논리연구 18-2(2015) pp. 197-215

Some Axiomatic Extensions of the Involutive Micanorm Logic **IMICAL***

Eunsuk Yang

【Abstract】 In this paper, we deal with standard completeness of some axiomatic extensions of the involutive micanorm logic **IMICAL**. More precisely, first, four involutive micanorm-based logics are introduced. Their algebraic structures are then defined, and their corresponding algebraic completeness is established. Next, standard completeness is established for two of them using construction in the style of Jenei-Montagna.

【Abstract】 fuzzy logic, (involutive) micanorm, algebraic completeness, standard completeness, **MICASIL, FIMICASIL, CnIMICAL, FCnIMICAL**

접수일자: 2015.03.02 심사 및 수정완료일: 2015.03.31 게재확정일: 2015.05.15.

* This paper was supported by research funds of Chonbuk National University in 2015. I must thank the anonymous referees for their helpful comments.

1. Introduction

The present author recently introduced *micanorms* (binary monotonic identity commutative aggregation operations on the real unit interval [0, 1]) and logics based on micanorms in Yang (2015). In particular, he provided standard completeness results for involutive such logics, which was a problem left open in Horčík (2011), using the Jenei-Montagna-style construction introduced in Esteva et al. (2002) and Jenei & Montagna (2002). After providing such completeness, he stated as follows in Remarks 2 and 3, respectively:

> The proof of standard completeness in Theorem 5 does not work for **IMICASIL** because the definition of $\odot$ does not satisfy contraction. ⋯ Let **FIMICASIL** be **IMICASIL** plus (FP). The proof in Theorem 5 instead works for **FIMICASIL**. We leave its proof to the interested reader(Yang (2015), p. 54).

> Wang defined a new monoid $\odot$ based on Wang's monoid $\circ_W$ for involution and provided standard completeness for **CnIUL** in Wang (2013). Since Yang's monoid $\circ_{Y''}$ is also Wang's monoid, we can also define such a monoid based on $\circ_{Y''}$ and provide standard completeness results for **CnIUL** and similarly for **IMICAL** and **CnIMICAL**(Yang (2015), p. 57).

Let ϕ^n stand for $((\cdots(\phi \ \& \ \phi) \ \& \ \cdots) \ \& \ \phi) \ \& \ \phi$, n ϕ's. The systems **IMICASIL**, **FIMICASIL**, and **CnIMICAL** are the involutive micanorm logic **IMICAL** with (S-INC) $\phi \to (\phi \ \& \ \phi)$, the **IMICASIL** with (FP) $\mathbf{t} \leftrightarrow \mathbf{f}$, and the **IMICAL** with (N-P) $\phi^n \to \phi^{n-1}$, $2 \leq n$, respectively. As the statements in Remark 2 show, although he insists that the proof in Theorem 5 (the standard completeness using the construction in the style of Jenei-Montagna) is applicable to **FIMICASIL**, he does not provide its proof, and similarly for **CnIMICAL** in Remark 3.

In this paper, we show that his insistence in Remark 2 is correct but that in Remark 3 is not. More exactly, we verify that the proof in Theorem 5 is applicable to **FIMICASIL**, but not to **CnIMICAL**. Instead, it can be applied to the **CnIMICAL** with (FP), **FCnIMICAL**. Note that the system **CnIMICAL** is a strengthening of the system **IMICASIL** in the sense that the former system is obtained from **IMICAL** adding additional axioms, in particular square decreasing axiom. Our results show that, while the systems **IMICASIL** and **CnIMICAL** are not standard complete, such systems with (FP), i.e., the systems **FIMICASIL** and **FCnIMICAL** are standard complete. This means that, w.r.t. standard completeness, the system **FCnIMICAL** has the same property (FP) as the system **FIMICASIL**. This is very natural in the sense that the standard negation 1 - x has the fixed-point 1/2, i.e., 1/2 = ~(1/2).[1)]

The paper is organized as follows. In Section 2, we present the axiomatizations of the systems **IMICASIL**, **FIMICASIL**, **CnIMICAL**, and **FCnIMICAL**. In Section 3, we define their corresponding algebraic structures, by subvarieties of the variety of commutative residuated lattices, and show that they are complete with respect to (w.r.t.) linearly ordered corresponding algebras. In Section 4, we establish standard completeness for **FIMICASIL** and **FCnIMICAL** using the method introduced in Yang (2015) together

1) For the basic reason of this kind of research, see the paper Yang (2015).

with the remark that this method is not applicable to **IMICASIL** and **CnIMICAL** (see Remark 5.5 below).

For convenience, we shall adopt notations and terminology similar to those in Cintula (2006), Esteva et al. (2002), Hájek (1998), Metcalfe & Montagna (2007), Yang (2009; 2013, 2014, 2015), and assume familiarity with them (together with the results found therein).

2. Syntax

We base some axiomatic extensions of the involutive micanorm logic **IMICAL** on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$, and

df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define $\phi^n{}_t$ as ϕ_t & $\cdots$ & ϕ_t, n factors, where $\phi_t := \phi \wedge \mathbf{t}$. For the rest of this paper, we use the customary notations and terminology, and the axiom systems to provide a consequence relation.

We start with the following axiomatization of **IMICAL**, the most basic fuzzy logic introduced here.

Definition 2.1 (Yang (2015)) **IMICAL** consists of the following axiom schemes and rules:

A1. $\phi \to \phi$ (self-implication, SI)

A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$ ($\wedge$-elimination, $\wedge$-E)

A3. $((\phi\to\psi)\wedge(\phi\to\chi)) \to (\phi\to(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi\to\chi)\wedge(\psi\to\chi)) \to ((\phi\vee\psi)\to\chi)$ ($\vee$-elimination, $\vee$-E)

A6. $\mathbf{F} \to \phi$ (ex falso quadlibet, EF)

A7. $(\phi \& \psi) \to (\psi \& \phi)$ (&-commutativity, &-C)

A8. $(\mathbf{t} \to \phi) \leftrightarrow \phi$ (push and pop, PP)

A9. $\phi \to (\psi \to (\psi \& \phi))$ (&-adjunction, &-Adj)

A10. $(\phi_t \& \psi_t) \to (\phi \wedge \psi)$ ($\& \wedge$)

A11. $(\psi \& (\phi \& (\phi \to (\psi \to \chi)))) \to \chi$ (residuation, Res')

A12. $((\phi \to (\phi \& (\phi \to \psi))) \& (\psi \to \chi)) \to (\phi \to \chi)$ (T')

A13. $((\delta \& \varepsilon) \to (\delta \& (\varepsilon \& (\phi \to \psi)_t))) \vee (\delta' \to (\varepsilon' \to ((\varepsilon' \& \delta') \& (\psi \to \phi)_t)))$ (PL)

A14. $\sim\sim\phi \to \phi$ (double negation elimination, DNE)

$\phi \to \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi \vdash \phi_t$ (adj$_u$)

$\phi \vdash (\delta \& \varepsilon) \to (\delta \& (\varepsilon \& \phi))$ (α)

$\phi \vdash \delta \to (\varepsilon \to ((\varepsilon \& \delta) \& \phi))$ (β)

Definition 2.2 A logic is an axiomatic extension (extension for short) of an arbitrary logic **L** if and only if (iff) it results from **L** by adding axiom schemes. In particular, the following are weakening-free, non-associative fuzzy logics that extend **IMICAL**:

- (Yang (2015)) Involutive square increasing micanorm logic

IMICASIL is **IMICAL** plus (S-INC) $\phi \to (\phi \& \phi)$.

- (Yang (2015)) Fixed-pointed involutive square increasing micanorm logic **FIMICASIL** is **IMICASIL** plus (FP) $\mathbf{t} \leftrightarrow \mathbf{f}$.
- (Yang (2015)) N-potent involutive micanorm logic **CnIMICAL** is **IMICAL** plus (N-P) $\phi^n \to \phi^{n-1}$, $2 \leq n$.
- Fixed-pointed n-potent involutive micanorm logic **FCnIMICAL** is **CnIMICAL** plus (FP).

For easy reference, we let Ls be the set of the weakening-free, non-associative fuzzy logics defined in Definition 2.

Definition 2.3 Ls = {**IMICASIL**, **FIMICASIL**, **CnIMICAL**, **FCnIMICAL**}

A *theory* over L ($\in$ Ls) is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using a rule of L. $T \vdash \phi$, more exactly $T \vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. A theory T is *inconsistent* if $T \vdash \mathbf{F}$; otherwise it is *consistent.*

The deduction theorem for L is as follows:

Proposition 2.4 (Cintula et al. (2013; 2015)) Let T be a theory, and ϕ, ψ formulas. $T \cup \{\phi\} \vdash_L \psi$ iff $T \vdash_L \gamma(\phi) \to \psi$ for some $\gamma \in \Pi(bDT^*)$.[2)]

[2)] For γ and $\Pi(bDT^*)$, see Cintula et al. (2013; 2015) and Yang (2015).

For convenience, "$\sim$," "$\wedge$," "$\vee$," and "$\rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

3. Semantics

Suitable algebraic structures for L ($\in$ Ls) are obtained as a subvariety of the variety of commutative monoidal residuated lattices.

Definition 3.1 (Yang (2015)) (i) A *pointed bounded commutative residuated lattice* is a structure **A** = (A, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *, $\rightarrow$) such that:

(Ⅰ) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded lattice with top element $\top$ and bottom element $\bot$.

(Ⅱ) (A, *, t) is a commutative monoid.

(Ⅲ) $y \leq x \rightarrow z$ iff $x * y \leq z$, for all x, y, z $\in$ A (residuation).

(ii) An *IMICAL-algebra* is a pointed bounded commutative residuated lattice satisfying

- $t \leq ((z*w)\rightarrow(z*(w*(x\rightarrow y)_t))) \vee (z'\rightarrow(w'\rightarrow((w'*z')*(y\rightarrow x)_t)))$, for all x, y, z, w, z', w' $\in$ A (PL^A).
- $t \leq {\sim}{\sim}x \rightarrow x$, for all x $\in$ A (DNE^A).

L-algebras the class of which characterizes L are defined as follows.

Definition 3.2 (L-algebras) An *IMICASIL-algebra* is an IMICAL-algebra satisfying: (S-INC^A) $t \leq x \rightarrow (x * x)$, for all $x \in A$; a *FIMICASIL-algebra* is an IMICASIL-algebra satisfying (FP^A) $t \leq t \leftrightarrow f$; a *CnIMICAL-algebra* is an IMICAL-algebra satisfying: (N-P^A) $t \leq x^n \rightarrow x^{n-1}$, $2 \leq n$, for all $x \in A$; a *FCnIMICAL-algebra* is a CnIMICAL-algebra satisfying (FP^A). We call all these algebras *L-algebras*.

An L-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function $v : FOR \rightarrow \mathcal{A}$ satisfying: $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$, $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$, $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$, $v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$, $v(\mathbf{F}) = \perp$, $v(\mathbf{f}) = f$, (and hence $v({\sim}\phi) = {\sim}v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = t$).

Definition 3.4 (Cintula (2006)) Let $\mathcal{A}$ be an L-algebra, T a theory, ϕ a formula, and K a class of L-algebras.

(i) (Tautology) ϕ is a *t-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq t$ for each $\phi \in T$. We denote the class of $\mathcal{A}$-models of T, by *Mod(T, $\mathcal{A}$)*.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \models_{\mathsf{K}} \phi$, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ $\{\phi\}$,

$\mathcal{A}$) for each $\mathcal{A} \in$ K.

Definition 3.5 (**L**-algebra, Cintula (2006)) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is an ***L**-algebra* iff, whenever ϕ is L-provable in T (i.e. T $\vdash_L \phi$, L an L logic), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. T $\vDash_{\{A\}}$ ϕ), $\mathcal{A}$ a corresponding L-algebra). By $MOD^{(l)}(L)$, we denote the class of (linearly ordered) **L**-algebras. Finally, we write T $\vDash^{(l)}_L \phi$ in place of T $\vDash_{MOD^{(l)}(L)} \phi$.

Theorem 3.6 (Strong completeness) Let T be a theory, and ϕ a formula. T $\vdash_L \phi$ iff T $\vDash_L \phi$ iff T $\vDash^1_L \phi$.

Proof: We obtain this theorem as a corollary of Theorem 3.1.8 in Cintula & Noguera (2011). □

4. Standard completeness

In this section, we provide standard completeness results for L $\in$ {**FIMICASIL**, **FCnIMICAL**} using the Jenei-Montagna-style construction in Eeteva et al. (2002) and Jenei & Montagna (2002).

We first show that finite or countable, linearly ordered **L**-algebras are embeddable into a standard algebra. (For convenience, we add the 'less than or equal to' relation symbol "$\leq$" to such algebras.) First, note the following results.

Theorem 5.1 (Yang (2015))

(i) For every finite or countable linearly ordered **MICAL**-algebra **A** = (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *, $\rightarrow$), there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the following conditions hold:

(I) X is densely ordered, and has a maximum Max, a minimum Min, and special elements e, ∂.

(II) (X, $\circ$, $\leq$, e) is a linearly ordered, monotonic, commutative groupoid with unit.

(III) $\circ$ is conjunctive and left-continuous w.r.t. the order topology on (X, $\leq$).

(IV) h is an embedding of the structure (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *) into (X, $\leq$, Max, Min, e, ∂, min, max, $\circ$), and for all m, n $\in$ A, h(m $\rightarrow$ n) is the residuum of h(m) and h(n) in (X, $\leq$, Max, Min, e, ∂, max, min, $\circ$).

(ii) For every finite or countable linearly ordered **IMICAL**-algebra **A** = (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *, $\rightarrow$), there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the conditions (I) to (IV) in (i) and the following condition hold:

(V) For all x $\in$ X, x is involutive, i.e., it satisfies (DNE^A).

Proposition 5.2 (i) For every finite or countable linearly ordered **FIMICASIL**-algebra **A** = (A, $\leq_A$, $\top$, $\bot$, t, f, $\wedge$, $\vee$, *, $\rightarrow$), there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the conditions (I) to (V) of (ii) in Theorem 5.1 and the following condition hold:

(A) $(X, \circ, \leq, e)$ is square increasing and fixed-pointed.

(ii) For every finite or countable linearly ordered **FCnIMICAL**-algebra $\mathbf{A} = (A, \leq_A, \top, \bot, t, f, \wedge, \vee, *, \rightarrow)$, there is a countable ordered set X, a binary operation $\circ$, and a map h from A into X such that the conditions (I) to (V) of (ii) in Theorem 5.1 and the following condition hold:

(B) $(X, \circ, \leq, e)$ is n-potent and fixed-pointed.

Proof: For convenience, we assume A as a subset of $\mathbf{Q} \cap [0, 1]$ with finite or countable elements, where 0 and 1 are least and greatest elements, respectively, and some e and any ∂ are special elements, each of which corresponds to $\top, \bot, t, f$, respectively.

We first note that, for **MICAL**, a linearly ordered, monotonic groupoid with unit $(X, \circ, \leq, e)$ is defined as follows:

$$X = \{(m, x): m \in A \setminus \{0\ (= \bot)\} \text{ and } x \in \mathbf{Q} \cap (0, m]\} \cup \{(0, 0)\};$$

for $(m, x), (n, y) \in X$,

$(m, x) \leq (n, y)$ iff either $m <_A n$, or $m =_A n$ and $x \leq y$;

$$(m,x) \circ (n,y) = \begin{cases} \max\{(m,x), (n,y)\} & \text{if } m*n =_A m \vee n,\ m \neq_A n, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \\ \min\{(m,x), (n,y)\} & \text{if } m * n = m \wedge z, \text{ and } (m, x) \leq e \text{ or } (n, y) \leq e; \end{cases}$$

$(m * n, m * n)$ otherwise.

For convenience, we henceforth drop the index A in $\leq_A$ and $=_A$, if we need not distinguish them. Context should clarify the intention.

We next note that, for **IMICAL**, m^+ denotes the successor of m if it exists, otherwise $m^+ = m$, for each $m \in A$; since the negation in A, defined as $\sim m := m \rightarrow \partial$ is involutive, we have that: $m = (\sim n)^+$ iff $n = (\sim m)^+$; moreover, if $m < m^+$, then $(\sim(m^+))^+ = \sim m$. Here, we use Y below in place of the X above. Let $(Y, \leq)$ be the linearly ordered set, defined by

$$Y = \{(m, m): m \in A\} \cup$$
$$\{(m, x): \exists m' \in A \text{ such that } m = m'^+ > m', \text{ and } x \in Q \cap (0, m)\},$$

and $\leq$ being the corresponding lexicographic ordering as above. It is clear that $(Y, \leq)$ is a subset of the ordered set $(X, \leq)$ defined as above with the same bounds and special elements e $(= (t, t))$ and ∂ $(= (f, f))$. Notice that Y is closed under $\circ$ and that $\leq$ is a linear order with maximum (1, 1), minimum (0, 0), and special elements e and ∂. Furthermore, $\leq$ is dense. This proves (I).

For condition (II), we need to define a new operation $\odot$ on Y, based on $\circ$, as follows:

$$(m,x)\odot(n,y) = \min\{\partial,(m,x)\circ(n,y)\} \text{ if } m = (\sim n)^+ \text{ and } p/q+p'/q' \leq 1,$$
$$\text{where } x = mp/q \text{ and } y=np'/q',$$

$$\begin{array}{ll} & \text{or } m < (\sim n)^+; \\ (m,x) \circ (n,y) & \text{otherwise.} \end{array}$$

The operation $\odot$ satisfies conditions (II) to (V) (see Theorem 5 in Yang (2015)).

Thus, for (i), we need to prove (A), i.e., $(X, \odot, \leq, e)$ is square increasing and fixed-pointed. We first prove the square increasingness of $\odot$, i.e., $(m, x) \leq (m, x) \odot (m, x)$. Let $m \leq t$. Since $t = f$, $m < (\sim m)^+$ and thus $(m, x) \odot (m, x) = \min\{\partial, (m, x) \circ (m, x)\}$; therefore, $(m, x) \leq (m, x) \odot (m, x)$ since $\min\{\partial, (m, x) \circ (m, x)\} = (m, x) \circ (m, x)$ and $m = m * m$. Let $m > t$. Since $t = f$, $m > (\sim m)^+$ and thus $(m, x) \odot (m, x) = (m, x) \circ (m, x)$; therefore, $(m, x) \leq (m, x) \odot (m, x)$ since $(m, x) \leq (m, x) \circ (m, x)$. We next prove fixed-point, i.e., $e = \partial$. It directly follows from the fact that $t = f$ and so $(t, t) = (f, f)$.

For (ii), we need to prove (B), i.e., $(X, \odot, \leq, e)$ is n-potent and fixed-pointed. We first prove the n-potency of $\odot$, i.e., $(m, x)^n \leq (m, x)^{n-1}$. First, consider the case $m^2 = m$. Let $m \leq t$. Since $t = f$, $m < (\sim m)^+$ and thus $(m, x) \odot (m, x) = \min\{\partial, (m, x) \circ (m, x)\} = (m, x) \circ (m, x)$. Then, we have $(m, x) = (m, x) \odot (m, x)$ since $(m, x) \circ (m, x) = (m, x)$; therefore, $(m, x)^n = (m, x)^{n-1}$. Let $m > t$. Since $t = f$, $m > (\sim m)^+$ and thus $(m, x) \odot (m, x) = (m, x) \circ (m, x)$. Hence, we have $(m, x) \odot (m, x) = (m, m)$ and $((m, x) \odot (m, x)) \odot (m, x) = (m, m) \odot (m, x) = (m, m) = (m, x) \odot (m, x)$; therefore, $(m, x)^n = (m, x)^{n-1}$. Let $m^2 \neq m$. Its proof is almost the same as Theorem 3.3 (e) in

Wang (2012) since $(m, x) \odot (m, x) = (m, x) \circ (m, x)$. The proof of fixed-point is the same as in (i). □

Proposition 5.3 Every countable linearly ordered **L**-algebra can be embedded into a standard algebra.

Proof: In an analogy to the proof of Theorem 3.2 in Jenei & Montagna (2002), we prove this. Let X, **A**, etc. be as in Proposition 5.2. Since $(X, \preceq)$ is a countable, dense, linearly-ordered set with maximum and minimum, it is order isomorphic to $(\mathbf{Q} \cap [0, 1], \leq)$. Let g be such an isomorphism. If (I) to (V) and (A) ((B) resp) hold, letting for $\alpha, \beta \in [0, 1]$, $\alpha \odot' \beta = g(g^{-1}(\alpha) \odot g^{-1}(\beta))$, and, for all $m \in A$, $h'(m) = g(h(m))$, we obtain that $\mathbf{Q} \cap [0, 1], \leq, 1, 0, e, \partial, \odot', h'$ satisfy the conditions (I) to (V) and (A) ((B) resp) of Proposition 5.2 whenever X, $\preceq$, Max, Min, e, ∂, $\odot$, and h do. Thus, without loss of generality, we can assume that $X = \mathbf{Q} \cap [0, 1]$ and $\preceq = \leq$.

Now we define for $\alpha, \beta \in [0, 1]$,

$$\alpha \odot'' \beta = \sup_{x \in X: x \leq \alpha} \sup_{y \in X: y \leq \beta} x \odot y.$$

Commutativity of $\odot''$ follows from that of $\odot$. Its monotonicity, identity, fixed-point, and square increasingness (n-potency resp) are easy consequences of the definition. Furthermore, it follows from the definition that $\odot''$ is conjunctive, i.e., $0 \odot'' 1 = 0$.

We prove left-continuity. Suppose that $<\alpha_n: n \in \mathbf{N}>$, $<\beta_n: n$

$\in \mathbf{N}>$ are increasing sequences of reals in [0, 1] such that $\sup\{\alpha_n: n \in \mathbf{N}\} = \alpha$ and $\sup\{\beta_n: n \in \mathbf{N}\} = \beta$. By the monotonicity of $\odot''$, $\sup\{\alpha_n \odot'' \beta_n\} = \alpha \odot'' \beta$. Since the restriction of $\odot''$ to $\mathbf{Q} \cap [0, 1]$ is left-continuous, we obtain

$$\begin{aligned}\alpha \odot'' \beta &= \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q \leq \alpha, r \leq \beta\} \\ &= \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\}.\end{aligned}$$

For each $q < \alpha$, $r < \beta$, there is n such that $q < \alpha_n$ and $r < \beta_n$. Thus,

$$\sup\{\alpha_n \odot'' \beta_n: n \in \mathbf{N}\} \geq \sup\{q \odot'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\} = \alpha \odot'' \beta.$$

Hence, $\odot''$ is a left-continuous involutive micanorm on [0, 1].

It is an easy consequence of the definition that $\odot''$ extends $\odot$. By (I) to (V) and (A) ((B) resp), h is an embedding of $(A, \leq_A, \top, \bot, t, f, \wedge, \vee, *)$ into $([0, 1], \leq, 1, 0, e, \partial, \min, \max, \odot'')$. Moreover, $\odot''$ has a residuum, calling it $\rightharpoonup$.

We finally prove that for $x, y \in A$, $h(x \rightarrow y) = h(x) \rightharpoonup h(y)$. By (IV), $h(x \rightarrow y)$ is the residuum of $h(x)$ and $h(y)$ in $(\mathbf{Q} \cap [0, 1], \leq, 1, 0, e, \partial, \min, \max, \odot)$. Thus

$$h(x) \odot'' h(x \rightarrow y) = h(x) \odot h(x \rightarrow y) \leq h(y).$$

Suppose toward contradiction that there is $\alpha > h(x \rightarrow y)$ such that $\alpha \odot'' h(x) \leq h(y)$. Since $\mathbf{Q} \cap [0, 1]$ is dense in [0, 1], there is $q \in \mathbf{Q} \cap [0, 1]$ such that $h(x \to y) < q \leq \alpha$. Hence $q \odot'' h(x) = q \odot h(x) \leq h(y)$, contradicting (IV). □

Theorem 5.4 (Strong standard completeness) For L $\in$ {**FIMICASIL**, **FCnIMICAL**}, the following are equivalent:

(1) $T \vdash_L \phi$.

(2) For every standard **L**-algebra and evaluation v, if $v(\psi) \geq e$ for all $\psi \in T$, then $v(\phi) \geq e$.

Proof: (1) to (2) follows from definition. We prove (2) to (1). Let ϕ be a formula such that $T \nvdash_L \phi$, **A** a linearly ordered **L**-algebra, and v an evaluation in **A** such that $v(\psi) \geq t$ for all $\psi \in T$ and $v(\phi) < t$. Let h' be the embedding of **A** into the standard **L**-algebra as in proof of Proposition 5.3. Then, $h' \odot v$ is an evaluation into the standard **L**-algebra such that $h' \odot v(\psi) \geq e$ and yet $h' \odot v(\phi) < e$. □

Remark 5.5 The proof of standard completeness in Theorem 5.4 does not work for **IMICASIL** and **CnIMICAL** because the definition of $\odot$ does not satisfy square increasingness. We recall the example introduced in Yang (2015). Consider the following case: $\partial = {\sim}m < m = ({\sim}m)^{+} = e$ and $p/q + p/q \leq 1$, where $x = mp/q$. Since $m = m * m$, we have $(m, x) \odot (m, x) = \min\{\partial, (m, x) \circ (m, x)\} = \partial \prec (m, x)$; therefore, $(m, x) \nleq (m, x) \odot (m, x)$.

5. Concluding remark

We investigated (not merely algebraic completeness for **IMICASIL**, **FIMICASIL**, **CnIMICAL**, and **FCnIMICAL** but also) standard completeness for **FIMICASIL** and **FCnIMICAL**. We further noted that the proof of standard completeness does not work for **IMICASIL** and **CnIMICAL**. This shows that the insistence in Remark 2 in Yang (2015) is correct but that in Remark 3 in Yang (2015) is not.

References

Cintula, P. (2006), "Weakly Implicative (Fuzzy) Logics I: Basic properties", *Archive for Mathematical Logic* 45, pp. 673-704.

Cintula, P., Horĉík, R., and Noguera, C. (2013), "Non-associative substructural logics and their semilinear extensions: axiomatization and completeness properties", *Review of Symbolic Logic* 12, pp. 394-423.

Cintula, P., Horĉík, R., and Noguera, C. (2015), "The quest for the basic fuzzy logic", *Mathematical Fuzzy Logic*, P. Hájek (Ed.), Springer, pp. 245-290.

Cintula, P. and Noguera, C. (2011), "A general framework for mathematical fuzzy logic", *Handbook of Mathematical Fuzzy Logic*, vol 1, P. Cintula, P. Hájek, and C. Noguera (Eds.), London, College publications, pp. 103-207.

Esteva, F., Gispert, L., Godo, L., and Montagna, F. (2002), "On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic", *Studia Logica* 71, pp. 393-420.

Hájek, P. (1998), *Metamathematics of Fuzzy Logic*, Amsterdam, Kluwer.

Horĉík, R. (2011), "Algebraic semantics: semilinear FL-algebras", *Handbook of Mathematical Fuzzy Logic*, vol 1, P. Cintula, P. Hájek, and C. Noguera (Eds.), London, College publications, pp. 283-353.

Jenei, S. and Montagna, F. (2002), "A Proof of Standard completeness for Esteva and Godo's Logic MTL", *Studia*

Logica 70, pp. 183-192.

Metcalfe, G., and Montagna, F. (2007), “Substructural Fuzzy Logics”, *Journal of Symbolic Logic* 72, pp. 834-864.

Yang, E. (2009), “On the standard completeness of an axiomatic extension of the uninorm logic”, *Korean Journal of Logic* 12 (2), pp. 115-139.

Yang, E. (2013), “Standard completeness for **MTL**”, *Korean Journal of Logic* 16 (3), pp. 437-452.

Yang, E. (2014), “An Axiomatic Extension of the Uninorm Logic Revisited”, *Korean Journal of Logic* 17 (2), pp. 323-348.

Yang, E. (2015), “Weakening-free, non-associative fuzzy logics: Micanorm-based logics”, *Fuzzy Sets and Systems* 276, pp. 43-58.

전북대학교 철학과, 비판적사고와논술연구소

Department of Philosophy & Institute of Critical Thinking and Writing, Chonbuk National University

eunsyang@jbnu.ac.kr

누승적 미카놈 논리 IMICAL의 몇몇 공리적 확장

양 은 석

이 글에서 우리는 누승적 미카놈 논리 IMICAL의 몇몇 공리적 확장 체계의 표준 완전성을 다룬다. 이를 위하여, 먼저 누승적 미카놈에 바탕을 둔 네 논리 체계를 소개한다. 각 체계에 상응하는 대수적 구조를 정의한 후, 이들 체계가 대수적으로 완전하다는 것을 보인다. 다음으로, 이 논리 체계들 중 두 체계가 표준적으로 완전하다는 것 즉 단위 실수 [0,1]에서 완전하다는 것을 제네이-몬테그나 방식의 구성을 사용하여 보인다.

주요어: 퍼지 논리, (누승적) 미카놈, 대수적 완전성, 표준 완전성