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# THE RIESZ DECOMPOSITION THEOREM FOR SKEW SYMMETRIC OPERATORS

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Abstract. An operator $T$ on a complex Hilbert space $\mathcal{H}$ is called skew symmetric if $T$ can be represented as a skew symmetric matrix relative to some orthonormal basis for $\mathcal{H}$. In this note, we explore the structure of skew symmetric operators with disconnected spectra. Using the classical Riesz decomposition theorem, we give a decomposition of certain skew symmetric operators with disconnected spectra. Several corollaries and illustrating examples are provided.

## 1. Introduction

Throughout this paper, we denote by $\mathcal{H}$ a complex separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$.

**Definition 1.1.** A map $C$ on $\mathcal{H}$ is called an *antiunitary operator* if $C$ is conjugate-linear, invertible and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. If, in addition, $C^{-1} = C$, then $C$ is called a *conjugation*.

**Definition 1.2** ([18])**.** An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *skew symmetric* if there exists a conjugation $C$ on $\mathcal{H}$ such that $CTC = -T^*$. $T$ is said to be *complex symmetric* if $CTC = T^*$ for some conjugation $C$ on $\mathcal{H}$.

Using [5, Lem. 1], one can see that $T \in \mathcal{B}(\mathcal{H})$ is skew symmetric if and only if there exists an orthonormal basis (ONB for short) $\{e_n\}$ of $\mathcal{H}$ such that $\langle Te_n, e_m\rangle = -\langle Te_m, e_n\rangle$ for all $m, n$; that is, $T$ admits a skew symmetric matrix representation with respect to $\{e_n\}$. Thus skew symmetric operators can be viewed as an infinite dimensional analogue of skew symmetric matrices. The most obvious examples of skew symmetric operators on finite dimensional spaces are those Jordan blocks with odd orders (see [14, Ex. 1.7]).

The following lemma contains some elementary facts about skew symmetric operators.

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**Lemma 1.3** ([14]). *Let $C$ be a conjugation on $\mathcal{H}$. Denote $S_C(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}) : CXC = -X^*\}$. Then*

(i) *if $A, B \in \mathcal{B}(\mathcal{H})$, $CAC = A^*$ and $CBC = B^*$, then $[A, B] \triangleq AB - BA \in S_C(\mathcal{H})$;*

(ii) *if $T \in S_C(\mathcal{H})$, then $CT^{2n}C = (T^{2n})^*$ for all $n \in \mathbb{N}$;*

(iii) *the class $S_C(\mathcal{H})$ is norm-closed and forms a Lie algebra under the commutator bracket $[\cdot, \cdot]$;*

(iv) *if $T \in S_C(\mathcal{H})$, then $\sigma(T) = -\sigma(T)$.*

The primary motivation for the study of skew symmetric operators lies in its connections to complex symmetric operators, which have received much attention in the last decade [3, 4, 5, 6, 7, 8, 9, 10, 11, 22, 23]. By Lemma 1.3(i), one can use complex symmetric operators to construct new skew symmetric operators. By [5, Prop. 3], all truncated Toeplitz operators are complex symmetric with respect to the same conjugation. Then it follows from Lemma 1.3(i) that any commutator of two truncated Toeplitz operators is skew symmetric. In particular, if $T$ is complex symmetric, then $T^*T - TT^*$ is skew symmetric. In view of the description of skew symmetric normal operators [14, Thm. 1.10], this provides a new approach to describing complex symmetric operators. In a recent paper [11], one can see such an application to Toeplitz operators.

Another motivation for the study of skew symmetric operators lies in the connection between skew symmetric operators and anti-automorphisms of singly generated $C^*$-algebras. Recall that an *anti-automorphism* of a $C^*$-algebra $\mathcal{A}$ is a vector space isomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ with $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for $a, b \in \mathcal{A}$. An anti-automorphism $\rho$ is said to be *involutory* if $\rho^{-1} = \rho$. Involutory anti-automorphisms play an important role in the study of the real structure of $C^*$-algebras [2, 17]. It is not necessary that each $C^*$-algebra possesses an involutory anti-automorphism on it [15]. However, each $C^*$-algebra generated by a skew symmetric operator admits an involutory anti-automorphism on it (see [20, Cor. 3.2]).

Recently, there has been growing interest in skew symmetric operators, and some interesting results have been obtained [13, 14, 18, 19, 20, 21]. In particular, skew symmetric normal operators, partial isometries, compact operators and weighted shifts are classified [13, 14, 21].

The aim of this note is to explore the structure of skew symmetric operators with disconnected spectra. If $T \in \mathcal{B}(\mathcal{H})$ is skew symmetric and $CTC = -T^*$ for some conjugation $C$, then $C(T - \alpha)C = -(T + \alpha)^*$ for each $\alpha \in \mathbb{C}$. This means that the behavior of $T$ at $\alpha$ is very like that at $-\alpha$. In particular, $T - \alpha$ is invertible if and only if $T + \alpha$ is invertible. So $\sigma(T) = -\sigma(T)$. By [14, Thm. 1.10], a normal operator $A$ is skew symmetric if and only if

$$A \cong 0 \oplus N \oplus (-N) \tag{1.1}$$

for some normal operator $N$. All these facts motivate us to explore the central symmetry of the spectra of skew symmetric operators. Since a general skew symmetric operator need not be reducible, one can not expect a general decomposition like (1.1) for skew symmetric operators. So, in this note, we employ the idea of the classical Riesz decomposition theorem, and explore the structure of skew symmetric operators with disconnected spectra. To proceed, let us first recall some familiar notation and terminology.

Let $T \in \mathcal{B}(\mathcal{H})$. If $\sigma$ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain $\Omega$ such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$. We let $E(\sigma; T)$ denote the *Riesz idempotent* of $T$ corresponding to $\sigma$, i.e.,

$$E(\sigma; T) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda - T)^{-1} \mathrm{d}\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to $\Omega$ in the sense of complex variable theory. In this case, we denote $\mathcal{H}(\sigma; T) = \mathrm{ran} E(\sigma; T)$. Since $E(\sigma; T)T = TE(\sigma; T)$, one can see $T(\mathcal{H}(\sigma; T)) \subseteq \mathcal{H}(\sigma; T)$. Denote by $T_\sigma$ the restriction of $T$ to $\mathcal{H}(\sigma; T)$.

The following result is the Riesz decomposition theorem.

**Theorem 1.4** ([16], Thm. 2.10)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$, where $\sigma_1, \sigma_2$ are clopen subsets of $\sigma(T)$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Then $\mathcal{H}(\sigma_1; T) + \mathcal{H}(\sigma_2; T) = \mathcal{H}, \mathcal{H}(\sigma_1; T) \cap \mathcal{H}(\sigma_2; T) = \{0\}$ and $\sigma(T_{\sigma_i}) = \sigma_i, i = 1, 2;$ moreover, $T$ can be represented as*

$$T = \begin{bmatrix} T_{\sigma_1} & 0 \\ 0 & T_{\sigma_2} \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_2; T). \end{matrix}$$

*Remark* 1.5. In Theorem 1.4, we remark that $\mathcal{H}(\sigma_1; T)$ is not orthogonal to $\mathcal{H}(\sigma_2; T)$ in general, that is, $\mathcal{H}(\sigma_1; T) \neq \mathcal{H}(\sigma_2; T)^\perp$.

Let $T \in \mathcal{B}(\mathcal{H})$ be skew symmetric and $\sigma$ be a clopen subset of $\sigma(T)$. Denote $\sigma_- = -\sigma$ and $\sigma' = \sigma(T) \setminus \sigma$. Since $\sigma(T) = -\sigma(T)$, both $\sigma_-$ and $\sigma'$ are clopen subsets of $\sigma(T)$. In this note, we are mainly interested in the following natural questions:

(a) What is the internal connection between $T_\sigma$ and $T_{\sigma_-}$?
(b) Does there exist a decomposition of $T$ like (1.1)?
(c) If $\sigma = \sigma_-$, then does it follow that $T_\sigma$ is skew symmetric?
(d) If $T_\sigma$ is skew symmetric, then does it follow that $T_{\sigma'}$ is skew symmetric?

In this note, we exhibit the connection between $T_\sigma$ and $T_{\sigma_-}$, and give a decomposition of $T$ in the case that $\sigma_- = \sigma'$. We answer (c) and (d) negatively by constructing an example (see Example 3.6). To state our main result, we need some other notation and terminology.

**Definition 1.6** ([1], page 95)**.** Let $T \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called a *transpose* of $T$ if $A = CT^*C$ for some conjugation $C$ on $\mathcal{H}$.

The concept "transpose" of an operator is in fact a generalization of that for matrices. If $T \in \mathcal{B}(\mathcal{H})$ is normal, then $T$ is complex symmetric and there exists a conjugation $C$ on $\mathcal{H}$ such that $T = CT^*C$ ([5]). It follows that $T$ is a transpose of itself. In general, an operator has more than one transpose [20, Ex. 2.2]. However, one can check that any two transposes of an operator are unitarily equivalent ([10]). We often write $T^t$ to denote a transpose of $T$. In general, there is no ambiguity especially when we write $T \cong T^t$. It is easy to check that $\sigma(T) = \sigma(T^t)$. By [20, Lem. 3.7], $T \oplus (-T^t)$ is always skew symmetric for $T \in \mathcal{B}(\mathcal{H})$. Conversely, by (1.1), each skew symmetric normal operator $A$ has the form

$$A = 0 \oplus N \oplus (-N^t). \tag{1.2}$$

Two operators $A, B \in \mathcal{B}(\mathcal{H})$ are said to be *similar*, denoted by $A \sim B$, if there exists invertible $X \in \mathcal{B}(\mathcal{H})$ such that $AX = XB$; if, in addition, $X$ is unitary, then $A, B$ are said to be *unitarily equivalent*, denoted by $A \cong B$.

The main result of this note is the following theorem, which describes the structure of skew symmetric operators with disconnected spectra.

**Theorem 1.7** (Main Theorem)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be skew symmetric and $\sigma$ be a clopen subset of $\sigma(T)$. Then*

(i) *$\sigma_-$ is also a clopen subset of $\sigma(T)$ and $T_{\sigma_-} \sim (-T^t_\sigma)$;*

(ii) *if $\sigma \cap \sigma_- = \emptyset$, then $\sigma \cup \sigma_-$ is a clopen subset of $\sigma(T)$ and*
$$T_{\sigma\cup\sigma_-} \sim T_\sigma \oplus (-T^t_\sigma);$$

(iii) *if $\sigma_- = \sigma(T) \setminus \sigma$, then there exists a conjugation $C$ on $\mathcal{H}(\sigma; T)$ and $E \in S_C(\mathcal{H}(\sigma; T))$ such that*

$$T \cong \begin{bmatrix} T_\sigma & E \\ 0 & -CT^*_\sigma C \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T). \end{matrix} \tag{1.3}$$

Let $T \in \mathcal{B}(\mathcal{H})$ have the form (1.3). Then it is easy to check that

$$D \triangleq \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}$$

is a conjugation on $\mathcal{H}(\sigma; T)^{(2)}$. Given a Hilbert space $\mathcal{K}$ and a cardinal number $d$, $\mathcal{K}^{(d)}$ denotes the direct sum of $d$ copies of $\mathcal{K}$. With respect to $D$, all the following three operators

$$\begin{bmatrix} T_\sigma & 0 \\ 0 & -CT^*_\sigma C \end{bmatrix}, \quad \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} T_\sigma & E \\ 0 & -CT^*_\sigma C \end{bmatrix}$$

are skew symmetric.

*Remark* 1.8. (i) The similarity $\sim$ in Theorem 1.7(i) can not be replaced by stricter unitary equivalence (see Example 3.4).

(ii) In Theorem 1.7(ii), the operator $T_{\sigma\cup\sigma_-}$ is similar to a skew symmetric operator of the form $A \oplus (-A^t)$. The similarity can not be replaced by unitary equivalence since $T_{\sigma\cup\sigma_-}$ need not be skew symmetric (see Example 3.6). In fact, even if $T_{\sigma\cup\sigma_-}$ is skew symmetric, $\sim$ can not be replaced by $\cong$ (see Remark 3.5).

The proof of Theorem 1.7 shall be provided in Section 2. In Section 3, we shall give several immediate corollaries and two illustrating examples.

## 2. Proof of Main Theorem

We first make some preparation.

**Lemma 2.1.** *Let $T \in \mathcal{B}(\mathcal{H})$ and*

$$T = \begin{bmatrix} A & F \\ 0 & B \end{bmatrix} \begin{matrix} M \\ M^{\perp}, \end{matrix}$$

*where $M$ is a nontrivial subspace of $\mathcal{H}$, $A \in \mathcal{B}(M)$ and $B \in \mathcal{B}(M^{\perp})$. If $\sigma(A) \cap \sigma(B) = \emptyset$, then $\sigma \triangleq \sigma(A)$ is a clopen subset of $\sigma(T)$ and $\mathcal{H}(\sigma; T) = M$.*

*Proof.* Since $\sigma(A) \cap \sigma(B) = \emptyset$, by [12, Cor. 3.22], $T \sim A \oplus B$. Then $\sigma(T) = \sigma(A) \cup \sigma(B)$ and $\sigma = \sigma(A)$ is a clopen subset of $\sigma(T)$.

For each $\lambda \in \mathbb{C} \setminus \sigma(T)$, one can check that

$$(T - \lambda)^{-1} = \begin{bmatrix} (A-\lambda)^{-1} & (\lambda - A)^{-1} F (B-\lambda)^{-1} \\ 0 & (B-\lambda)^{-1} \end{bmatrix}.$$

Choose an analytic Cauchy domain $\Omega$ such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$. Since $(B - \lambda)^{-1}$ is analytic on $\Omega^{-}$ and $\sigma(A) \subset \Omega$, one can see that

$$E(\sigma; T) = \frac{1}{2\pi \mathrm{i}} \int_{\partial \Omega} (\lambda - T)^{-1} \mathrm{d}\lambda = \begin{bmatrix} I_1 & * \\ 0 & 0 \end{bmatrix},$$

where $I_1$ is the identity operator on $M$. So $\mathcal{H}(\sigma; T) = \mathrm{ran} E(\sigma; T) = M$ and $T_\sigma = T|_M = A$. □

**Lemma 2.2.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $\sigma$ be a clopen subset of $\sigma(T)$. Then $T$ can be written as*

$$T = \begin{bmatrix} T_\sigma & F \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^{\perp}, \end{matrix}$$

*where $\sigma(B) = \sigma'$ and $B \sim T_{\sigma'}$.*

*Proof.* Since $\mathcal{H}(\sigma; T)$ is an invariant subspace of $T$, $T$ can be written as

$$T = \begin{bmatrix} T_\sigma & F \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^{\perp}, \end{matrix}$$

where $B \in \mathcal{B}(\mathcal{H}(\sigma; T)^{\perp})$ and $F : \mathcal{H}(\sigma; T)^{\perp} \to \mathcal{H}(\sigma; T)$. So when $x \in \mathcal{H}(\sigma; T)$ and $y \in \mathcal{H}(\sigma; T)^{\perp}$, one can see $Tx = T_\sigma x$ and $Ty = Fy + By$.

Since $\mathrm{ran} E(\sigma; T) = \mathcal{H}(\sigma; T)$ and $E(\sigma; T)x = x$ for $x \in \mathcal{H}(\sigma; T)$, $E(\sigma; T)$ admits the following matrix representation

$$E(\sigma; T) = \begin{bmatrix} I_1 & E \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^{\perp}, \end{matrix}$$

where $I_1$ is the identity operator on $\mathcal{H}(\sigma; T)$. Noting that $E(\sigma; T)T = TE(\sigma; T)$, a matrix computation shows that

$$T_\sigma E = F + EB. \tag{2.1}$$

By the Riesz decomposition theorem, $E(\sigma';T) = I - E(\sigma;T)$ and hence

$$E(\sigma';T) = \begin{bmatrix} 0 & -E \\ 0 & I_2 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^{\perp}, \end{matrix}$$

where $I_2$ is the identity operator on $\mathcal{H}(\sigma;T)^{\perp}$. Thus $\mathcal{H}(\sigma';T) = \{x - Ex : x \in \mathcal{H}(\sigma;T)^{\perp}\}$.

Define

$$\begin{aligned} S : \mathcal{H}(\sigma;T)^{\perp} &\longrightarrow \mathcal{H}(\sigma';T), \\ x &\longmapsto x - Ex. \end{aligned}$$

It is obvious that $S$ is linear and surjective. Moreover, for each $x \in \mathcal{H}(\sigma;T)^{\perp}$,

$$\|x\| \leq \sqrt{\|x\|^2 + \|Ex\|^2} = \|x - Ex\| \leq (1 + \|E\|)\|x\|.$$

So $S$ is bounded and invertible. Now we shall check that $SB = T_{\sigma'}S$.

Fix an $x \in \mathcal{H}(\sigma;T)^{\perp}$. Noting that $Ex \in \mathcal{H}(\sigma;T)$, we have

$$\begin{aligned} T_{\sigma'}Sx &= T_{\sigma'}(x - Ex) \\ &= T(x - Ex) \\ &= Tx - TEx \\ &= Bx + Fx - T_{\sigma}Ex \qquad \text{by (2.1)} \\ &= Bx - EBx = SBx. \end{aligned}$$

It follows that $SB = T_{\sigma'}S$ and hence $T_{\sigma'} \sim B$. □

By Lemma 2.2 and [12, Cor. 3.22], the following result is clear.

**Corollary 2.3.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $\sigma$ be a clopen subset of $\mathbb{C}$. Then $T \sim T_{\sigma} \oplus T_{\sigma'}$.*

**Corollary 2.4.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $\sigma$ be a clopen subset of $\sigma(T)$. Then, with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}(\sigma;T) \oplus \mathcal{H}(\sigma;T)^{\perp}$, $E(\sigma;T)$ and $E(\sigma';T)$ can be represented as*

$$E(\sigma;T) = \begin{bmatrix} I_1 & E \\ 0 & 0 \end{bmatrix}, \quad E(\sigma';T) = \begin{bmatrix} 0 & -E \\ 0 & I_2 \end{bmatrix},$$

*where $E : \mathcal{H}(\sigma;T)^{\perp} \to \mathcal{H}(\sigma;T)$, $I_1, I_2$ are respectively the identity operators on $\mathcal{H}(\sigma_1;T)$ and $\mathcal{H}(\sigma_1;T)^{\perp}$.*

**Proposition 2.5.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $C$ be a conjugation on $\mathcal{H}$ satisfying $CTC = -T^*$. If $\sigma$ is a nonempty clopen subset of $\sigma(T)$, then $CE(\sigma;T) = E(\sigma;-T)^*C$.*

*Proof.* Assume that $\Omega$ is an analytic Cauchy domain satisfying $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$, $\Gamma = \partial\Omega$ is positively oriented with respect to $\Omega$ in the sense of complex variable theory. Then

$$E(\sigma;T) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (\lambda - T)^{-1} \mathrm{d}\lambda$$

$$
\begin{aligned}
&= \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda + CT^*C)^{-1} \mathrm{d}\lambda \\
&= \frac{1}{2\pi \mathrm{i}} \int_\Gamma C(\overline{\lambda} + T^*)^{-1} C \mathrm{d}\lambda.
\end{aligned}
$$

For convenience, we directly assume that $\Gamma$ is connected. The proof for general case is similar. Given a partition $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n (= \lambda_0)$ of $\Gamma$, one can verify that

$$
\begin{aligned}
\sum_{i=0}^{n-1} C(\overline{\lambda}_i + T^*)^{-1} C(\lambda_{i+1} - \lambda_i) &= C\left(\sum_{i=0}^{n-1} (\overline{\lambda}_i + T^*)^{-1} (\overline{\lambda_{i+1}} - \overline{\lambda}_i)\right) C \\
&= C\left(\sum_{i=0}^{n-1} (\lambda_i + T)^{-1} (\lambda_{i+1} - \lambda_i)\right)^* C.
\end{aligned}
$$

Since $\sigma(T) = -\sigma(T)$, it follows that $\sigma$ is also a clopen subset of $\sigma(-T)$. Hereby we deduce that

$$
\begin{aligned}
E(\sigma; T) &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma C(\overline{\lambda} + T^*)^{-1} C \mathrm{d}\lambda \\
&= \frac{1}{2\pi \mathrm{i}} C \left( \int_\Gamma (\lambda + T)^{-1} \mathrm{d}\lambda \right)^* C \\
&= C \left( \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda + T)^{-1} \mathrm{d}\lambda \right)^* C \\
&= CE(\sigma; -T)^* C.
\end{aligned}
$$

□

**Lemma 2.6.** *Let $T \in \mathcal{B}(\mathcal{H})$ and assume that $\sigma(T) = -\sigma(T)$. If $\sigma$ is a nonempty clopen subset of $\sigma(T)$, then $\sigma_-$ is also a clopen subset of $\sigma(T)$ and $E(\sigma; -T) = E(\sigma_-; T)$.*

*Proof.* Set $\sigma_1 = \sigma_-$ and $\sigma_2 = \sigma(T) \setminus \sigma_-$. Then $\sigma_1, \sigma_2$ are clopen subsets of $\sigma(T)$, $\sigma(T) = \sigma_1 \cup \sigma_2$ and $\sigma_1 \cap \sigma_2 = \emptyset$. By Theorem 1.4, $T$ can be written as

$$
T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_2; T), \end{matrix}
$$

where $\sigma(A) = \sigma_1$ and $\sigma(B) = \sigma_2$. So

$$
E(\sigma_1; T) = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_2; T), \end{matrix} \tag{2.2}
$$

where $I_1$ is the identity operator on $\mathcal{H}(\sigma_1; T)$. Note that

$$
-T = \begin{bmatrix} -A & 0 \\ 0 & -B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_2; T). \end{matrix}
$$

Note that $\sigma(-A) = -\sigma_1 = \sigma$ is a clopen subset of $\sigma(-T)$ and $\sigma(-A) \cap \sigma(-B) = \emptyset$. one can see that

$$
E(\sigma; -T) = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_2; T). \end{matrix}
$$

In view of (2.2), we have $E(\sigma;-T)=E(\sigma_-;T)$. □

Now we are going to prove Main Theorem.

*Proof of Theorem 1.7.* Assume that $D$ is a conjugation on $\mathcal{H}$ such that $DTD=-T^*$.

(i) By Proposition 2.5 and Lemma 2.6, we have

$$DE(\sigma;T)D=E(\sigma;-T)^*=E(\sigma_-;T)^*.$$

Since $DE(\sigma;T)TE(\sigma;T)D=(DE(\sigma;T)D)(DTD)(DE(\sigma;T)D)$, it follows that

$$DE(\sigma;T)TE(\sigma;T)D=-E(\sigma_-;T)^*T^*E(\sigma_-;T)^*. \tag{2.3}$$

By Corollary 2.4, we may assume that

$$T=\begin{bmatrix}T_{\sigma_-} & F\\ 0 & B\end{bmatrix}\begin{matrix}\mathcal{H}(\sigma_-;T)\\ \mathcal{H}(\sigma_-;T)^\perp\end{matrix},\quad E(\sigma_-;T)=\begin{bmatrix}I_1 & G\\ 0 & 0\end{bmatrix}\begin{matrix}\mathcal{H}(\sigma_-;T)\\ \mathcal{H}(\sigma_-;T)^\perp,\end{matrix} \tag{2.4}$$

where $I_1$ is the identity operator on $\mathcal{H}(\sigma_-;T)$.

Denote by $P$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}(\sigma_-;T)$. So $P$ can be represented as

$$P=\begin{bmatrix}I_1 & 0\\ 0 & 0\end{bmatrix}\begin{matrix}\mathcal{H}(\sigma_-;T)\\ \mathcal{H}(\sigma_-;T)^\perp.\end{matrix} \tag{2.5}$$

By (2.4), each $z\in \mathrm{ran}E(\sigma_-;T)^*$ has the form $z=x+G^*x$ for some $x\in\mathcal{H}(\sigma_-;T)$. For such $z$, define $P_1z=x$. Note that $G^*x\in\mathcal{H}(\sigma_-;T)^\perp$ for $x\in\mathcal{H}(\sigma_-;T)$. One can deduce that $P_1:\mathrm{ran}E(\sigma_-;T)^*\to\mathcal{H}(\sigma_-;T)$ is an invertible bounded linear operator.

On the other hand, since $E(\sigma_-;T)^*$ is idempotent and $D=D^{-1}$, one can see that $D(\mathrm{ran}E(\sigma;T))=\mathrm{ran}E(\sigma_-;T)^*$. Thus the map

$$\begin{aligned}D_1:\mathcal{H}(\sigma;T)&\longrightarrow \mathrm{ran}E(\sigma_-;T)^*\\ x&\longmapsto Dx\end{aligned}$$

is conjugate-linear and invertible. Choose a conjugation $C$ on $\mathcal{H}(\sigma;T)$ and set $S=P_1D_1C$. Then $S$ is an invertible bounded linear operator from $\mathcal{H}(\sigma;T)$ onto $\mathcal{H}(\sigma_-;T)$. Now we shall check that $S(CT_\sigma C)=-T_{\sigma_-}^*S$.

Now fix an $x\in\mathcal{H}(\sigma;T)$. Compute to see that

$$\begin{aligned}S(CT_\sigma C)x&=P_1D_1T_\sigma Cx=(PD)\Big(E(\sigma;T)TE(\sigma;T)\Big)Cx \qquad \text{by (2.3)}\\ &=-P\Big(E(\sigma_-;T)TE(\sigma_-;T)\Big)^*DCx.\end{aligned}$$

By (2.4), one can check that

$$\Big(E(\sigma_-;T)TE(\sigma_-;T)\Big)^*=\begin{bmatrix}T_{\sigma_-}^* & 0\\ G^*T_{\sigma_-}^* & 0\end{bmatrix}\begin{matrix}\mathcal{H}(\sigma_-;T)\\ \mathcal{H}(\sigma_-;T)^\perp.\end{matrix}$$

It follows from (2.5) that

$$\begin{aligned} S(CT_\sigma C)x &= -\begin{bmatrix} T^*_{\sigma_-} & 0 \\ 0 & 0 \end{bmatrix} DCx \\ &= -\begin{bmatrix} T^*_{\sigma_-} & 0 \\ 0 & 0 \end{bmatrix} PDCx = -T^*_{\sigma_-}Sx. \end{aligned}$$

Since $x \in \mathcal{H}(\sigma;T)$ is arbitrary, we obtain $S(CT_\sigma C) = -T^*_{\sigma_-}S$. So $CT_\sigma C \sim (-T^*_{\sigma_-})$, that is, $T_{\sigma_-} \sim (-T^t_\sigma)$. This proves (i).

(ii) Denote $\Delta = \sigma \cup \sigma_-$ and $\Gamma = \Delta'$. Then $\Gamma$ is also a clopen subset of $\sigma(T)$. By Corollary 2.3, $T \sim T_\sigma \oplus T_{\sigma_-} \oplus T_\Gamma$ and $T_\Delta \sim T_\sigma \oplus T_{\sigma_-}$. By (i), $T_{\sigma_-} \sim (-T^t_\sigma)$ and hence $T_\Delta \sim T_\sigma \oplus (-T^t_\sigma)$.

(iii) Since $\sigma$ is a nonempty clopen subset of $\sigma(T)$, by Lemma 2.1, $T$ can be written as

$$T = \begin{bmatrix} T_\sigma & F_1 \\ 0 & B_1 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^\perp, \end{matrix}$$

where $\sigma(T_\sigma) = \sigma$ and $\sigma(B_1) = \sigma(T) \setminus \sigma = \sigma_-$. Thus

$$E(\sigma;T) = \begin{bmatrix} I_2 & * \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^\perp \end{matrix}$$

and

$$E(\sigma_-;T) = I - E(\sigma;T) = \begin{bmatrix} 0 & * \\ 0 & I_3 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^\perp, \end{matrix} \tag{2.6}$$

where $I_2$ and $I_3$ are identity operators on $\mathcal{H}(\sigma;T)$ and $\mathcal{H}(\sigma;T)^\perp$ respectively.

By Proposition 2.5, $DE(\sigma;T)D = E(\sigma;-T)^* = E(\sigma_-;T)^*$. Then, by (2.6),

$$D(\mathcal{H}(\sigma;T)) = D(\mathrm{ran}E(\sigma;T)) = \mathrm{ran}E(\sigma_-;T)^* = \mathcal{H}(\sigma;T)^\perp.$$

So $D$ admits the following matrix representation

$$D = \begin{bmatrix} 0 & D_2 \\ D_1 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T)^\perp, \end{matrix}$$

where $D_1, D_2$ are antiunitary operators. Thus $\dim \mathcal{H}(\sigma;T) = \dim \mathcal{H}(\sigma;T)^\perp$ and, up to unitary equivalence, we may assume that $\mathcal{H}(\sigma;T) = \mathcal{H}(\sigma;T)^\perp$. So $D_1, D_2$ are antiunitary operators on $\mathcal{H}(\sigma;T)$. Since $D^{-1} = D$, one can see that $D_2 = D_1^{-1}$ and

$$T = \begin{bmatrix} T_\sigma & F_1 \\ 0 & B_1 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma;T) \\ \mathcal{H}(\sigma;T). \end{matrix}$$

Choose a conjugation $C$ on $\mathcal{H}(\sigma;T)$ and set $U = D_1C$. Then $U \in \mathcal{B}(\mathcal{H}(\sigma;T))$ is unitary and $D_1 = UC$. Set $V = I_2 \oplus U$. Then $V$ acting on $\mathcal{H}(\sigma;T)^{(2)}$ is unitary,

$$V^*TV = \begin{bmatrix} T_\sigma & F_1U \\ 0 & U^*B_1U \end{bmatrix} \triangleq \begin{bmatrix} T_\sigma & E \\ 0 & B_2 \end{bmatrix} \quad \text{and} \quad C_1 \triangleq V^*DV = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}$$

is a conjugation on $\mathcal{H}(\sigma;T)^{(2)}$. Since $DT = -T^*D$, it follows that

$$C_1(V^*TV) = -(V^*TV)^*C_1.$$

It follows from a direct matrical calculation that $B_2 = -CT_\sigma^*C$ and $E = -CE^*C$, whence we conclude that

$$T \cong \begin{bmatrix} T_\sigma & E \\ 0 & -CT_\sigma^*C \end{bmatrix}.$$

This completes the proof. □

## 3. Corollaries and examples

In this section we give several corollaries of Theorem 1.7 and illustrating examples.

**Corollary 3.1.** *Let $T \in \mathcal{B}(\mathcal{H})$. If $\sigma(T)$ is finite and $0 \notin \sigma(T)$, then $T$ is skew symmetric if and only if $T$ is unitarily equivalent to an operator of the form*

$$\begin{bmatrix} A & E \\ 0 & -CA^*C \end{bmatrix},$$

*where $C$ is a conjugation on some Hilbert space $\mathcal{K}$, $A \in \mathcal{B}(\mathcal{K})$ and $E \in S_C(\mathcal{K})$.*

*Proof.* We need only prove the necessity. Assume that $T$ is skew symmetric. Then $\sigma(T) = -\sigma(T)$. Set $\sigma = \{\lambda \in \sigma(T) : \lambda > 0 \quad \text{or} \quad \text{Im}\lambda > 0\}$. Then $\sigma_- \cap \sigma = \emptyset$ and $\sigma(T) = \sigma \cup \sigma_-$, since $0 \notin \sigma(T)$. Noting that $\sigma(T)$ is finite, $\sigma$ is a clopen subset of $\sigma(T)$. By Theorem 1.7(iii), the desired result follows readily. □

**Corollary 3.2.** *If $T \in \mathcal{B}(\mathbb{C}^n)$ is invertible and skew symmetric, then $n$ is even.*

**Corollary 3.3.** *Each skew symmetric operator on $\mathbb{C}^2$ is normal and hence reducible.*

*Proof.* Assume that $T \in \mathcal{B}(\mathbb{C}^2)$ is skew symmetric. If $T$ is invertible, then, by Corollary 3.1,

$$T \cong \begin{bmatrix} A & E \\ 0 & -CA^*C \end{bmatrix} \begin{matrix} \mathbb{C} \\ \mathbb{C}, \end{matrix}$$

where $E \in \mathcal{B}(\mathbb{C})$ is skew symmetric. Noting that $\sigma(E) = -\sigma(E)$ and $E \in \mathcal{B}(\mathbb{C})$, we obtain $E = 0$. So $T$ is normal.

If $T$ is not invertible, then $0 \in \sigma(T)$. Noting that $\sigma(T) = -\sigma(T)$ has at most two points, we deduce that $\sigma(T) = \{0\}$. Hence

$$T = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}$$

with respect to some ONB of $\mathbb{C}^2$. By [21, Lem. 2.11], $\lambda = 0$. Then $T$ is normal. This completes the proof. □

Now we give an example to show that the similarity $\sim$ in Theorem 1.7(i) can not be replaced by unitary equivalence.

**Example 3.4.** Let $\{e_i\}_{i=1}^{\infty}$ be an ONB of $\mathcal{H}$ and $S$ be the unilateral shift given by

$$Se_i = e_{i+1}, \quad \forall i \geq 1.$$

Set $A = -(S+2)^*$, $B = S+2$, $E = e_1 \otimes e_2 - e_2 \otimes e_1$ and $F = AE - EB$. So $A, B, E, F \in \mathcal{B}(\mathcal{H})$. For $x \in \mathcal{H}$ with $x = \sum_{i=1}^{\infty} \alpha_i e_i$, define

$$Cx = \sum_{i=1}^{\infty} \overline{\alpha_i} e_i.$$

Then $C$ is a conjugation on $\mathcal{H}$. Moreover, one can check that $CSC = S$ and

$$B = -CA^*C, \quad CEC = E = -E^*, \quad CFC = -F^*. \tag{3.1}$$

Define an operator $T \in \mathcal{B}(\mathcal{H}^{(2)})$ as

$$T = \begin{bmatrix} A & F \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2, \end{matrix}$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$.

Denote $\sigma = \{z \in \mathbb{C} : |z+2| \leq 1\}$. It is obvious that $\sigma(A) = \sigma$, $\sigma(B) = -\sigma$ and $\sigma \cap \sigma_- = \emptyset$. So $\sigma(T) = \sigma \cup \sigma_-$ and $\sigma$ is a clopen subset of $\sigma(T)$. Then, by Lemma 2.1, $A = T_\sigma$. One can directly check that $T$ is skew symmetric with respect to the following conjugation

$$D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2. \end{matrix}$$

In the remaining we shall show that $T_{\sigma_-}$ is not unitarily equivalent to $(-T_\sigma^t)$. For a proof by contradiction, we assume that $T_{\sigma_-} \cong (-T_\sigma^t)$. Then, by (3.1), we have

$$B = -A^t = -T_\sigma^t \cong T_{\sigma_-},$$

that is, $B \cong T_{\sigma_-}$. Noting that $B - 2I_2$ is isometric, it follows that $T_{\sigma_-} - 2I_{\sigma_-}$ is isometric, where $I_{\sigma_-}$ is the identity operator on $\mathcal{H}^{(2)}(\sigma_-; T)$.

By Lemma 2.1, $\mathcal{H}_1 = \mathcal{H}^{(2)}(\sigma; T)$ and $E(\sigma; T)$ can be written as

$$E(\sigma; T) = \begin{bmatrix} I_1 & E_1 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2, \end{matrix}$$

where $I_1$ is the identity operator on $\mathcal{H}_1$. Since $E(\sigma;T)T = TE(\sigma;T)$, a matrical computation shows that $AE_1 - E_1B = F$ and hence $A(E_1 - E) - (E_1 - E)B = 0$. Noting that $\sigma(A) \cap \sigma(B) = \emptyset$, it follows from Rosenblum's Theorem [12, Cor. 3.20] that $E_1 - E = 0$, that is, $E_1 = E$. So

$$E(\sigma; T) = \begin{bmatrix} I_1 & E \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} \quad \text{and} \quad E(\sigma'; T) = \begin{bmatrix} 0 & -E \\ 0 & I_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2, \end{matrix}$$

where $I_2$ is the identity operator on $\mathcal{H}_2$. So each $z \in \mathrm{ran} E(\sigma'; T)$ has the form

$$z = \begin{pmatrix} -Ex \\ x \end{pmatrix}$$

for some $x \in \mathcal{H}_2$. Let $x = e_2$. Then

$$\|z\| = \left\| \begin{pmatrix} -Ee_2 \\ e_2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -e_1 \\ e_2 \end{pmatrix} \right\| = \sqrt{2}.$$

Since $AE - EB = F$, we have

$$\begin{aligned} \|(T'_\sigma - 2I_{\sigma'})z\| &= \|Tz - 2z\| \\ &= \left\| \begin{bmatrix} A-2 & F \\ 0 & B-2 \end{bmatrix} \begin{pmatrix} -Ex \\ x \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} (2-A)Ex + Fx \\ Sx \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} 2Ex + (F - AE)x \\ Sx \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} 2Ex - EBx \\ Sx \end{pmatrix} \right\| = \left\| \begin{pmatrix} -ESx \\ Sx \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} -ESe_2 \\ Se_2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ e_3 \end{pmatrix} \right\| = 1. \end{aligned}$$

So $T'_\sigma - 2I_{\sigma'}$ is not isometric, a contradiction.

*Remark* 3.5. Note that the unilateral shift $S$ has no eigenvalues and

$$\vee\{\ker(S^*)^n : n \geq 1\} = \mathcal{H}.$$

Since $S$ is irreducible, one can check that the operator $T$ in Example 3.4 is not irreducible.

Now we give an example to show that the similarity $\sim$ in Theorem 1.7(ii) can not be replaced by unitary equivalence.

**Example 3.6.** Let $\{e_i\}_{i=1}^4$ be the canonical ONB of $\mathbb{C}^4$. For $x \in \mathbb{C}^4$ with $x = \sum_{i=1}^4 \alpha_i e_i$, define

$$Tx = \frac{\alpha_1 + \alpha_3}{\sqrt{2}} e_1 + \Big( \frac{\alpha_1}{\sqrt{8}} - \frac{\alpha_2}{\sqrt{2}} - \frac{\alpha_3}{\sqrt{8}} + \frac{\alpha_4}{2} \Big) e_2.$$

Then $T \in \mathcal{B}(\mathbb{C}^4)$ and, with respect to $\{e_i\}_{i=1}^4$, $T$ can be represented as

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{8}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{8}} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$T^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{8}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{8}} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4. \end{matrix}$$

Set

$$f_1 = -\frac{e_1}{\sqrt{8}} + \frac{e_2}{\sqrt{2}} + \frac{e_3}{\sqrt{8}} - \frac{e_4}{2}, \quad f_2 = \frac{e_1}{\sqrt{2}} + \frac{e_3}{\sqrt{2}}.$$

Then

$$T^* = f_2 \otimes e_1 - f_1 \otimes e_2, \quad T = e_1 \otimes f_2 - e_2 \otimes f_1.$$

Note that $\{e_1, e_2\}$, $\{f_1, f_2\}$ are both orthonomal subsets of $\mathbb{C}^4$, and $\langle f_2, e_1\rangle = \frac{1}{\sqrt{2}} = \langle f_1, e_2\rangle$. Then, by [13, Cor. 2.6], $T$ is skew symmetric.

Denote $M = \vee\{e_1, e_2\}$ and

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{8}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{matrix} e_1 \\ e_2. \end{matrix}$$

Then $T$ can be written as

$$T = \begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} \begin{matrix} M \\ M^{\perp}, \end{matrix}$$

where $E \neq 0$. Denote $\sigma = \{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\}$. Then $\sigma(T) = \sigma \cup \{0\}$, $\sigma(A) = \sigma$ and, by Corollary 2.1, $T_\sigma = A$. Note that $\sigma' = \{0\}$. It follows from Lemma 2.2 that $T_{\sigma'} = 0$. So $T_{\sigma'}$ is skew symmetric. Easy to see that $T_\sigma$ is not normal. In view of Corollary 3.3, $T_\sigma$ is not skew symmetric.

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