J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2015.22.1.57
Volume 22, Number 1 (February 2015), Pages 57–63

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

SPLIT HYPERHOLOMORPHIC FUNCTION IN CLIFFORD ANALYSIS

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Abstract. We define a hyperholomorphic function with values in split quaternions, provide split hyperholomorphic mappings on $\Omega \subset \mathbb{C}^2$ and research the properties of split hyperholomorphic functions.

1. Introduction

A set of quaternions can be represented as

$$\mathcal{H} = \{z = x_0 + e_1x_1 + e_2x_2 + e_3x_3 : x_k \in \mathbb{R}, k = 0, 1, 2, 3\},$$

where $e_1^2 = e_2^2 = e_3^2 = -1$ and $e_1e_2e_3 = -1$, which is non-commutative division algebra. A set of split quaternions can be expressed as

$$\mathcal{S} = \{z = x_0 + e_1x_1 + e_2x_2 + e_3x_3 : x_k \in \mathbb{R}, k = 0, 1, 2, 3\},$$

where $e_1^2 = -1$, $e_2^2 = e_3^2 = 1$ and $e_1e_2e_3 = 1$, which is also non-commutative. On the other hand, unlike quaternion algebra, a set of split quaternions contains zero divisors, nilpotent elements and non-trivial idempotents. Because split quaternions are used to express Lorentzian rotations, studies of the geometric and physical applications of split quaternions require solving split quaternionic equations (see [6], [9]). Deavours [3] generated regular functions in a quaternion analysis and provided the Cauchy-Fueter integral formulas for regular quaternion functions. Carmody [1, 2] investigated the properties of hyperbolic quaternions, octonions, and sedenions, and Sangwine and Bihan [10] provided a new polar representation of quaternions that is represented by a pair of complex numbers in the Cayley-Dickson form.

We shall denote by $\mathbb{C}$ and $\mathbb{R}$, respectively, the field of complex numbers and the field of real numbers. We [4, 5] showed that any complex-valued harmonic function

Received by the editors October 14, 2014. Accepted November 22, 2014.
2010 *Mathematics Subject Classification.* 30G35, 32W50, 32A99.
Key words and phrases. split quaternion, split hyperholomorphic function, Clifford analysis.
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f_1 in a pseudoconvex domain D of $\mathbb{C}^2 \times \mathbb{C}^2$ has a conjugate function f_2 in D such that the quaternion-valued function $f_1 + f_2 j$ is hyperholomorphic in D and gave a regeneration theorem in a quaternion analysis in view of complex and Clifford analysis method. We define a split hyperholomorphic function with values in split quaternions and examine the properties of split hyperholomorphic functions based on [7].

2. Preliminary

The split quaternionic field $\mathcal{S}$ is a four-dimensional non-commutative $\mathbb{R}$-field generated by four base elements e_0, e_1, e_2, and e_3 with the following non-commutative multiplication rules :

$$e_1^2 = -1,\ e_2^2 = e_3^2 = 1,\ e_k e_l = -e_l e_k,\ \overline{e_k} = -e_k\ (k \neq l, k \neq 0, l \neq 0),$$

$$e_1 e_2 = e_3,\ e_2 e_3 = -e_1,\ e_3 e_1 = e_2.$$

The element e_0 is the identity of $\mathcal{S}$, and e_1 identifies the imaginary unit $i = \sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers. A split quaternion z is given by

$$z = \sum_{k=0}^{3} e_k x_k = z_1 + z_2 e_2,$$

where $z_1 = x_0 + e_1 x_1$, $z_2 = x_2 + e_1 x_3$, $\overline{z_1} = x_0 - e_1 x_1$ and $\overline{z_2} = x_2 - e_1 x_3$ are complex numbers in $\mathbb{C}$ and x_k $(k = 0, 1, 2, 3)$ are real numbers.

The multiplications of two pure split quaternions $\tilde{z} = e_1 x_1 + e_2 x_2 + e_3 x_3$ and $\tilde{w} = e_1 y_1 + e_2 y_2 + e_3 y_3$ $(y_k \in \mathbb{R},\ k = 1, 2, 3)$ is defined as follows:

$$\begin{aligned} \tilde{z} \cdot \tilde{w} &:= -x_1 y_1 + x_2 y_2 + x_3 y_3, \\ \tilde{z} \times \tilde{w} &:= \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \end{aligned}$$

For pure split quaternions $\tilde{z}$, $\tilde{w}$ and $\tilde{t}$, the cross product satisfies two rules as follows:

$$\tilde{z} \times \tilde{w} = -\tilde{w} \times \tilde{z},$$

$$\tilde{z} \times (\tilde{w} \times \tilde{t}) + \tilde{w} \times (\tilde{t} \times \tilde{z}) + \tilde{t} \times (\tilde{z} \times \tilde{w}) = 0.$$

The split quaternionic conjugate z^*, the multiplicative modulus $M(z)$ and the inverse z^{-1} of z in $\mathcal{S}$ are defined as

$$z^* = \sum_{k=0}^{3} \overline{e_k} x_k = \overline{z_1} - z_2 e_2,$$

$$M(z) := zz^* = z^*z = x_0^2 + x_1^2 - x_2^2 - x_3^2 = |z_1|^2 - |z_2|^2,$$

$$z^{-1} = \frac{z^*}{M(z)} \ (M(z) \neq 0).$$

We let

$$J = \frac{e_1x_1 + e_2x_2 + e_3x_3}{\sqrt{-x_1^2 + x_2^2 + x_3^2}} \text{ with } J^2 = e_0 = id.$$

The split quaternion number z of $\mathcal{S}$ is

$$z = \xi_0 + J\xi_1,$$

where $\xi_0 = x_0$ and $\xi_1 = \sqrt{-x_1^2 + x_2^2 + x_3^2}$. Then the split quaternionic conjugate number of z is $z^* = \xi_0 - J\xi_1$, and the multiplicative modulus of z is $M(z) = \xi_0^2 - \xi_1^2$. Let Ω be an open set in $\mathbb{C}^2$ and consider a function f defined on Ω with values in $\mathcal{S}$:

$$f = \sum_{k=0}^{3} u_k e_k = u + Jv$$

$$z = (\xi_0, \xi_1) \in \Omega \longmapsto f(z) = u(z) + Jv(z) \in \mathcal{S},$$

where $u = u_0$ and $v = \frac{\tilde{z}\cdot\tilde{f} + \tilde{z}\times\tilde{f}}{\sqrt{-x_1^2+x_2^2+x_3^2}}$ with $\tilde{f} = \sum_{k=1}^{3} u_k e_k$.

We give differential operators as

$$D := \frac{1}{2}\left(\frac{\partial}{\partial\xi_0} - J\frac{\partial}{\partial\xi_1}\right) \text{ and } D^* = \frac{1}{2}\left(\frac{\partial}{\partial\xi_0} + J\frac{\partial}{\partial\xi_1}\right),$$

where $\frac{\partial}{\partial\xi_0} = \frac{\partial}{\partial x_0}$ and

$$\frac{\partial}{\partial\xi_1} = \frac{\tilde{z}\cdot\tilde{D^*} + \tilde{z}\times\tilde{D^*}}{\sqrt{-x_1^2 + x_2^2 + x_3^2}},$$

where $\tilde{D^*} = \sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k}$. Then the Coulomb operator (see [8]) is

$$M(D) = DD^* = D^*D = \frac{1}{4}\sum_{k=0}^{3}\frac{\partial^2}{\partial x_k^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial\xi_0^2} - \frac{\partial^2}{\partial\xi_1^2}\right).$$

Definition 2.1. Let Ω be an open set in $\mathbb{C}^2$. A function $f(z) = f_1(z) + f_2(z)e_2$ is said to be an $L(R)$-*split hyperholomorphic function* on Ω if the following two conditions are satisfied:

(1) $f_1(z)$ and $f_2(z)$ are continuously differential functions on Ω, and
(2) $D^*f(z) = 0$ $(f(z)D^* = 0)$ on Ω.

In this paper, we consider a L-split hyperholomorphic function on Ω in $\mathbb{C}^2$.

3. Split Hyperholomorphic Function

Let $\xi_0 = r\cosh\theta$ and $\xi_1 = r\sinh\theta$ with $r^2 = |zz^*|$. Then any $z = \xi_0 + J\xi_1$ can be expressed as $z = r(\cosh\theta + J\sinh\theta)$, where θ is the angle between the vector $z \in \mathbb{C}^2$ and the real axis.

Theorem 3.1. *Let Ω be a domain of holomorphy in $\mathbb{C}^2$. If $u(r,\theta)$ is a split quaternion function satisfying $M(D)f = 0$ on Ω, then there exists a split hyper-conjugate quaternion function $v(r,\theta)$ satisfying $M(D)f = 0$ such that $u(r,\theta) + Jv(r,\theta)$ is a split hyperholomorphic function on Ω.*

Proof. We put

$$\phi(r,\theta) := -\frac{1}{r}\frac{\partial u}{\partial \theta}dr - r\frac{\partial u}{\partial r}d\theta.$$

We operate the operator ∂ from the left-hand side of $\phi(r,\theta)$ on Ω.

$$\begin{aligned}\partial\phi(r,\theta) &= \left(\frac{\partial}{\partial r}dr + \frac{\partial}{\partial\theta}d\theta\right)\left(-\frac{1}{r}\frac{\partial u}{\partial\theta}dr - r\frac{\partial u}{\partial r}d\theta\right)\\ &= \left(-\frac{\partial u}{\partial r} - r\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial\theta^2}\right)dr\wedge d\theta.\end{aligned}$$

Since $\frac{\partial f}{\partial r} = \cosh\theta\ \frac{\partial f}{\partial \xi_0} + \sinh\theta\ \frac{\partial f}{\partial\xi_1}$, $\frac{\partial^2 f}{\partial r^2} = \cosh^2\theta\ \frac{\partial^2 f}{\partial\xi_0^2} + 2\sinh\theta\cosh\theta\ \frac{\partial^2 f}{\partial\xi_0\partial\xi_1} + \sinh^2\theta\ \frac{\partial^2 f}{\partial\xi_1^2}$ and $\frac{\partial^2 f}{\partial\theta^2} = r\frac{\partial f}{\partial r} + r^2\Big(\sinh^2\theta\ \frac{\partial^2 f}{\partial\xi_0^2} + 2\sinh\theta\cosh\theta\ \frac{\partial^2 f}{\partial\xi_0\partial\xi_1} + \cosh^2\theta\ \frac{\partial^2 f}{\partial\xi_1^2}\Big)$, we get $\partial\phi(r,\theta)$ is zero. Since Ω is a domain of holomorphy, the ∂-closed form $\phi(r,\theta)$ is a ∂-exact form on Ω. Hence, there exists a split hyper-conjugate quaternion function $v(r,\theta)$ satisfying $M(D)f = 0$ on Ω such that $u(r,\theta) + Jv(r,\theta)$ is a split hyperholomorphic function on Ω. □

Example 3.2. If the split quaternion function

$$u(r,\theta) = r^n\cosh(n\theta) + (r + \frac{1}{r})\cosh\theta$$

in a domain of holomorphy $\Omega \subset \mathbb{C}^2 - \{0\}$ is known, then a split hyper-conjugate quaternion function $v(r,\theta)$ of $u(r,\theta)$ on Ω can be found. That is,

$$v(r,\theta) = -r^n\sinh(n\theta) - (r - \frac{1}{r})\sinh\theta$$

and $f(r,\theta) = u(r,\theta) + Jv(r,\theta)$ is a split hyperholomorphic function satisfying $M(D)f = 0$ on Ω.

Theorem 3.3. *Let Ω be an open set in $\mathbb{C}^2$ and f be a split quaternion function satisfying $M(D)f = 0$ on Ω. Then the multiplicative modulus of Df is*

$$M(Df) = \left(\frac{\partial u}{\partial \xi_0}\right)^2 - \left(\frac{\partial u}{\partial \xi_1}\right)^2 = \left(\frac{\partial v}{\partial \xi_1}\right)^2 - \left(\frac{\partial v}{\partial \xi_0}\right)^2.$$

Proof. For $f = u + Jv$ and $\overline{f} = u - Jv$,

$$\begin{aligned} M(Df) &= DfD^*\overline{f} \\ &= \frac{1}{4}\Big\{\Big(\frac{\partial u}{\partial \xi_0}\frac{\partial u}{\partial \xi_0} - 2\frac{\partial u}{\partial \xi_0}\frac{\partial v}{\partial \xi_1} - \frac{\partial \overline{v}}{\partial \xi_0}\frac{\partial v}{\partial \xi_0} + \frac{\partial \overline{v}}{\partial \xi_0}\frac{\partial u}{\partial \xi_1} + \frac{\partial u}{\partial \xi_1}\frac{\partial v}{\partial \xi_0} - \frac{\partial u}{\partial \xi_1}\frac{\partial u}{\partial \xi_1} \\ &\quad + \frac{\partial v}{\partial \xi_1}\frac{\partial v}{\partial \xi_1}\Big) + J\Big(-\frac{\partial v}{\partial \xi_0}\frac{\partial v}{\partial \xi_1} + \frac{\partial u}{\partial \xi_1}\frac{\partial v}{\partial \xi_1} + \frac{\partial \overline{v}}{\partial \xi_1}\frac{\partial v}{\partial \xi_0} - \frac{\partial \overline{v}}{\partial \xi_1}\frac{\partial u}{\partial \xi_1}\Big)\Big\}, \end{aligned}$$

where $\overline{v} = \frac{\tilde{z}\cdot\tilde{f} - \tilde{z}\times\tilde{f}}{\sqrt{-x_1^2+x_2^2+x_3^2}}$. Since $\frac{\partial u}{\partial \xi_0} = -\frac{\partial v}{\partial \xi_1}$ and $\frac{\partial v}{\partial \xi_0} = -\frac{\partial u}{\partial \xi_1}$, we have

$$M(Df) = \frac{1}{4}\Big(4\frac{\partial u}{\partial \xi_0}\frac{\partial u}{\partial \xi_0} - 4\frac{\partial u}{\partial \xi_1}\frac{\partial u}{\partial \xi_1}\Big) = \left(\frac{\partial u}{\partial \xi_0}\right)^2 - \left(\frac{\partial u}{\partial \xi_1}\right)^2 = \left(\frac{\partial v}{\partial \xi_1}\right)^2 - \left(\frac{\partial v}{\partial \xi_0}\right)^2.$$

□

Theorem 3.4. *Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a polar coordinates mapping defined by*

$$f(r,\theta) = (r\cosh\theta, r\sinh\theta).$$

Then the determinant of this mapping is

$$det\Delta_{\mathbb{R}} f(r,\theta) = 1,$$

where $\Delta_{\mathbb{R}} f := \frac{\partial(u,v)}{\partial(\xi_0,\xi_1)}$.

Proof. The chain rule gives

$$\begin{aligned} \frac{\partial u}{\partial \xi_0} &= \cosh\theta\,\frac{\partial u}{\partial r} - \frac{1}{r}\sinh\theta\,\frac{\partial u}{\partial \theta}, \quad -\frac{\partial v}{\partial \xi_1} = \sinh\theta\,\frac{\partial v}{\partial r} - \frac{1}{r}\cosh\theta\,\frac{\partial v}{\partial \theta}, \\ \frac{\partial u}{\partial \xi_1} &= -\sinh\theta\,\frac{\partial u}{\partial r} + \frac{1}{r}\cosh\theta\,\frac{\partial u}{\partial \theta}, \quad -\frac{\partial v}{\partial \xi_0} = -\cosh\theta\,\frac{\partial v}{\partial r} + \frac{1}{r}\sinh\theta\,\frac{\partial v}{\partial \theta}. \end{aligned}$$

Then

$$\begin{aligned} \Delta_{\mathbb{R}} f(r,\theta) &= \begin{pmatrix} \cosh\theta & -\frac{1}{r}\sinh\theta \\ -\sinh\theta & \frac{1}{r}\cosh\theta \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} \\ \frac{\partial u}{\partial \theta} & \frac{\partial v}{\partial \theta} \end{pmatrix} \\ &= \begin{pmatrix} \cosh\theta & -\frac{1}{r}\sinh\theta \\ -\sinh\theta & \frac{1}{r}\cosh\theta \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ r\sinh\theta & r\cosh\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

□

Theorem 3.5. *Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be a polar coordinates mapping defined by*

$$f(r,\theta) = (e^r \cosh\theta, e^r \sinh\theta).$$

Then the determinant of this mapping is

$$det\Delta_{\mathbb{R}} f(r,\theta) = \frac{1}{r}e^{2r}.$$

Proof. We can prove as above Theorem 3.4. □

Theorem 3.6. *Let Ω be an open set in $\mathbb{C}^2$ and f be a split hyperholomorphic function on Ω. Then there exists a differentiable function φ on Ω such that the vector field*

$$f(\xi_0,\xi_1) = (u(\xi_0,\xi_1), v(\xi_0,\xi_1)) = \left(\frac{\partial}{\partial\xi_0}\varphi(\xi_0,\xi_1), -\frac{\partial}{\partial\xi_1}\varphi(\xi_0,\xi_1)\right).$$

Proof. We let any point (ξ_0', ξ_1') on Ω. Consider

$$\varphi(\xi_0,\xi_1) = \int_{\xi_0'}^{\xi_0} u(t,\xi_1)dt + \mu(\xi_1),$$

where $\mu(\xi_1)$ is a split quaternion-valued function. By the fundamental theorem of calculus, we can find

$$\frac{\partial}{\partial\xi_0}\varphi(\xi_0,\xi_1) = \frac{\partial}{\partial\xi_0}\int_{\xi_0'}^{\xi_0} u(t,\xi_1)dt + \frac{\partial}{\partial\xi_0}\mu(\xi_1) = u(\xi_0,\xi_1).$$

Since f is a split hyperholomorphic function on Ω and differentiating with respect to ξ_1, we obtain

$$\begin{aligned}
\frac{\partial}{\partial\xi_1}\varphi(\xi_0,\xi_1) &= \int_{\xi_0'}^{\xi_0} \frac{\partial}{\partial\xi_1}u(t,\xi_1)dt + \frac{\partial}{\partial\xi_1}\mu(\xi_1) \\
&= -\int_{\xi_0'}^{\xi_0} \frac{\partial}{\partial\xi_0}v(t,\xi_1)dt + \frac{\partial}{\partial\xi_1}\mu(\xi_1) \\
&= -\int_{\xi_0'}^{\xi_0} \frac{\partial}{\partial\xi_0}\sum_{k=0}^{3} e_k v_k(t,\xi_1)dt + \frac{\partial}{\partial\xi_1}\mu(\xi_1) \\
&= \sum_{k=0}^{3} e_k(-v_k(\xi_0,\xi_1) + v_k(\xi_0',\xi_1)) + \frac{\partial}{\partial\xi_1}\mu(\xi_1) \\
&= -v(\xi_0,\xi_1) + v(\xi_0',\xi_1) + \frac{\partial}{\partial\xi_1}\mu(\xi_1),
\end{aligned}$$

where $v_0 = -\frac{x_1u_1-x_2u_2-x_3u_3}{\sqrt{-x_1^2+x_2^2+x_3^2}}$, $v_1 = -\frac{x_2u_3-x_3u_2}{\sqrt{-x_1^2+x_2^2+x_3^2}}$, $v_2 = -\frac{x_1u_3-x_3u_1}{\sqrt{-x_1^2+x_2^2+x_3^2}}$ and $v_3 = -\frac{x_2u_1-x_1u_2}{\sqrt{-x_1^2+x_2^2+x_3^2}}$. Putting $\mu(\xi_1) = -\int v(\xi_0',\xi_1)d\xi_1$ and then we have $\frac{\partial}{\partial\xi_1}\varphi(\xi_0,\xi_1) = -v(\xi_0,\xi_1)$. □

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