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DERIVATIVE FORMULAE FOR MODULAR FORMS AND THEIR PROPERTIES

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Abstract. In this paper, by using the modular forms of weight nk ($2 \leq n \in \mathbb{N}$ and $k \in \mathbb{Z}$), we construct a formula which generates modular forms of weight $2nk+4$. This formula consist of some known results in [14] and [4]. Moreover, we obtain Fourier expansion of these modular forms. We also give some properties of an operator related to the derivative formula. Finally, by using the function j_4, we obtain the Fourier coefficients of modular forms with weight 4.

1. Introduction, definitions and notations

Throughout our paper, we use the following standard notations:

$\mathbb{N} = \{1, 2, \ldots\}$, as usual, $\mathbb{Z}$ denotes the set of integers and $\mathbb{C}$ denotes the set of complex numbers. Let $k \in \mathbb{Z}$ and

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

The set of all Möbius transformations of the form

$$z' = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, is called the *modular group* and is denoted by Γ (cf. [1]).

A function f said to be a *modular form of weight* $2k$ if it satisfies the following conditions (cf. [1], [14]):

(i) f is analytic in $\mathbb{H}$.

(ii)

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z),$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

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(c) The function f has a Fourier expansion as follows:

$$f(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z}.$$

The constant term $c(0)$ is called the value of f at $i\infty$, denoted by $f(i\infty)$. If $c(0) = 0$, then the function f is called a *cusp form* (cf. [1]).

If $2 \leq k \in \mathbb{N}$, then the Eisenstein series $G_{2k}(z)$ is defined by

$$G_{2k}(z) = \sum_{0 \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(mz+n)^{2k}} \tag{1.1}$$

(cf. [1], [15], [16]).

For $k = 1$, the series in (1.1) is no longer absolutely convergent.

We define the function $G_2(z)$, for $z \in \mathbb{H}$, as follows:

$$G_2(z) = 2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}, \tag{1.2}$$

where

$$\sigma(n) = \sum_{d|n} d$$

(cf. [1], [15], [16]).

Let $q = e^{2\pi i z}$. The series in (1.2) is an absolutely convergent power series for $|q| < 1$. So $G_2(z)$ is analytic in $\mathbb{H}$ (cf. [1], [15], [16]).

The normalized Eisenstein series is defined by

$$E_{2k}(z) = \frac{1}{2\zeta(2k)} G_{2k}(z),$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

for $\operatorname{Re} s > 1$ (cf. [1], [15], [16]).

In Section 2, we use the Eisestein series $E_2(z)$ in the derivative formula. Also we can give the following proposition for $E_2(z)$.

Proposition 1 ([7]). *There is a holomorphic function E_2 which has Fourier expansion at ∞*

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z},$$

which satisfies the following transformation formula:

$$z^{-2} E_2\left(-\frac{1}{z}\right) = E_2(z) + \frac{12}{2\pi i z}.$$

Many authors have studied on formulae generating the modular forms and obtained the modular forms by using the derivative of modular forms.

Koblitz raised the following question concerning the normalized Eisenstein series $E_2(z)$ and modular forms ([11]):

Theorem 1. *Let $f(z)$ be a modular form of weight k for Γ. If*

$$h(z) = \frac{1}{2\pi i}\frac{d}{dz}f(z) - \frac{k}{12}f(z),$$

then $h(z)$ is a modular form of weight $k+2$ for Γ.

Sebbar ([13]) investigated the modular differential equations associated with Eisenstein series.

One particular second order linear differential equation that has drawn the attention of Klein, Hurwitz and Van der Pol (see for detail [6], [10], [17]) takes the form

$$y'' + \frac{\pi^2}{36}E_4(z)y = 0. \tag{1.3}$$

Van der Pol ([17]) gave the relation between equation (1.3) and Riccati equation as follows:

$$\frac{6}{i\pi}u' + u^2 = E_4, \tag{1.4}$$

and one has $u = -E_2$ as a solution to (1.4) by using Ramanujan's identities ([12]).

A. Sebbar *et al.* investigated differential equations similar to (1.4) and they studied the Riccati equation, or the corresponding linear ordinary differential equation, for $k = 1, \ldots, 6$, of the form

$$\frac{k}{i\pi}u' + u^2 = E_4$$

and the corresponding second order ordinary differential equation

$$y'' + \frac{\pi^2}{k^2}E_4(z)y = 0.$$

(See for detail [13].)

Therefore, they arrived at the following proposition:

Proposition 2 ([13]). *The Riccati equation*

$$\frac{6}{i\pi}u' + u^2 = E_4(z)$$

has $u = -E_2(z)$ as a solution, and the corresponding linear differential equation

$$y'' + \frac{\pi^2}{36}E_4(z)y = 0$$

has a solution

$$y = (\Delta(z))^{-\frac{1}{12}} = (\eta(z))^{-2},$$

where $\eta(z)$ is a Dedekind eta function defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

for $q = e^{2\pi i z}$ and $z \in \mathbb{H}$.

Also, they proved the following theorems:

Theorem 2 ([13]). *The function*

$$y = E_4'(z)(\Delta(z))^{-\frac{1}{2}}$$

is a solution to

$$y'' + \pi^2 E_4(z) y = 0.$$

Theorem 3 ([13]). *The function*

$$u = \frac{1}{i\pi}\frac{E_4''}{E_4'} - E_2$$

is a solution to the Riccati equation

$$\frac{1}{i\pi}u' + u^2 = E_4.$$

Silverman gave a derivative formula related to modular forms as follows:

Theorem 4 ([14]). *Let k be any integer.*

(i) *Let f be a modular form of weight $2k$. Then,*

$$g = (2k+1)\left(\frac{df}{dz}\right)^2 - 2k \cdot f \cdot \frac{d^2 f}{dz^2}$$

is a modular form of weight $4k+4$.

(ii) *If f is a modular form, then g is a cusp form.*

(iii) *If f is the Eisenstein series $G_4(z)$, then*

$$g(z) = \frac{1}{2^4 3^3 5^2 \pi}\Delta(z).$$

Similarly, if $f(z) = G_6(z)$, then there exists a constant number c such that

$$g(z) = cG_4(z)\Delta(z).$$

Aygunes *et al.* constructed a sequence $(f_n(z))_{n\in\mathbb{N}}$ of modular functions of weight $2^n k + 4(2^{n-1}-1)$ in the following theorem:

Theorem 5 ([4]). *Let $n \in \mathbb{N}$. If $f_n(z)$ is a modular function of weight $2^n k + 4(2^{n-1}-1)$, then the function $f_{n+1}(z)$ is given by*

$$f_{n+1}(z) = (2^n k + 4(2^{n-1}-1)+1)\left(\frac{d}{dz}f_n(z)\right)^2 - (2^n k + 4(2^{n-1}-1))f_n(z)\left(\frac{d^2}{dz^2}f_n(z)\right),$$

where $f_{n+1}(z)$ is a modular form of weight $2^{n+1}k + 4(2^n - 1)$.

In Section 2, we take the modular forms of weight nk, for $n \in \mathbb{N}$ and then we obtain the modular forms of weight $2nk+4$ by using derivative formula.

In Section 3, we extend the derivative formula for any real number c as follows:

$$\mathcal{F}_c(f(z)) = (c+1)(f'(z))^2 - cf(z)f''(z)$$

In the above equation, we define an operator related to generalized derivative formula and we observe the properties of operator $\mathcal{F}_c$.

Many authors investigated the operators related to modular forms and cusp forms.

Let M_k and C_k be a space of modular forms and cusp forms, respectively. Hecke introduced a sequence of linear operators T_n, which map the linear operators M_k onto itself. Hecke's operators are defined as follows ([3], [5]):

Definition 1. *For a fixed integer k and any $n = 1, 2, \ldots$, the operator T_n is defined on M_k by the equation*

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

In the special case when $n = p$ is a prime, we have

$$(T_p f)(z) = p^{k-1} f(pz) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right).$$

Aygunes and Simsek applied the Hecke operators to the generalized Dedekind eta functions and they obtained the following theorem ([3]):

Theorem 6. *Let p be a prime number. If $N \mid g$, then we have*

$$T_p(\log \eta_{g,h}(z,N)) = \frac{\pi i(p-1)}{12} + \log \eta_{g,h}(pz,N) + p \log \eta_{g,h}\left(\frac{z}{p}, N\right).$$

Generalized Dedekind eta functions are defined by

$$\eta_{g,h}(z,N) = \alpha_{g,h}(N) \exp\left(\pi i z \overline{B}_2\left(\frac{g}{N}\right)\right) \prod_{\substack{m \equiv g(N) \\ m>0}} (1-\zeta_N^h q_N^m) \prod_{\substack{m \equiv -g(N) \\ m>0}} (1-\zeta_N^{-h} q_N^m)$$

for $z \in \mathbb{H}$ and $N, g, h \in \mathbb{Z}$, where

$$\zeta_N = \exp\left(\frac{2\pi i}{N}\right),$$

and

$$q_N = \exp\left(\frac{2\pi i z}{N}\right)$$

If $g \equiv 0, h \not\equiv 0$ (N), then

$$\alpha_{g,h}(N) = \exp\left(\pi i \overline{B}_1\left(\frac{h}{N}\right)\right)(1-\zeta_N^{-h});$$

otherwise

$$\alpha_{g,h}(N) = 1.$$

$\overline{B}_1$ and $\overline{B}_2$ in the formulae are Bernoulli functions.

In the following theorem, Aygunes ([2]) observed the properties of operator $\mathcal{H}_c$, for any real number c, where M_4 is a space of modular forms with weight k and H_{2c} is a space of functions which satisfies the functional equations of

$$h\left(\frac{a\tau + b}{c\tau + d}\right) = 2kc(c\tau + d) + (c\tau + d)^2 h(\tau).$$

Theorem 7 ([2])**.** *Let c be any real number. Define the operator $\mathcal{H}_c$ as follows*:

$$\mathcal{H}_c(h) = -2c \cdot \frac{dh}{d\tau} + h^2,$$

where

$$\mathcal{H}_c : H_{2c} \longrightarrow M_4.$$

Then, we have the following properties:
(i) $\forall\ h_1, h_2 \in H_{2c}$,

$$\mathcal{H}_c(h_1 + h_2) - \mathcal{H}_c(h_1) - \mathcal{H}_c(h_2) = 2h_1 \cdot h_2.$$

(ii) $\forall\ \lambda \in \mathbb{R}$,

$$\mathcal{H}_c(\lambda h) = \lambda^2 \mathcal{H}_{\frac{c}{\lambda}}(h).$$

(iii) $\forall\ c_1, c_2 \in \mathbb{R}$,

$$\mathcal{H}_{c_1+c_2}(h) = \mathcal{H}_{c_1\sqrt{2}}\left(\frac{h}{\sqrt{2}}\right) + \mathcal{H}_{c_2\sqrt{2}}\left(\frac{h}{\sqrt{2}}\right).$$

In Section 4, we consider the function $j_4(z)$ defined by

$$j_4(z) = \frac{\theta_3\left(\frac{z}{2}\right)}{\theta_4\left(\frac{z}{2}\right)}.$$

(See for detail [8] and [9].)

Since $j_4(z)$ is a modular function, we obtain a modular form with weight 4 by using the formula of

$$\mathcal{F}_0(j_4(z)) = \left(\frac{d}{dz} j_4(z)\right)^2.$$

Finally, we calculate the Fourier coefficients of the above function $\mathcal{F}_0(j_4(z))$.

2. A derivative formula related to modular forms

In this section, we give a generalized derivative formula which consist of the formula

$$g(z) = (2k+1)(f'(z))^2 - 2kf(z)f''(z). \tag{2.1}$$

The formula in (2.1) generates the modular forms of weight $4k+4$ for any modular form f of weight $2k$.

Theorem 8. *Let $2 \leq n \in \mathbb{N}$. If $f(z)$ is a modular form of weight nk, then the function $F(z)$, for*

$$\frac{d}{dz}f(z) = f'(z)$$

and

$$\frac{d^2}{dz^2}f(z) = f''(z),$$

is given by

$$F(z) = (nk+1)(f'(z))^2 - nkf(z)f''(z),$$

where $F(z)$ is a modular form of weight $2nk+4$.

Proof. Let

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k}f(z).$$

Thus we get

$$\frac{d}{dz}f\left(\frac{az+b}{cz+d}\right) = \frac{a(cz+d)-c(az+b)}{(cz+d)^2}f'\left(\frac{az+b}{cz+d}\right)$$

and

$$\frac{d}{dz}f\left(\frac{az+b}{cz+d}\right) = nkc(cz+d)^{nk-1}f(z) + (cz+d)^{nk}f'(z).$$

Since $ad-bc=1$, we have

$$f'\left(\frac{az+b}{cz+d}\right) = nkc(cz+d)^{nk+1}f(z) + (cz+d)^{nk+2}f'(z). \tag{2.2}$$

On the other hand,

$$\frac{d^2}{dz^2}f\left(\frac{az+b}{cz+d}\right) = -\frac{2c}{(cz+d)^3}f'\left(\frac{az+b}{cz+d}\right) + \frac{1}{(cz+d)^4}f''\left(\frac{az+b}{cz+d}\right).$$

We obtain

$$\begin{aligned}\frac{d^2}{dz^2}f\left(\frac{az+b}{cz+d}\right) &= nk(nk-1)c^2(cz+d)^{nk-2}f(z)\\ &\quad + 2nkc(cz+d)^{nk-1}f'(z) + (cz+d)^{nk}f''(z)\end{aligned}$$

or

$$-2c(cz+d)f'\left(\frac{az+b}{cz+d}\right) + f''\left(\frac{az+b}{cz+d}\right)$$

$$= nk(nk-1)c^2(cz+d)^{nk+2}f(z) + 2nkc(cz+d)^{nk+3}f'(z) \\ + (cz+d)^{nk+4}f''(z).$$

By using (2.2), we have

$$\begin{aligned} f''\left(\frac{az+b}{cz+d}\right) &= nk(nk-1)c^2(cz+d)^{nk+2}f(z) \\ &\quad + 2nkc(cz+d)^{nk+3}f'(z) + (cz+d)^{nk+4}f''(z) \\ &\quad + 2c(cz+d)\left\{nkc(cz+d)^{nk+1}f(z) + (cz+d)^{nk+2}f'(z)\right\} \end{aligned}$$

or

$$\begin{aligned} f''\left(\frac{az+b}{cz+d}\right) &= nk(nk+1)c^2(cz+d)^{nk+2}f(z) \\ &\quad + 2c(nk+1)(cz+d)^{nk+3}f'(z) + (cz+d)^{nk+4}f''(z). \end{aligned} \tag{2.3}$$

By using (2.2) and (2.3), we get

$$(nk+1)\left(f'\left(\frac{az+b}{cz+d}\right)\right)^2 = (nk+1)\cdot\left\{\begin{array}{c} n^2k^2c^2(cz+d)^{2nk+2}\left(f(z)\right)^2 \\ +2nkc(cz+d)^{2nk+3}f(z)f'(z) \\ +(cz+d)^{2nk+4}\left(f'(z)\right)^2 \end{array}\right\}$$

and

$$\begin{aligned} -nkf\left(\frac{az+b}{cz+d}\right)f''\left(\frac{az+b}{cz+d}\right) = &-n^2k^2(nk+1)c^2(cz+d)^{2nk+2}(f(z))^2 \\ &-2nkc(nk+1)(cz+d)^{2nk+3}f'(z)f(z) \\ &-nk(cz+d)^{2nk+4}f''(z)f(z). \end{aligned}$$

Consequently, we arrive

$$\begin{aligned} F\left(\frac{az+b}{cz+d}\right) &= (cz+d)^{2nk+4}\left\{(nk+1)(f'(z))^2 - nkf(z)f''(z)\right\} \\ &= (cz+d)^{2nk+4}F(z). \end{aligned}$$

□

Lemma 1. *Let $q = e^{2\pi iz}$. If f has the Fourier expansion*

$$f(z) = \sum_{m=0}^{\infty} c(m)q^m,$$

then F has the Fourier expansion given by

$$\begin{aligned} F(z) = &\left\{4\pi^2nk\cdot c(0)\cdot c(1)\right\}q \\ &+ 4\pi^2nk\sum_{m=2}^{\infty}\left\{\sum_{j=0}^{m-1}(j+1)^2\cdot c(j+1)\cdot c(m-j-1)\right\}q^m \\ &- 4\pi^2(nk+1)\sum_{m=2}^{\infty}\left\{\sum_{j=0}^{m-2}(j+1)(m-j-1)\cdot c(j+1)\cdot c(m-j-1)\right\}q^m. \end{aligned}$$

Proof. Let $f(z)$ be a modular form. Then $f(z)$ is a periodic function and given by

$$f(z) = \sum_{m=0}^{\infty} c(m)e^{2\pi imz} \tag{2.4}$$

for $z \in \mathbb{H}$.

We take the first and second derivative of the above series:

$$f'(z) = 2\pi i \sum_{m=1}^{\infty} mc(m)e^{2\pi imz} \tag{2.5}$$

and

$$f''(z) = -4\pi^2 \sum_{m=1}^{\infty} m^2c(m)e^{2\pi imz}. \tag{2.6}$$

By using (2.4), (2.5) and (2.6), we have

$$\begin{aligned} F(z) = (nk+1)&\left(2\pi i \sum_{m=1}^{\infty} mc(m)e^{2\pi imz}\right)^2 \\ &- nk\left(\sum_{m=0}^{\infty} c(m)e^{2\pi imz}\right)\left(-4\pi^2 \sum_{m=1}^{\infty} m^2c(m)e^{2\pi imz}\right) \end{aligned}$$

or

$$\begin{aligned} F(z) = &-4\pi^2(nk+1)q^2\left(\sum_{m=0}^{\infty}(m+1)c(m+1)q^m\right)^2 \\ &+ 4\pi^2nkq\left(\sum_{m=0}^{\infty} c(m)q^m\right)\left(\sum_{m=0}^{\infty}(m+1)^2c(m+1)q^m\right), \end{aligned}$$

where $q = e^{2\pi iz}$.

By using Cauchy product, we obtain the desired result. □

Theorem 9. *If $f(z)$ is a modular form, then*

$$F(z) = (nk+1)(f'(z))^2 - nkf(z)f''(z)$$

is a cusp form.

Proof. If we take $z = i\infty$ in Lemma 1, we obtain

$$F(i\infty) = 0.$$

From Theorem 8, we know that $F(z)$ is a modular form of weight $2nk + 4$. Hence, $F(z)$ is a cusp form. □

In Theorem 4(iii), the derivative formula was applied to the Eisenstein series $G_4(z)$ and $G_6(z)$. In the following theorem, we substitute $f = E_2(z)$ into equation (2.1).

Theorem 10. *If f is the Eisenstein series $E_2(z)$ in* (2.1)*, then*

$$g(z)=\frac{\pi^2}{36}\left\{(E_2(z))^4-6(E_2(z))^2E_4(z)-3(E_4(z))^2+8E_2(z)E_6(z)\right\}.$$

Proof. For the proof, we use the following well-known results ([12], [13]):

$$\frac{d}{dz}E_2(z)=\frac{\pi i}{6}\left\{(E_2(z))^2-E_4(z)\right\} \tag{2.7}$$

and

$$\frac{d}{dz}E_4(z)=\frac{2\pi i}{3}\left\{E_2(z)E_4(z)-E_6(z)\right\}. \tag{2.8}$$

If we take the derivative in (2.7), we have

$$\frac{d^2}{dz^2}E_2(z)=\frac{\pi i}{6}\left\{2E_2(z)E_2'(z)-E_4'(z)\right\}. \tag{2.9}$$

By substituting (2.7) and (2.8) into (2.9), we get

$$\frac{d^2}{dz^2}E_2(z)=\frac{\pi i}{6}\left\{\begin{array}{c}2E_2(z)\frac{\pi i}{6}\left((E_2(z))^2-E_4(z)\right)\\ -\frac{2\pi i}{3}\left(E_2(z)E_4(z)-E_6(z)\right)\end{array}\right\}$$

or

$$\frac{d^2}{dz^2}E_2(z)=-\frac{\pi^2}{18}\left\{(E_2(z))^3-3E_2(z)E_4(z)+2E_6(z)\right\}. \tag{2.10}$$

By substituting (2.7) and (2.10) into (2.1), we arrive at the desired result. □

Theorem 11. *If f is the Eisenstein series $E_4(z)$ in* (2.1)*, then*

$$g(z)=\frac{20\pi^2}{9}\frac{1}{1728}\Delta(z).$$

Proof. One can see the proof in [4]. □

Theorem 12. *If f is the Eisenstein series $E_6(z)$ in* (2.1)*, then*

$$g(z)=-7\pi^2\frac{1}{1728}E_4(z)\Delta(z).$$

Proof. One can see the proof in [4]. □

3. An operator related to generalized derivative formula

In this section, we define an operator related to generalized derivative formula given by

$$g_c(z)=(c+1)(f'(z))^2-cf(z)f''(z), \tag{3.1}$$

where c is any real number.

Let A and B be any set of functions. For any constant c, the operator depending on the function $f(z)$ is defined by

$$\mathcal{F}_c(f(z))=(c+1)(f'(z))^2-cf(z)f''(z), \tag{3.2}$$

where

$$\mathcal{F}_c : A \longrightarrow B.$$

Theorem 13. *We have the following properties*:

(i) $\forall\ f_1, f_2 \in A$,

$$\mathcal{F}_c(f_1+f_2) - \mathcal{F}_c(f_1) - \mathcal{F}_c(f_2) = 2(c+1)f_1 f_2 - cf_2 f_1'' - cf_1 f_2''.$$

(ii) $\forall\ \lambda \in \mathbb{R}$,

$$\mathcal{F}_c(\lambda f) = \lambda^2 \mathcal{F}_c(f).$$

(iii) $\forall\ c_1, c_2 \in \mathbb{R}$,

$$\mathcal{F}_{c_1+c_2}(f) = \frac{1}{2}\left\{\mathcal{F}_{2c_1}(f) + \mathcal{F}_{2c_2}(f)\right\}.$$

Proof. For the proof of (i), we have

$$\mathcal{F}_c(f_1+f_2) = (c+1)\left(f_1' + f_2'\right)^2 - c(f_1+f_2)(f_1''+f_2'')$$

or

$$\begin{aligned}\mathcal{F}_c(f_1+f_2) = {} & (c+1)\left\{(f_1')^2 + (f_2')^2\right\} + 2(c+1)f_1 f_2 \\ & - cf_1 f_1'' - cf_2 f_2'' - cf_2 f_1'' - cf_1 f_2''.\end{aligned}$$

Hence, we arrive at the desired result.

For the proof of (ii), we have

$$\mathcal{F}_c(\lambda f) = (c+1)((\lambda f)')^2 - c\,(\lambda f)\,(\lambda f)''$$

or

$$\mathcal{F}_c(\lambda f) = (c+1)\lambda^2 f' + c\lambda^2 f f''.$$

Therefore, we arrive

$$\mathcal{F}_c(\lambda f) = \lambda^2\left\{(c+1)f' + cff''\right\} = \lambda^2 \mathcal{F}_c(f).$$

For the proof of (iii), we have

$$2\mathcal{F}_{c_1+c_2}(f) = 2(c_1+c_2+1)(f')^2 - 2(c_1+c_2)ff''$$

or

$$2\mathcal{F}_{c_1+c_2}(f) = (2c_1+1)\,(f')^2 - 2c_1 ff'' + (2c_2+1)\,(f')^2 - 2c_2 ff''.$$

Therefore, we obtain the desired result. □

Remark 1. *Let A be a space of modular forms of weight $2k$, for $k \in \mathbb{Z}$. If the operator $\mathcal{F}_{2k}$ is given by*

$$\mathcal{F}_{2k}(f(z)) = (2k+1)(f'(z))^2 - 2kf(z)f''(z),$$

then B is a space of modular forms of weight $4k+4$.

4. Fourier coefficients of function j_4

Kim and Koo defined the function $j_4(z)$ by means of a quotient of Jacobi theta series ([8]) and they derived recursion formulas for the Fourier coefficients of j_4 and $N(j_4)$ (= the normalized operator), respectively ([9]).

Let $\mu, v \in \mathbb{R}$ and $z \in \mathbb{H}$. Define that

$$\Theta_{\mu,v}(z) = \sum_{n\in\mathbb{Z}} \exp\left\{\pi i \left(n + \frac{1}{2}\mu\right)^2 z + \pi i n v\right\}.$$

This series uniformly converges for $\mathrm{Im}(z) \geq \eta > 0$, and hence defines a holomorphic function on $\mathbb{H}$.

Theorem 14 ([9])**.** *If $z \in \mathbb{H}$, then*

$$\Theta_{\mu,v}(z) = \frac{\exp\left(-\frac{1}{2}\pi i \mu v\right)}{(-iz)^{\frac{1}{2}}} \Theta_{v,-\mu}\left(-\frac{1}{z}\right).$$

Jacobi theta functions $\theta_2, \theta_3, \theta_4$ are defined by

$$\theta_2(z) = \Theta_{1,0}(z) = \sum_{n\in\mathbb{Z}} q_2^{\left(n+\frac{1}{2}\right)^2},$$

$$\theta_3(z) = \Theta_{0,0}(z) = \sum_{n\in\mathbb{Z}} q_2^{n^2},$$

$$\theta_4(z) = \Theta_{0,1}(z) = \sum_{n\in\mathbb{Z}} (-1)^n q_2^{n^2},$$

where $q_2 = \exp(\pi i z)$.

For $z \in \mathbb{H}$, we have the following transformation formulas ([9]):

(i)

$$\theta_2(z+1) = \exp\left(\frac{\pi i}{4}\right)\theta_2(z),$$

$$\theta_3(z+1) = \theta_4(z),$$

$$\theta_4(z+1) = \theta_3(z).$$

(ii)

$$\theta_2\left(-\frac{1}{z}\right) = (-iz)^{\frac{1}{2}}\theta_4(z),$$

$$\theta_3\left(-\frac{1}{z}\right) = (-iz)^{\frac{1}{2}}\theta_3(z),$$

$$\theta_4\left(-\frac{1}{z}\right) = (-iz)^{\frac{1}{2}}\theta_2(z).$$

By using the above formulas, we have the following transformation formulas ([8]):

(iii)

$$\theta_2(z+4) = -\theta_2(z),$$

$$\theta_3\left(\frac{z+2}{2}\right)=\theta_4\left(\frac{z}{2}\right),$$
$$\theta_3\left(\frac{z+4}{2}\right)=\theta_3\left(\frac{z}{2}\right),$$
$$\theta_4\left(\frac{z+2}{2}\right)=\theta_3\left(\frac{z}{2}\right),$$
$$\theta_4\left(\frac{z+4}{2}\right)=\theta_4\left(\frac{z}{2}\right).$$

(iv)

$$\theta_2\left(-\frac{2}{z}\right)=\left(-\frac{iz}{2}\right)^{\frac{1}{2}}\theta_4\left(\frac{z}{2}\right),$$
$$\theta_3\left(-\frac{1}{2z}\right)=(-2iz)^{\frac{1}{2}}\theta_3(2z),$$
$$\theta_3\left(-\frac{2}{z}\right)=\left(-\frac{iz}{2}\right)^{\frac{1}{2}}\theta_3\left(\frac{z}{2}\right),$$
$$\theta_4\left(-\frac{1}{2z}\right)=(-2iz)^{\frac{1}{2}}\theta_2\left(2z\right).$$

Define that

$$j_4(z)=\frac{\theta_3\left(\frac{z}{2}\right)}{\theta_4\left(\frac{z}{2}\right)}=1+4q_4+8q_4^2+16q_4^3+32q_4^4+56q_4^5+\cdots,$$

where $q_4=\exp\left(\frac{\pi iz}{2}\right)$.

Then Kim and Koo concluded the following proposition:

Proposition 3 ([9]). *Let*

$$j_4(z)=\sum_{m\geq 0}b_m q_4^m. \tag{4.1}$$

Then for $k\geq 1$,

$$b_{4k-1}=\frac{1}{b_1}\left(2\sum_{0\leq j<k}b_jb_{2k-j}+b_k^2+\sum_{2\leq j<2k-1}(-1)^jb_jb_{4k-j}+\frac{b_{2k}^2}{2}\right),$$
$$b_{4k}=2\sum_{0\leq j<k}b_jb_{2k-j}+b_k^2,$$
$$b_{4k+1}=\frac{1}{b_1}\left(2\sum_{0\leq j\leq k}b_jb_{2k-j+1}+\sum_{2\leq j\leq 2k}(-1)^jb_jb_{4k-j+2}+\frac{b_{2k+1}^2}{2}\right),$$
$$b_{4k+2}=2\sum_{0\leq j\leq k}b_jb_{2k-j+1},$$

with the initial values $b_0=1, b_1=4$ *and* $b_2=8$.

We take the first and second derivative of $j_4(z)$ in (4.1).

$$\frac{d}{dz}j_4(z) = j_4'(z) = \sum_{m\geq 1} mb_m q_4^{m-1} \tag{4.2}$$

and

$$\frac{d^2}{dz^2}j_4(z) = j_4''(z) = \sum_{m\geq 2} m(m-1)b_m q_4^{m-2}. \tag{4.3}$$

For $c = 0$, substituting $f(z) = j_4(z)$ into equation (3.2), we have

$$\mathcal{F}_0(j_4(z)) = \left(\sum_{m\geq 1} mb_m q_4^{m-1}\right)^2$$

or

$$\mathcal{F}_0(j_4(z)) = \left(\sum_{m\geq 0} (m+1)b_{m+1} q_4^{m}\right)^2.$$

By using Cauchy product, we have

$$\mathcal{F}_0(j_4(z)) = \sum_{m\geq 0}\left(\sum_{k=0}^{m}(k+1)(m-k+1)b_{k+1}b_{m-k+1}\right)q_4^m.$$

Hence, we arrive at the following theorem:

Theorem 15. *The Fourier coefficients c_m of modular form $\mathcal{F}_0(j_4(z))$ with weight 4 are given by*

$$c_m = \sum_{k=0}^{m}(k+1)(m-k+1)b_{k+1}b_{m-k+1}.$$

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