)$ is ε_E-concentrated on E. Then

$$M_0^2 mes_{A_k}(T) mes_{\nu_k}(E) \geq (1 - \varepsilon_E - \varepsilon_T)^2.$$

Proof. Since $||f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} = \|(\mathcal{H}^W)^{-1}(f)\|_{L^2_{A_k}(\mathbb{R}^d)^W} = 1$ and $\varepsilon_E + \varepsilon_T < 1$, the measures of E and T must both be non-zero. Indeed, if not, then the ε_E-concentration of f implies that

$$||f - P_T f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} = ||f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} = 1 \leq \varepsilon_T,$$

which contradicts with $\varepsilon_T < 1$, likewise for $(\mathcal{H}^W)^{-1}(f)$. If at least one of $mes_{A_k}(T)$ and $mes_{\nu_k}(E)$ is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both E and T have finite positive measures. Since $||Q_T||_2 = 1$, it follows that

$$\begin{aligned} ||f - Q_T P_E f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} &\leq ||f - Q_T f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} + ||Q_T f - Q_T P_E f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} \\ &\leq \varepsilon_T + ||Q_T||_2 ||f - P_E f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} \\ &\leq \varepsilon_E + \varepsilon_T \end{aligned}$$

and thus,

$$||Q_T P_E f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} \geq ||f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} - ||f - Q_T P_E f||_{L^2_{\nu_k}(\mathbb{R}^d)^W} \geq 1 - \varepsilon_E - \varepsilon_T.$$

Hence $||Q_T P_E||_2 \geq 1 - \varepsilon_E - \varepsilon_T$. (3.26) and Lemma 2 yields the desired inequality. □

Let $B_{L^p_{\nu_k}(\mathbb{R}^d)^W}(T)$, $1 \leq p \leq 2$, the subspace of all $g \in L^p_{\nu_k}(\mathbb{R}^d)^W$ such that $Q_T g = g$. We say that f is ε-bandlimited to T if there is a $g \in B_{L^p_{\nu_k}(\mathbb{R}^d)^W}(T)$ with $||f - g||_{L^p_{\nu_k}(\mathbb{R}^d)^W} < \varepsilon \|f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}$. Here we denote by $\|P_E\|_p$ the operator norm of P_E on $L^p_{\nu_k}(\mathbb{R}^d)^W$ and by $\|P_E\|_{p,T}$ the operator norm of $P_E : B_{L^p_{\nu_k}(\mathbb{R}^d)^W}(T) \to L^p_{\nu_k}(\mathbb{R}^d)^W$. Corresponding to (3.26) and Lemma 2 in the $L^2_{\nu_k}(\mathbb{R}^d)$ case, we can obtain the following.

Lemma 3. *Let E and T be measurable sets of $\mathbb{R}^d$. For $p \in [1, 2]$, we have*

$$\|P_E\|_{p,T} \leq M_0^{3-p}\Big(mes_{A_k}(T) mes_{\nu_k}(E)\Big)^{\frac{1}{p}}.$$

Proof. As above, if at least one of $mes_{\nu_k}(E)$ and $mes_{A_k}(T)$ is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both E and T have finite positive measures.

For $f \in B_{L^p_{\nu_k}(\mathbb{R}^d)^W}(T)$, we see that

$$f(y) = \int_T F_y(-x)(\mathcal{H}^W)^{-1}(f)(x) A_k(x) dx.$$

By (2.12), Hölder's inequality and Proposition 6

$$\begin{aligned} |f(y)| &\leq M_0\Big(mes_{A_k}(T)\Big)^{\frac{1}{p}} \|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\leq M_0^{3-p}\Big(mes_{A_k}(T)\Big)^{\frac{1}{p}} \|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W}. \end{aligned}$$

Therefore

$$\begin{aligned} ||P_E f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} &= \Big(\int_E |f(x)|^p d\nu_k(x)\Big)^{\frac{1}{p}} \\ &\leq M_0^{3-p}\Big(mes_{\nu_k}(E) mes_{A_k}(T)\Big)^{\frac{1}{p}} ||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}. \end{aligned}$$

Then, it follows that for $f \in B_{L^p_{\nu_k}(\mathbb{R}^d)^W}(T)$,

$$\frac{||P_E f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}}{||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}} \leq M_0^{3-p}\Big(mes_{A_k}(T) mes_{\nu_k}(E)\Big)^{\frac{1}{p}},$$

which implies the desired inequality. □

Proposition 8. *Let $f \in L^p_{\nu_k}(\mathbb{R}^d)^W$. If f is ε_E-concentrated to E and ε_T-bandlimited to T, then*

$$M_0^{3-p}\Big(mes_{A_k}(T)mes_{\nu_k}(E)\Big)^{\frac{1}{p}} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}.$$

Proof. Without loss of generality, we may suppose that $\|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W} = 1$. Since f is ε_E-concentrated to E, it follows that

$$\|P_E f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W} \geq \|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W} - \|f - P_E g\|_{L^p_{\nu_k}(\mathbb{R}^d)^W} \geq 1-\varepsilon_E.$$

Moreover, since f is ε_T-bandlimited, there is a $g \in B_{L^p_{\nu_k}(\mathbb{R}^d)^W}(T)$ with $\|f - P_E\|_{L^p_{\nu_k}(\mathbb{R}^d)^W} \leq \varepsilon_T$. Therefore, it follows that

$$\begin{aligned}||P_E g||_{L^p_{\nu_k}(\mathbb{R}^d)^W} &\geq ||P_E f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} - ||P_E(g-f)||_{L^p_{\nu_k}(\mathbb{R}^d)^W}\\ &\geq ||P_E f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} - \varepsilon_T \geq 1-\varepsilon_E-\varepsilon_T\end{aligned}$$

and

$$||g||_{L^p_{\nu_k}(\mathbb{R}^d)^W} \leq ||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} + \varepsilon_T = 1+\varepsilon_T.$$

Then, we see that

$$\frac{||P_E g||_{L^p_{\nu_k}(\mathbb{R}^d)^W}}{||g||_{L^p_{\nu_k}(\mathbb{R}^d)^W}} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}.$$

Hence $\|P_E\|_{p,T} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}$ and Lemma 3 yields the desired inequality. □

Proposition 9. *Let $f \in L^1_{\nu_k}(\mathbb{R}^d)^W \cap L^2_{\nu_k}(\mathbb{R}^d)^W$ with $\|f\|_{L^2_{\nu_k}(\mathbb{R}^d)^W} = 1$. If f is ε_E-concentrated to E in $L^1_{\nu_k}(\mathbb{R}^d)^W$-norm and $(\mathcal{H}^W)^{-1}(f)$ is ε_T-concentrated to T in $L^2_{A_k}$-norm, then*

$mes_{\nu_k}(E) \geq (1-\varepsilon_E)^2\|f\|^2_{L^1_{\nu_k}(\mathbb{R}^d)^W}$ *and* $M_0^2 mes_{A_k}(T)\|f\|^2_{L^1_{\nu_k}(\mathbb{R}^d)^W} \geq (1-\varepsilon_T^2)$.

In particular,

$$M_0^2 mes_{A_k}(T)mes_{\nu_k}(E) \geq (1-\varepsilon_E)^2(1-\varepsilon_T^2).$$

Proof. By the orthogonality of the projection operator P_T,

$$\|f\|_{L^2_{\nu_k}(\mathbb{R}^d)^W} = \|(\mathcal{H}^W)^{-1}(f)\|_{L^2_{A_k}(\mathbb{R}^d)^W} = 1$$

and $(\mathcal{H}^W)^{-1}(f)$ is ε_T-concentrated to T in $L^2_{A_k}$-norm, it follows that

$$\begin{aligned}\|P_T(\mathcal{H}^W(f))\|^2_{L^2_{A_k}(\mathbb{R}^d)^W} =& \|(\mathcal{H}^W)^{-1}(f)\|^2_{L^2_{A_k}(\mathbb{R}^d)^W}\\ &- \|(\mathcal{H}^W)^{-1}(f) - P_T((\mathcal{H}^W)^{-1}(f))\|^2_{L^2_{A_k}(\mathbb{R}^d)^W}\\ \geq& 1-\varepsilon_T^2,\end{aligned}$$

and thus,

$$\begin{aligned}1-\varepsilon_T^2 &\leq \int_T |\mathcal{H}^W)^{-1}(f)(\xi)|^2 A_k(\lambda)d\lambda \\ &\leq mes_{A_k}(T)||\mathcal{H}^W)^{-1}(f)||^2_{L^\infty_{A_k}(\mathbb{R}^d)^W} \leq M_0^2 mes_{A_k}(T)||f||^2_{L^1_{\nu_k}(\mathbb{R}^d)^W}.\end{aligned}$$

Similarly, f is ε_E-concentrated to E in $L^1_{\nu_k}(\mathbb{R}^d)^W$-norm,

$$(1-\varepsilon_E)\|f\|_{L^1_{\nu_k}(\mathbb{R}^d)^W} \leq \int_E |f(x)|d\nu_k(x) \leq \sqrt{mes_{\nu_k}(E)}.$$

Here we used the Cauchy-Schwarz inequality and the fact that $\|f\|_{L^2_{\nu_k}(\mathbb{R}^d)^W} = 1$. □

PROPOSITION 10. *Let E and T be measurable subsets of $\mathbb{R}^d$, and $f \in L^p_{\nu_k}(\mathbb{R}^d)^W$ for $p \in (1,2]$. If f is ε_E-concentrated to E in $L^p_{\nu_k}(\mathbb{R}^d)^W$-norm and $(\mathcal{H}^W)^{-1}(f)$ is ε_T-concentrated to T in $L^{p'}_{A_k}(\mathbb{R}^d)^W$-norm, then*

$$\begin{aligned}&M_0^{2-p}(mes_{A_k}(T)mes_{\nu_k}(E))^{\frac{1}{p'}} \\ &\qquad\geq \frac{(1-\varepsilon_T)\|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} - \varepsilon_E M_0^{2-p}||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}}{||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}}.\end{aligned}$$

Proof. Let $f \in L^p_{\nu_k}(\mathbb{R}^d)^W$ for $p \in (1,2]$. As above

$$\begin{aligned}&||(\mathcal{H}^W)^{-1}(f) - (\mathcal{H}^W)^{-1}(f)(Q_T P_E f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\quad\leq ||(\mathcal{H}^W)^{-1}(f) - (\mathcal{H}^W)^{-1}(Q_T f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\quad+ ||(\mathcal{H}^W)^{-1}(Q_T f) - (\mathcal{H}^W)^{-1}(Q_T P_E f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\quad\leq \varepsilon_T||(\mathcal{H}^W)^{-1}(f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} + M_0^{2-p}||f - P_E f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} \\ &\quad\leq \varepsilon_T||(\mathcal{H}^W)^{-1}(f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} + \varepsilon_E M_0^{2-p}||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}\end{aligned}$$

and thus,

$$\begin{aligned}&||(\mathcal{H}^W)^{-1}(Q_T P_E f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ \geq\ &||(\mathcal{H}^W)^{-1}(f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} - ||(\mathcal{H}^W)^{-1}(f) - (\mathcal{H}^W)^{-1}(Q_T P_E f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ \geq\ &(1-\varepsilon_T)||(\mathcal{H}^W)^{-1}(f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} - \varepsilon_E M_0^{2-p}||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}.\end{aligned}$$

On the other hand, it is easy to obtain

$$\frac{||(\mathcal{H}^W)^{-1}(Q_T P_E f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W}}{||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}} \leq M_0^{2-p}\Big(mes_{A_k}(T)mes_{\nu_k}(E)\Big)^{\frac{1}{p'}}.$$

Hence

$$\begin{aligned} &M_0^{2-p}(mes_{A_k}(T)mes_{\nu_k}(E))^{\frac{1}{p'}}||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} \\ &\qquad \geq (1-\varepsilon_T)\|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} - \varepsilon_E M_0^{2-p}||f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}, \end{aligned}$$

which gives the desired result. □

Proposition 11. *Let* $f \in L^1_{\nu_k}(\mathbb{R}^d)^W \cap L^p_{\nu_k}(\mathbb{R}^d)^W$, $p \in (1,2]$. *If* f *is* ε_E*-concentrated to* E *in* $L^1_{\nu_k}(\mathbb{R}^d)^W$*-norm and* $(\mathcal{H}^W)^{-1}(f)$ *is* ε_T*-concentrated to* T *in* $L^{p'}_{A_k}(\mathbb{R}^d)^W$*-norm, then*

$$M_0(mes_{A_k}(T)mes_{\mu_k}(E))^{\frac{1}{p'}} \geq (1-\varepsilon_E)(1-\varepsilon_T)\frac{\|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W}}{\|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W}}.$$

Proof. Let $f \in L^1_{\nu_k}(\mathbb{R}^d)^W \cap L^p_{\nu_k}(\mathbb{R}^d)^W$, $p \in (1,2]$. As $(\mathcal{H}^W)^{-1}(f)$ is ε_T-concentrated to T in $L^{p'}_{\nu_k}$-norm, it follows that

$$\begin{aligned} &\|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ \leq\;& \varepsilon_T\|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} + \Big(\int_T |(\mathcal{H}^W)^{-1}(f)(\lambda)|^{p'}A_k(\lambda)d\lambda\Big)^{\frac{1}{p'}} \\ \leq\;& \varepsilon_T\|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} + (mes_{A_k}(T))^{\frac{1}{p'}}\|(\mathcal{H}^W)^{-1}(f)\|_{L^{\infty}_{A_k}(\mathbb{R}^d)^W}. \end{aligned}$$

Thus from Proposition 3,

$$(3.27)\quad (1-\varepsilon_T)\|(\mathcal{H}^W)^{-1}(f)\|_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \leq M_0(mes_{A_k}(T))^{\frac{1}{p'}}\|f\|_{L^1_{\nu_k}(\mathbb{R}^d)^W}.$$

Similarly, using f is ε_E-concentrated to E in $L^1_{\nu_k}(\mathbb{R}^d)^W$-norm, and Hölder inequality, we obtain

$$(3.28)\qquad (1-\varepsilon_E)\|f\|_{L^1_{\nu_k}(\mathbb{R}^d)^W} \leq (mes_{\nu_k}(E))^{\frac{1}{p'}}\|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W}.$$

Combining (3.27) and (3.28), we obtain the result. □

Remark 2. *Recently Trimèche in [29], has proved that, when the Cherednik operators and the Heckman-Opdam theory are attached to the root system of type* B_2 *or* C_2*, the Heckman-Opdam kernel admits a Laplace type integral representation, and has a better estimate than* *the known one (2.12). In the same paper, Trimèche, reclaim that the estimates for the Heckman-Opdam kernel is also true for the operators attached to the root systems $A_{d-1}, B_d, C_d, BC_d, d \geq 3$. Thus, we deduce that in particular cases of the previous root systems, we can obtain the best estimates in our results by replacing the term M_0 by 1.*

We put for $t > 0$,

$$h_t(x) := (\mathcal{H}^W)^{-1}(e^{-t(||\lambda||^2+||\varrho||^2)})(x), \quad \text{for all}\, x \in \mathbb{R}^d.$$

LEMMA 4. *Let $2 \leq q < \infty$. We have*

$$||h_t||_{L^q_{A_k}(\mathbb{R}^d)} \leq \begin{cases} Ce^{-t||\varrho||^2}t^{-\frac{d+|\mathcal{R}_0^+|}{2q'}} & \text{if} \quad t > 1 \\ Ce^{-t||\varrho||^2}t^{-\frac{2\gamma+d}{2q'}} & \text{if} \quad t \leq 1. \end{cases}$$

Proof. Let $2 \leq q < \infty$. From Proposition 6, we have

$$||h_t||^{q'}_{L^q_{A_k}(\mathbb{R}^d)} \leq C \int_{\mathbb{R}^d} e^{-tq'(||\lambda||^2+||\varrho||^2)} d\nu_k(\lambda).$$

Using now the estimates

$$|C_k(\lambda)| \leq \begin{cases} C||\lambda||^{|\mathcal{R}_0^+|} & \text{if} \quad ||\lambda|| \leq K \\ C||\lambda||^{2\gamma} & \text{if} \quad ||\lambda|| > K. \end{cases}$$

We obtain

$$\begin{aligned} & ||h_t||^{q'}_{L^q_{A_k}(\mathbb{R}^d)} \\ \leq \; & C \int_{\mathbb{R}^d} e^{-tq'(||\lambda||^2+||\varrho||^2)} d\nu_k(\lambda) \\ \leq \; & Ce^{-tq'||\varrho||^2}\Big(\int_{||\lambda|| \leq K} e^{-tq'||\lambda||^2}||\lambda||^{|\mathcal{R}_0^+|}d\lambda + \int_{||\lambda|| \geq K} e^{-tq'||\lambda||^2}||\lambda||^{2\gamma}d\lambda\Big) \\ \leq \; & Ce^{-tq'||\varrho||^2}\Big([t^{-\frac{|\mathcal{R}_0^+|+d}{2}} \int_0^{tq'K^2} e^{-v}v^{\frac{|\mathcal{R}_0^+|+d-2}{2}}dv \\ & \qquad + t^{-\frac{2\gamma+d}{2}} \int_{tq'K^2}^{\infty} e^{-v}v^{\frac{2\gamma+d-2}{2}}dv]\Big) \end{aligned}$$

and the lemma will be proved from the above inequality. □

LEMMA 5. *Let $s > 0$, $p \in [1, 2]$, and $0 < a < \frac{d+|R_0^+|}{q}$, we have*

$$||\,||x||^{-a}\chi_{B(0,s)}||_{L^{p'}_{\nu_k}(\mathbb{R}^d)^W} \leq \begin{cases} Cs^{\frac{2\gamma+d}{p'}-a} & \text{if} \quad s > 1 \\ Cs^{\frac{d+|R_0^+|}{p'}-a} & \text{if} \quad s \leq 1. \end{cases}$$

Proof. Using the estimates

$$|C_k(\lambda)| \leq \begin{cases} C||\lambda||^{|\mathcal{R}_0^+|} & \text{if} \quad ||\lambda|| \leq K \\ C||\lambda||^{2\gamma} & \text{if} \quad ||\lambda|| > K. \end{cases}$$

A simple calculation give that

$$|| \, ||x||^{-a}\chi_{B(0,s)}||_{L^{p'}_{\nu_k}(\mathbb{R}^d)^W} \leq Cs^{-a}V(s),$$

where

$$V(s) \leq \begin{cases} Cs^{\frac{2\gamma+d}{p'}} & \text{if} \quad s > 1 \\ Cs^{\frac{d+|R_0^+|}{p'}} & \text{if} \quad s \leq 1. \end{cases}$$

So we obtain the result. □

On the following propositions, we assume that $2\gamma = |R_0^+|$.

Proposition 12. *Let* $1 < p \leq 2$ *and* $0 < a < \frac{2\gamma+d}{p'}$. *Then for all* $f \in L^p_{\nu_k}(\mathbb{R}^d)^W$ *and* $t > 0$,

$$||h_t(\mathcal{H}^W)^{-1}(f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \leq Ct^{-\frac{a(p'-1)}{2}}|| \, ||x||^a f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}. \tag{3.29}$$

Proof. Inequality (3.29) holds if $|| \, ||x||^a f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} = \infty$. Assume that $|| \, ||x||^a f||_{L^p_{\nu_k}(\mathbb{R}^d)^W} < \infty$. For $s > 0$ let $f_s = f\chi_{B(0,s)}$ and $f^s = f - f_s$. Then since, $|f^s(x)| \leq s^{-a}| \, ||x||^a f(x)|$,

$$\begin{aligned} &||h_t(\mathcal{H}^W)^{-1}(f\chi_{B(0,s)^c})||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\leq \quad ||h_t||_{L^\infty_{A_k}(\mathbb{R}^d)^W} \, ||(\mathcal{H}^W)^{-1}(f\chi_{B(0,s)^c})||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\leq \quad C||f\chi_{B(0,s)^c}||_{L^p_{\nu_k}(\mathbb{R}^d)^W} \\ &\leq \quad Cs^{-a}|| \, ||x||^a f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}. \end{aligned}$$

On the other hand, by Proposition 6 and Hölder's inequality

$$\begin{aligned} &||h_t(\mathcal{H}^W)^{-1}(f\chi_{B(0,s)})||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ &\leq \quad ||h_t||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \, ||(\mathcal{H}^W)^{-1}(f\chi_{B(0,s)})||_{L^\infty_{A_k}(\mathbb{R}^d)^W} \\ &\leq \quad M_0||h_t||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \, ||f\chi_{B(0,s)}||_{L^1_{\nu_k}(\mathbb{R}^d)^W} \\ &\leq \quad M_0||h_t||_{L^{p'}_{A_k}(\mathbb{R}^d)^W}|| \, ||x||^{-a}\chi_{B(0,s)}||_{L^{p'}_{\nu_k}(\mathbb{R}^d)^W}|| \, ||x||^a f||_{L^p_{\nu_k}(\mathbb{R}^d)^W}. \end{aligned}$$

Using Lemma 4 and Lemma 5, we obtain

$$\begin{aligned} & ||h_t(\mathcal{H}^W)^{-1}(f)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ \leq\ & ||h_t(\mathcal{H}^W)^{-1}(f_s)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} + ||h_t(\mathcal{H}^W)^{-1}(f^s)||_{L^{p'}_{A_k}(\mathbb{R}^d)^W} \\ \leq\ & Cs^{-a}(1+V(s)||h_t||_{L^{p'}_{\nu_k}(\mathbb{R}^d)^W})||\,||x||^a f||_{L^{p}_{\nu_k}(\mathbb{R}^d)^W}. \end{aligned}$$

Choosing $s = t^{\frac{p'-1}{2}}$, we obtain (3.29). □

Proposition 13. *Let $s > 0$. Then there exists a constant $C(d,k,s)$ such that for all f belongs to $L^1_{A_k}(\mathbb{R}^d)^W \bigcap L^2_{A_k}(\mathbb{R}^d)^W$,*

$$(3.30)\qquad ||f||^{2+\frac{4s}{2\gamma+d}}_{L^2_{A_k}(\mathbb{R}^d)^W} \leq C(d,k,s)||f||^{\frac{4s}{2\gamma+d}}_{L^1_{A_k}(\mathbb{R}^d)^W}||\,||\lambda||^s\mathcal{H}^W(f)||^2_{L^2_{\nu_k}(\mathbb{R}^d)^W}.$$

Proof. Let $A > 0$. From Plancherel's theorem we have

$$\begin{aligned} \|f\|^2_{L^2_{A_k}(\mathbb{R}^d)^W} &= \|\mathcal{H}^W(f)\|^2_{L^2_{\nu_k}(\mathbb{R}^d)^W} \\ &= ||\chi_{B(0,A)}\mathcal{H}^W(f)||^2_{L^2_{\nu_k}(\mathbb{R}^d)^W} + ||(1-\chi_{B(0,A)})\mathcal{H}^W(f)||^2_{L^2_{\nu_k}(\mathbb{R}^d)^W}. \end{aligned}$$

By a simple calculations we find

$$||\chi_{B(0,A)}\mathcal{H}^W(f)||^2_{L^2_{\nu_k}(\mathbb{R}^d)^W} \leq C(k,d)A^{2\gamma+d}\|f\|^2_{L^1_{A_k}(\mathbb{R}^d)^W}.$$

On the other hand

$$\begin{aligned} & ||(1-\chi_{B(0,A)})\mathcal{H}^W(f)||^2_{L^2_{\nu_k}(\mathbb{R}^d)^W} \\ \leq\ & A^{-2s}||(1-\chi_{B(0,A)})||\lambda||^s\mathcal{H}^W(f)||^2_{L^2_{\nu_k}(\mathbb{R}^d)^W} \\ \leq\ & A^{-2s}||\,||\lambda||^s\mathcal{H}^W(f)||^2_{L^2_{\nu_k}(\mathbb{R}^d)^W}. \end{aligned}$$

It follows then

$$\|f\|^2_{L^2_{A_k}(\mathbb{R}^d)^W} \leq C(k,d)A^{2\gamma+d}\|f\|^2_{L^1_{A_k}(\mathbb{R}^d)^W} + A^{-2s}||\,||\lambda||^s\mathcal{H}^W(f)||^2_{L^2_{\nu_k}(\mathbb{R}^d)^W}.$$

Minimizing the right hand side of that inequality over $A > 0$ gives

$$(3.31)\qquad ||f||^2_{L^2_{A_k}(\mathbb{R}^d)^W} \leq C(d,s,k)||f||^{\frac{4s}{2\gamma+d+2s}}_{L^1_{A_k}(\mathbb{R}^d)^W}||\,||\lambda||^s\mathcal{H}^W(f)||^{\frac{2(2\gamma+d)}{2s+2\gamma+d}}_{L^2_{\nu_k}(\mathbb{R}^d)^W}.$$

The desired result follows immediately from (3.31). □

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