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# Statistical Properties of Kumaraswamy Exponentiated Gamma Distribution

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**Abstract.** The Exponentiated Gamma (EG) distribution is one of the important families of distributions in lifetime tests. In this paper, a new generalized version of this distribution which is called kumaraswamy Exponentiated Gamma (KEG) distribution is introduced. A new distribution is more flexible and has some interesting properties. A comprehensive mathematical treatment of the KEG distribution is provided. We derive the $r^{th}$ moment and moment generating function of this distribution. Moreover, we discuss the maximum likelihood estimation of the distribution parameters. Finally, an application to real data sets is illustrated.

**Key Words:** Exponentiated gamma distribution, hazard function, Kumaraswamy distribution, maximum likelihood estimation, Moments

## 1. INTRODUCTION AND MOTIVATION

The gamma distribution is the most popular model for analyzing skewed data and hydrological processes. one of the important families of distributions in lifetime tests is the exponentiated gamma (EG) distribution. The exponentiated gamma (EG) distribution has been introduced by Gupta et al. (1998) which has cumulative distribution function (c.d.f.) and a probability density function (p.d.f.) of the form, respectively;

$$G(x;\lambda,\theta)=\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta}, \lambda>0, \theta>0,\ x\geq 0. \tag{1}$$

$$g(x;\lambda,\theta)=\theta\lambda^{2}xe^{-\lambda x}\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta-1}. \tag{2}$$

where $\lambda$ and $\theta$ are scale and shape parameters respectively.

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Shawky and Bakoban (2008) discussed the exponentiated gamma distribution as an important model of life time models and derived Bayesian and non-Bayesian estimators of the shape parameter, reliability and failure rate functions in the case of complete and type-II censored samples. Also order statistics from exponentiated gamma distribution and associated inference was discussed by Shawky and Bakoban (2009). Ghanizadeh, et al. (2011), deal with the estimation of parameters of the Exponentiated Gamma (EG) distribution with presence of $k$ outliers. The maximum likelihood and moment of the estimators were derived. These estimators are compared empirically using Monte Carlo simulation. Singh et al. (2011) proposed bayes estimators of the parameter of the exponentiated gamma distribution and associated reliability function under general entropy loss function for a censored sample. The proposed estimators were compared with the corresponding Bayes estimators obtained under squared error loss function and maximum likelihood estimators through their simulated risks. Khan and Kumar (2011) established the explicit expressions and some recurrence relations for single and product moments of lower generalized order statistics from exponentiated gamma distribution. Sanjay et el. (2011) proposed bayes estimators of the parameter of the exponentiated gamma distribution and associated reliability function under general entropy loss function for a censored sample. Navid and Muhammad (2012) introduced bayesian analysis of exponentiated gamma distribution under type II censored samples. Recentely, Parviz et al. (2013) discussed Classical and Bayesian estimation of parameters on the generalized exponentiated gamma distribution.

The Kumaraswamy distribution (Kumaraswamy, 1980) is not very common among statisticians and has been little explored in the literature. We refer to the Kum distribution to denote the Kumaraswamy distribution. Its cumulative distribution function (cdf) is defined by

$$F(x) = 1 - \left(1 - x^{a}\right)^{b}, 0 < x < 1, \tag{3}$$

where $a > 0$, and $b > 0$ are two additional parameters whose role is to introduce asymmetry and produce distributions with heavier tails. The Kum distribution does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution has been widely appreciated. However, in a very recent paper, Jones (2009) explored the background and genesis of the Kum distribution and, more importantly, made clear some similarities and differences between the beta and Kum distributions. He highlighted several advantages of the Kum distribution over the beta distribution: the normalizing constant is very simple; simple explicit formulae for the distribution and quantile functions which do not involve any special functions; a simple formula for random variate generation; explicit formulae for L-moments and simpler formulae for moments of order statistics. Further, according to Jones (2009), the beta distribution has the following advantages over the Kum distribution: simpler formulae for moments and moment generating function (mgf); a one-parameter sub-family of symmetric distributions; simpler moment estimation and more ways of generating the distribution via physical processes.

The probability density function (pdf) of the Kum distribution also has a simple form

$$f(x) = abx^{a-1}\left(1 - x^{a}\right)^{b-1}, \tag{4}$$

and it can be unimodal, increasing, decreasing or constant, depending in the same way on the values of its parameters like the beta distribution.
If $G(x)$ is the baseline cdf of a random variable, the cdf of the Kum-generalized distribution, say $Kum-G$ distribution, is defined by (Cordeiro and Castro, 2010)

$$F(x)=1-\left[1-G(x)^{a}\right]^{b} \qquad (5)$$

The density function corresponding to (5) is

$$f(x)=abg(x)G(x)^{a-1}\left[1-G(x)^{a}\right]^{b-1} \qquad (6)$$

where $g(x)=\frac{d}{dx}G(x)$. The density family (6) has many of the same properties of the class of beta -$G$ distributions (see Eugene et al. (2002)), but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. So, the new $Kum-G$ distribution is obtained by adding two parameters $a$ and $b$ to the quantile function of the $G$ distribution. This generalization contains distributions with unimodal and bathtub shaped hazard rate functions. It also contemplates a broad class of models with monotone risk functions. Some mathematical properties of the $Kum-G$ distribution derived by Cordeiro and Castro (2010) are usually much simpler than those properties of the beta $G$ distribution (Eugene et al., 2002).
In this note, we combine the works of Kumaraswamy (1980) and Cordeiro and de Castro (2011) to derive some mathematical properties of a new model, called the Kumaraswamy Exponentiated Gamma $(K_w-EG)$ distribution. Equivalently, as occurs with the beta-$G$ family of distributions. Special $Kw-G$ distributions can be generated as follows: the $K_w-$normal distribution is obtained by taking $G(x)$ in (1.5) to be the normal cumulative function. Analogously, the $K_w-$Weibull (Cordeiro et al.(2010)), General results for the Kumaraswamy- $G$ distribution (Nadarajah et al.(2011)). $K_w$-generalized gamma (Pascoa et al.(2011)), $K_w$-Birnbaum-Saunders (Saulo et al. (2012)) Beta-Linear Failure Rate Distribution and its Applications (see Jafari et al.(2012)) and $K_w-$Gumbel (Cordeiro et al. (2011)) distributions are obtained by taking $G(x)$ to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new $Kw-G$ distribution can be generated from a specified $G$ distribution.
A physical interpretation of the $K_w-G$ distribution given by (5) and (6) (for $a$ and $b$ positive integers) is as follows. Suppose a system is made of $b$ independent components and that each component is made up of $a$ independent subcomponents. Suppose the system fails if any of the $b$ components fails and that each component fails if all of the $a$ subcomponents fail. Let $X_{j1},X_{j2},...,X_{ja}$ denote the life times of the subcomponents with in the $j_{th}$ component, $j=1,...,b$ with common (cdf) $G$. Let $X_j$ denote the lifetime of the $j_{th}$ component, $j=1,...,b$ and let $X$ denote the lifetime of the entire system. Then the (cdf) of $X$ is given by

$$\begin{aligned} P(X \le x) &= 1 - P(X_1 > x, X_2 > x, \ldots, X_b > x) \\ &= 1 - \left[P(X_1 > x)\right]^b = 1 - \left\{1 - P(X_1 \le x\right\}^b \\ &= 1 - \left\{1 - P(X_{11} \le x, X_{12} \le x, \ldots, X_{1a} \le x\right\}^b \\ &= 1 - \left\{1 - P\left[X_{11} \le x\right]^a\right\}^b = 1 - \left\{1 - G^a(x)\right\}^b . \end{aligned} \tag{7}$$

So, it follows that the $K_w - G$ distribution given by (3) and (4) is precisely the time to failure distribution of the entire system.

The rest of the article is organized as follows. In Section 2, we define the cumulative, density and hazard functions of the $K_w - EG$ distribution and some special cases. Quantile function, median, moments, moment generating function discussed in Section 3. Section 4 included the order statistics. The least squares and weighted least squares estimators are introduced in Section 5. Maximum likelihood estimation is performed and the observed information matrix is determined in Section 6. Section 7 gives applications involving a real data set.

## 2. KUMARASWAMY EXPONENTIATED GAMMA DISTRIBUTION

Let $G(x, \lambda, \theta)$ is the exponentiated gamma cumulative distribution with parameters $\lambda$ and $\theta$, then the Equation (5) yields the Kumaraswamy exponentiated gamma $(KEG)$ cumulative distribution

$$F_{X\mid\{\lambda,\theta,a,b\}}(x) = 1 - \left[1 - \left(1 - e^{-\lambda x}(1 + \lambda x)\right)^{a\theta}\right]^b, \tag{8}$$

where $\lambda > 0$ is a scale parameter and the other positive parameters and $\theta$, $a$ and $b$ are shape parameters.

The corresponding pdf, hazard rate (HR) and reversed hazard rate (RHR) function are respectively,

$$f_{X\mid\{\lambda,\theta,a,b\}}(x) = a\,b\,\theta\lambda^2 x\, e^{-\lambda x}\left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{\theta a - 1}\left[1 - \left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{a\theta}\right]^{b-1}, \tag{9}$$

$$\begin{aligned} h_{X\mid\{\lambda,\theta,a,b\}}(x) &= \frac{f(x, \lambda, \theta, a, b)}{1 - F(x, \lambda, \theta, a, b)} \\ &= \frac{a\ b\ \theta\lambda^2 x\ e^{-\lambda x}\left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{\theta a - 1}}{\left[1 - \left[1 - e^{-\lambda x}(1 + \lambda x)\right]^{a\ \theta}\right]}, \end{aligned} \tag{10}$$

and

$$\tau_{X|\{\lambda,\theta,a,b\}}(x)=\frac{f(x,\lambda,\theta,a,b)}{F(x,\lambda,\theta,a,b)}$$

$$=\frac{a\ b\ \theta\lambda^{2}x\ e^{-\lambda x}\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta a-1}\left[1-\left[1-e^{-\lambda x}(1+\lambda x)\right]^{a\ \theta}\right]^{b-1}}{1-\left[1-\left(1-e^{-\lambda x}(1+\lambda x)\right)^{a\theta}\right]^{b}}. \qquad (12)$$

The probability density function in Equation (9) does not involve any complicated function. If $X$ is a random variable with pdf (9), we write $X\sim KEG(a,b,\lambda,\theta)$. If $a=b=1$, we get exponentiated gamma distribution, also when the shape parameter $a=b=\theta=\lambda=1$, we get the gamma distribution with shape parameter $\alpha=2$ and scale parameter $\beta=1$ , i.e., $G(2,1)$. For more details about this distribution, see Shawky and Bakoban (2008, 2009).

In Figures 1 and 2, we plot the KEG density and hazard rate function for selected parameter values respectively. Also in Figure 3, we plot the KEG CDF for selected parameter values.

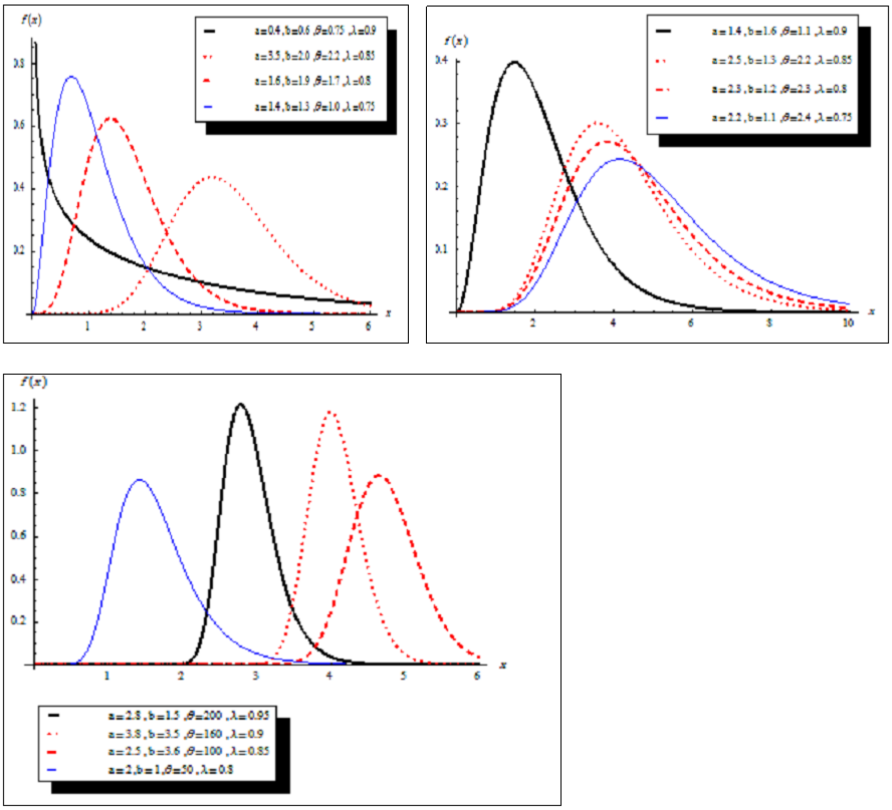


**Figure 1.** Plots of the KEG density for selected parameter values

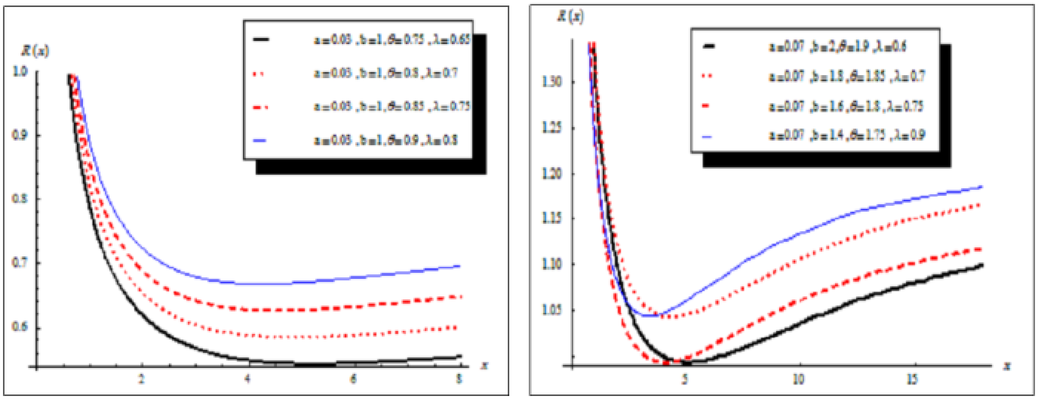


**Figure 2.** Plots of the KEG hazard rate for selected parameter values

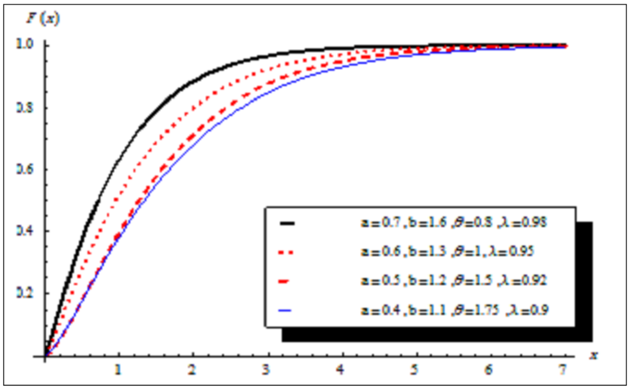


**Figure 3.** Plots of the KEG cdf for selected parameter values

### 2.1 Expansion for the density function

In this Subsection, we present two formulae for the cdf of the $KEG$ distribution depending if the parameter $b>0$ is real non- integer or integer. First, if $|z|<1$ and $b>0$ is real non- integer, we have

$$(1-z)^{b-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{b-1}{j}z^{j}=\frac{(-1)^{j}\Gamma(b)}{j!\Gamma(b-j)}z^{j}. \tag{13}$$

Using the expansion (12) in (8), the cdf of the $KEG$ distribution when $b>0$ is real non-integer follows

$$F_{X\mid\{\lambda,\theta,a,b\}}(x)=1-\sum_{j=0}^{\infty}(-1)^{j}\binom{b}{j}\left(1-e^{-\lambda x}(1+\lambda x)\right)^{a\theta j}.$$

when $b>0$ is integer, using the expansion (12) in (8), we get

$$F_{X|\{\lambda,\theta,a,b\}}(x) = 1-\sum_{j=0}^{b}(-1)^{j}\binom{b}{j}\left(1-e^{-\lambda x}(1+\lambda x)\right)^{a\theta j}, \tag{14}$$

also using the power series of (12) the pdf (9) becomes

$$f_{X|\{\lambda,\theta,a,b\}}(x) = \sum_{j=0}^{\infty}(-1)^{j}\binom{b-1}{j} a\ b\ \theta\lambda^{2} x\ e^{-\lambda x}\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta a(j+1)-1}, \tag{15}$$

again, by using (12) in the last factor of each summand in (14) we obtain

$$\begin{aligned} f_{X|\{\lambda,\theta,a,b\}}(x) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{\theta a(j+1)-1}{k} a\ b\ \theta\lambda^{2} x\ e^{-\lambda(k+1)x}(1+\lambda x)^{k} \\ &= w_{j,k}\, x\ e^{-\lambda(k+1)x}(1+\lambda x)^{k}, \end{aligned} \tag{16}$$

where

$$w_{j,k} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{\theta a(j+1)-1}{k} a\, b\, \theta\lambda^{2}.$$

## 3. STATISTOCAL PROPERTIES

This Section is devoted to studying statistical properties of the $(KEG)$ distribution, specifically quantile function, moments and moment generating function

### 3.1 Quantile function and simulation

The quantile function corresponding to (8) is $F(x_q) = P(X \le x_q)$ where $(x_q)_{(KEG)} = F^{-1}(u),$ is given by the following relation

$$e^{-\lambda x_{(q)}}(1+\lambda x_{(q)}) = 1-\left[1-(1-q)^{\frac{1}{b}}\right]^{\frac{1}{\theta a}} \tag{17}$$

Simulating the KEG random variable is straightforward. Let U be a uniform variate on the unit interval (0, 1). Thus, by means of the inverse transformation method, we consider the random variable $X$ given by the relation

$$e^{-\lambda x_{(i)}}(1+\lambda x_{(i)}) = 1-\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{\theta a}}. \tag{18}$$

### 3.2 Moments

In this Subsection we discuss the $r^{th}$ moment for $(KEG)$ distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

**Theorem 1**. If $X$ has $KEG\ (\Phi, x)$ $, \Phi = (\lambda, \theta, a, b)$ then the $r^{th}$ moment of $X$ is given by the following

$$\mu_r(x) = w_{j,k} \sum_{m=-0}^{k} \binom{k}{m} \lambda^m \left(\frac{\lambda}{k+1}\right)^{r+m+2} \Gamma(r+m+2), \tag{19}$$

where

$$w_{j,k} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{b-1}{j} \binom{\theta a(j+1)-1}{k} a\, b\, \theta \lambda^2$$

**Proof**.
Let $X$ be a random variable with density function (16). The $r^{th}$ ordinary moment of the $(KEG)$ distribution is given by

$$\begin{aligned} \mu_r(x) &= E(X^{r)} = \textstyle\int_0^\infty x^r f(x, \Phi) dx \\ &= w_{j,k} \textstyle\int_0^\infty x^{r+1} e^{-\lambda(k+1)x} (1+\lambda x)^k\, dx \end{aligned} \tag{20}$$

using the binomial series expansion we have

$$(1+\lambda x)^k = \sum_{m=0}^{k} \binom{k}{m} (\lambda x)^m$$

thus equation (20) becomes

$$\mu_r(x) = E(X^{r)} = w_{j,k} \sum_{m=-0}^{k} \binom{k}{m} \lambda^m \textstyle\int_0^\infty x^{r+m+1} e^{-\lambda(k+1)x} dx$$

let $\lambda(k+1)x = t$ then

$$\begin{aligned} \mu_r(x) &= w_{j,k} \sum_{m=-0}^{k} \binom{k}{m} \lambda^m \left(\frac{\lambda}{k+1}\right)^{r+m+2} \textstyle\int_0^\infty t^{r+m+1} e^{-t} dt \\ &= w_{j,k} \sum_{m=-0}^{k} \binom{k}{m} \lambda^m \left(\frac{\lambda}{k+1}\right)^{r+m+2} \Gamma(r+m+2) \end{aligned} \tag{21}$$

which completes the proof.

Based on the first four moments of the $(KEG)$ distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the $(KEG)$ distribution can obtained as

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^{\frac{3}{2}}}, \tag{22}$$

and

$$k(\Phi)=\frac{\mu_4(\theta)-4\mu_1(\theta)\mu_3(\theta)+6\mu_1^2(\theta)\mu_2(\theta)-3\mu_1^4(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^2}. \tag{23}$$

### 3.3 Moment generating function

In this Subsection we derived the moment generating function of $(KEG)$ distribution.

**Theorem 2.** If $X$ has $(KEG)$ distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t)=w_{j,k}\sum_{m=0}^{k}\binom{k}{m}\lambda^2\left(\frac{1}{(k+1)-t)}\right)^{m+2}\Gamma(m+2). \tag{24}$$

**Proof**.
We start with the well known definition of the moment generating function given by

$$\begin{aligned}M_X(t)&=E(e^{tX})=\int_0^\infty e^{tX}f_{KEG}(x,\Phi)dx\\ &=w_{j,k}\int_0^\infty x\ e^{-x(\lambda(k+1)-t)}(1+\lambda x)^k\,dx \qquad (25)\\ &=w_{j,k}\sum_{m=0}^{k}\binom{k}{m}\lambda^m\int_0^\infty x^{m+1}e^{-x(\lambda(k+1)-t)}dx\end{aligned}$$

Let $x(\lambda(k+1)-t)=z$ then

$$M_X(t)=w_{j,k}\sum_{m=0}^{k}\binom{k}{m}\lambda^m\left(\frac{1}{(\lambda(k+1)-t)}\right)^{m+2}\int_0^\infty z^{m+1}e^{-z}dz$$

$$=w_{j,k}\sum_{m=0}^{k}\binom{k}{m}\lambda^2\left(\frac{1}{(k+1)-t)}\right)^{m+2}\Gamma(m+2).$$

which completes the proof.

## 4. DISTRIBUTION OF THE ORDER STATISTICS

In this Section, we derive closed form expressions for the pdfs of The $r^{th}$ order statistic of the $(KEG)$ distribution, also, the measures of skewness and kurtosis of the distribution of the $r^{th}$ order statistic in a sample of size $n$ for different choices of $n;r$ are presented in this Section. Let $X_1,X_2,...,X_n$ be a simple random sample from (KEG) distribution with pdf and cdf given by (10) and (14), respectively.

$$f_{r\,:\,n}(x,;\Phi)=\frac{1}{B(r,n-r+1)}\left[F(x,\Phi)\right]^{r-1}\left[1-F(x,\Phi)\right]^{n-r}f(x,\Phi) \tag{26}$$

Let $X_1, X_2, ..., X_n$ denote the order statistics obtained from this sample. We now give the probability density function of $X_{r\,:\,n}$, say $f_{r\,:\,n}(x,\Phi)$ and the moments of $X_{r\,:\,n}$ $, r = 1, 2, ..., n$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r\,:\,n}$ are presented. The probability density function of $X_{r\,:\,n}$ is given by

$$f_{r\,:\,n}(x,;\Phi) = \frac{1}{B(r, n-r+1)}\left[F(x,\Phi)\right]^{r-1}\left[1-F(x,\Phi)\right]^{n-r} f(x,\Phi) \tag{27}$$

where $F(x,\Phi)$ and $f(x,\Phi)$ are the cdf and pdf of the $(KEG)$ distribution given by (8), (9), respectively, and since $0 < F(x,\Phi) < 1$, for $x > 0$, by using the binomial series expansion of $\left[1 - F(x,\Phi)\right]^{n-r}$, given by

$$\left[1-F(x,\Phi)\right]^{n-r} = \sum_{j=0}^{n-r}(-1)^j \binom{n-r}{j}\left[F(x,\Phi)\right]^j,$$

we have

$$f_{r\,:\,n}(x,\Phi) = \sum_{j=0}^{n-r}(-1)^j \binom{n-r}{j}\left[F(x,\Phi)\right]^{r+j-1} f(x,\Phi), \tag{28}$$

substituting from (8) and (9) into (28), we can express the $k^{th}$ ordinary moment of the $r^{th}$ order statistics $X_{r\,:\,n}$ say $E(X^k_{r\,:\,n})$ as a liner combination of the $k^{th}$ moments of the $(KEG)$ distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r\,:\,n}$ can be calculated.

## 5. LEAST SQUARES AND WEIGHTED LEAST SQUARES ESTIMATORS

In this Section we provide the regression based method estimators of the unknown parameters of the Kumaraswamy exponentiated Lomax, which was originally suggested by Swain, Venkatraman and Wilson (1988) to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $Y_1, ..., Y_n$ is a random sample of size $n$ from a distribution function $G(.)$ and suppose $Y_{(i)}$ ; $i = 1, 2, ..., n$ denotes the ordered sample. The proposed method uses the distribution of $G(Y_{(i)})$. For a sample of size $n$, we have

$$E\left(G(Y_{(j)})\right) = \frac{j}{n+1}, V\left(G(Y_{(j)})\right) = \frac{j(n-j+1)}{(n+1)^2(n+2)},$$

$$Cov\left(G(Y_{(j)}), G(Y_{(k)})\right) = \frac{j(n-k+1)}{(n+1)^2(n+2)}, \text{ for } j < k,$$

see Johnson, Kotz and Balakrishnan (1995). Using the expectations and the variances, two variants of the least squares methods can be used.

**Method 1 (Least Squares Estimators)**. Obtain the estimators by minimizing

$$\sum_{j=1}^{n}\left(G(Y_{(j)}-\frac{j}{n+1}\right)^{2}, \tag{29}$$

with respect to the unknown parameters. Therefore in case of $KEG$ distribution the least squares estimators of $\lambda,\theta,a$ and $b$, say $\hat{\lambda}_{LSE}$, $\hat{\theta}_{LSE}$, $\hat{a}_{LSE}$ and $\hat{b}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\left[1-\left[1-\left(1-e^{-\lambda x}(1+\lambda x)\right)^{a\theta}\right]^{b}-\frac{j}{n+1}\right]^{2}$$

with respect to $\lambda,\theta,\alpha,\ a$ and $b$.

**Method 2 (Weighted Least Squares Estimators).** The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n}w_{j}\left(G(Y_{(j)}-\frac{j}{n+1}\right)^{2}, \tag{30}$$

with respect to the unknown parameters, where

$$w_{j}=\frac{1}{V\left(G(Y_{(j)})\right)}=\frac{(n+1)^{2}(n+2)}{j(n-j+1)}.$$

Therefore, in case of $KEG$ distribution the weighted least squares estimators of $\lambda,\theta,a$ and $b$, say $\hat{\lambda}_{WLSE}$, $\hat{\theta}_{WLSE}$, $\hat{a}_{WLSE}$ and $\hat{b}_{WLSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}w_{j}\left[1-\left[1-\left(1-e^{-\lambda x}(1+\lambda x)\right)^{a\theta}\right]^{b}-\frac{j}{n+1}\right]^{2}$$

with respect to the unknown parameters only.

## 6. ESTIMATION AND INFERENCE

In this Section, we determine the maximum likelihood estimates (MLEs) of the parameters of the $(KEG)$ distribution from complete samples only. Let $X_1,X_2,\ldots,X_n$ be a random sample of size $n$ from $KEG\ (\lambda,\theta,a,b)$. The likelihood function for the vector of parameters $\Phi=(\lambda,\theta,a,b)$ can be written as

$$\begin{aligned}Lf(x_{(i)},\Phi)&=\Pi_{i=1}^{n}f(x_{(i)},\Phi)\\&=\left(ab\theta\lambda^{2}\right)^{n}\ \Pi_{i=1}^{n}x_{i}\ e^{-\lambda\sum_{i=1}^{n}x_{i}}\,\Pi_{i=1}^{n}\left[1-e^{-\lambda x_{i}}(1+\lambda x_{i})\right]^{\theta a-1}\\&\quad\times\Pi_{i=1}^{n}\left\{1-\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta a}\right\}^{b-1}.\end{aligned} \tag{31}$$

Taking the log-likelihood function for the vector of parameters $\Phi = (\lambda, \theta, a, b)$ we get

$$\log L = n\log\theta + 2n\log\lambda + n\log a + n\log b$$
$$+\sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n}x_{(i)} + (\theta a-1)\sum_{i=1}^{n}\log\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right] \quad (32)$$
$$+(b-1)\sum_{i=1}^{n}\log\left\{1-\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta a}\right\},$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (32). The components of the score vector are given by

$$\frac{\partial \log L}{\partial\lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n}x_i + (\theta a-1)\sum_{i=1}^{n}\frac{\lambda x_i^2 e^{-\lambda x_i}}{\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]}$$
$$-\theta a(b-1)\sum_{i=1}^{n}\frac{\lambda x_i^2 e^{-\lambda x_i}\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]^{\theta a-1}}{\left\{1-\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]^{\theta a}\right\}}, \quad (33)$$

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} + a\sum_{i=1}^{n}\log\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]$$
$$-a(b-1)\sum_{i=1}^{n}\frac{\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]^{\theta a}\log\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]}{\left\{1-\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]^{\theta a}\right\}}, \quad (34)$$

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} + \theta\sum_{i=1}^{n}\log\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]$$
$$-\theta(b-1)\sum_{i=1}^{n}\frac{\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]^{\theta a}\log\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]}{\left\{1-\left[1-e^{-\lambda x_i}(1+\lambda x_i)\right]^{\theta a}\right\}}, \quad (35)$$

and

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left\{1-\left[1-e^{-\lambda x}(1+\lambda x)\right]^{\theta a}\right\}. \quad (36)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (33) - (36) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. Also, all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix}\hat{\lambda}\\ \hat{\theta}\\ \hat{a}\\ \hat{b}\end{pmatrix} \sim N\left[\begin{pmatrix}\lambda\\ \theta\\ a\\ b\end{pmatrix}, \begin{pmatrix}\widehat{V_{\lambda\lambda}} & \widehat{V_{\lambda\theta}} & \widehat{V_{\lambda a}} & \widehat{V_{\lambda b}}\\ \widehat{V_{\theta\lambda}} & \widehat{V_{\theta\theta}} & \widehat{V_{\theta a}} & \widehat{V_{\theta b}}\\ \widehat{V_{a\lambda}} & \widehat{V_{a\theta}} & \widehat{V_{aa}} & \widehat{V_{ab}}\\ \widehat{V_{b\lambda}} & \widehat{V_{b\theta}} & \widehat{V_{ba}} & \widehat{V_{bb}}\end{pmatrix}\right]. \quad (37)$$

$$V^{-1} = -E\left[\begin{matrix}V_{\lambda\lambda} & V_{\lambda\theta} & V_{\lambda a} & V_{\lambda b}\\ V_{\theta\lambda} & V_{\theta\theta} & V_{\theta a} & V_{\theta b}\\ V_{a\lambda} & V_{a\theta} & V_{aa} & V_{ab}\\ V_{b\lambda} & V_{b\theta} & V_{ba} & V_{bb}\end{matrix}\right]$$

where

$$V_{\lambda\lambda} = \frac{\partial^2 L}{\partial\lambda^2}, V_{\theta\theta} = \frac{\partial^2 L}{\partial\theta^2}, V_{aa} = \frac{\partial^2 L}{\partial a^2}$$

$$V_{\lambda a} = \frac{\partial^2 L}{\partial a\partial\lambda}, V_{a\beta} = \frac{\partial^2 L}{\partial a\partial b}, V_{a\theta} = \frac{\partial^2 L}{\partial a\partial\theta}.$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariances of these ML estimators for $\hat{\lambda}, \hat{\theta}, \hat{a}$ and $\hat{b}$ Using (6.7), we approximate $100(1-\gamma)\%$ confidence intervals for $\lambda, \theta, a$ and $b$ are determined respectively as

$$\hat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\lambda\lambda}}},\ \hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\theta\theta}}},\ \hat{a} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{aa}}} \text{ and } \hat{b} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{bb}}}$$

where $z_\gamma$ is the upper $100\gamma_{the}$ percentile of the standard normal distribution. We noticed from Table 1 that all Mean Square Errors (MSEs) decrease as the sample size increases, w hile they increase with increasing of the true parameter.

## 7. APPLICATIONS TO REAL DATA SET

In this Section we fit KEG to two real data sets and compare the fitness with the generalized inverse Weibull (GIW), inverse Weibull (IW) and Lindley Geometric (LG) distributions, whose densities are given by

$$f_{IW}(x; \lambda, \theta) = \theta\lambda^{\theta}x^{-\theta-1}e^{(\frac{\lambda}{x})^{\theta}}, x > 0, \lambda, \theta > 0,$$

$$f_{GIW}(x; \lambda, \theta, \beta) = \beta\theta\lambda^{\beta\theta}x^{-\beta\theta-1}e^{(\frac{\lambda}{x})^{\beta\theta}}, x > 0, \lambda, \theta, \beta > 0,$$

$$f_{LG}(x; \theta, p) = \frac{\theta^2}{\theta+1}(1-p)(1+x)e^{-\theta x}(1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x})^{-2} x > 0, \theta > 0, 0 < p < 1.$$

Respectively. Specifically, we consider two data sets. The first set of data represents the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). See Table 2. The second set given in Table 3 represents the waiting times (in minutes) before service of 100 bank customers this data is examined and analyzed by Ghitany et al. (2008).

**Table 1.** The MSE of the MLEs.

| KEG($a,b,\theta,\lambda$) | Sample size(n) | MSE($\hat{a}$) | MSE($\hat{b}$) | MSE($\hat{\theta}$) | MSE ($\hat{\lambda}$) |
|---|---|---|---|---|---|
| KEG(1,0.6,0.75,0.7) | 15 | 0.0344 | 0.5208 | 0.0193 | 0.3734 |
| | 25 | 0.0257 | 0.5073 | 0.0144 | 0.1520 |
| | 35 | 0.0239 | 0.5001 | 0.0135 | 0.1464 |
| | 45 | 0.0214 | 0.3735 | 0.012 | 0.142 |
| | 55 | 0.0077 | 0.4137 | 0.0043 | 0.1037 |
| | 65 | 0.0101 | 0.3666 | 0.0057 | 0.1231 |
| | 75 | 0.0095 | 0.2993 | 0.0054 | 0.1306 |
| KEG(1.5,0.9,1,0.7) | 15 | 0.6342 | 0.4139 | 0.2819 | 0.3588 |
| | 25 | 0.4574 | 0.2886 | 0.2033 | 0.2603 |
| | 35 | 0.3864 | 0.1333 | 0.1717 | 0.2121 |
| | 45 | 0.1997 | 0.1061 | 0.1279 | 0.2072 |
| | 55 | 0.1891 | 0.0977 | 0.084 | 0.1983 |
| | 65 | 0.1626 | 0.0472 | 0.0723 | 0.1814 |
| | 75 | 0.059 | 0.0381 | 0.0115 | 0.1028 |
| KEG(2,1.5,2.5,0.8) | 15 | 0.1346 | 0.8564 | 0.2103 | 0.0932 |
| | 25 | 0.0833 | 0.5589 | 0.1715 | 0.0719 |
| | 35 | 0.0508 | 0.3447 | 0.0793 | 0.0698 |
| | 45 | 0.0356 | 0.2953 | 0.0557 | 0.0482 |
| | 55 | 0.0334 | 0.2730 | 0.0522 | 0.0325 |
| | 65 | 0.0313 | 0.2348 | 0.0488 | 0.0221 |
| | 75 | 0.0300 | 0.1877 | 0.0372 | 0.0156 |

**Table 2.** Remission times (in months) of a random sample of 128 bladder cancer patients.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6.97 | 4.98 | 3.52 | 2.23 | 0.20 | 23.63 | 13.11 | 8.66 | 6.94 | 4.87 | 3.48 | 2.09 | 0.08 |
| 5.09 | 3.64 | 2.46 | 0.50 | 25.74 | 13.80 | 9.22 | 7.09 | 5.06 | 3.57 | 2.26 | 0.40 | 13.29 |
| 2.62 | 0.81 | 26.31 | 14.76 | 9.74 | 7.28 | 5.17 | 3.70 | 2.54 | 0.51 | 25.82 | 14.24 | 9.47 |
| 0.90 | 34.26 | 14.83 | 10.34 | 7.39 | 5.32 | 3.88 | 2.64 | 32.15 | 14.77 | 10.06 | 7.32 | 5.32 |
| 16.62 | 10.75 | 7.62 | 5.41 | 4.23 | 2.69 | 1.05 | 36.66 | 15.96 | 10.66 | 7.59 | 5.34 | 4.18 |
| 11.25 | 7.66 | 5.49 | 4.33 | 2.83 | 1.26 | 46.12 | 17.12 | 7.63 | 5.41 | 4.26 | 2.75 | 1.19 |
| 11.79 | 7.93 | 5.71 | 4.34 | 3.02 | 1.40 | 17.36 | 11.64 | 7.87 | 5.62 | 2.87 | 1.35 | 79.05 |
| 2.02 | 12.02 | 8.37 | 6.25 | 4.50 | 3.25 | 1.76 | 19.13 | 11.98 | 8.26 | 5.85 | 4.40 | 1.46 |
| 6.93 | 3.36 | 2.07 | 21.73 | 12.07 | 6.76 | 3.36 | 2.02 | 20.28 | 12.03 | 8.53 | 6.54 | 4.51 |
| 8.65 | 3.31 | 18.10 | 17.14 | 43.01 | 2.69 | 3.82 | 7.26 | 9.02 | 22.69 | 12.63 | | |

**Table 3.** Waiting times (in minutes) before service of 100 bank customers.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2.7 | 2.6 | 2.1 | 1.9 | 1.9 | 1.8 | 1.5 | 1.3 | 0.8 | 0.8 |
| 4.2 | 4.2 | 4.1 | 4.0 | 3.6 | 3.5 | 3.3 | 3.2 | 3.1 | 2.9 |
| 4.9 | 4.9 | 4.8 | 4.7 | 4.7 | 4.6 | 4.4 | 4.4 | 4.3 | 4.3 |
| 6.3 | 6.2 | 6.2 | 6.2 | 6.1 | 5.7 | 5.7 | 5.5 | 5.3 | 5.0 |
| 8.0 | 7.7 | 7.6 | 7.4 | 7.1 | 7.1 | 7.1 | 7.1 | 6.9 | 6.7 |
| 9.6 | 9.5 | 8.9 | 8.9 | 8.8 | 8.8 | 8.6 | 8.6 | 8.6 | 8.2 |
| 11.5 | 11.2 | 11.2 | 11.1 | 11.0 | 11.0 | 10.9 | 10.7 | 9.8 | 9.7 |
| 13.9 | 13.7 | 13.6 | 13.3 | 13.1 | 13.0 | 12.9 | 12.5 | 12.4 | 11.9 |
| 19.0 | 18.9 | 18.4 | 18.2 | 18.1 | 17.3 | 17.3 | 15.4 | 15.4 | 14.1 |
| 38.5 | 33.1 | 31.6 | 27.0 | 23.0 | 21.9 | 21.4 | 21.3 | 20.6 | 19.9 |

**Table 4.** Maximum-likelihood estimates, AIC, BIC and AICC values, and Kolmogorov-Smirnov statistics for the 128 bladder cancer patients data.

| Model | MLEs | | | | | | Measures | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | λ | θ | β | p | a | b | K_S | -2logL | AIC | BIC | AICC |
| **KEG** | 0.218 | 0.683 | | | 0.855 | 0.688 | *0.085* | *829.144* | *837.144* | *848.552* | *837.469* |
| **GIW** | 0.75 | 0.34 | 1.797 | | | | 0.369 | 990.362 | 996.362 | 1004.918 | 996.555 |
| **IW** | 16.142 | 0.464 | | | | | 0.503 | 1000.238 | 1004.238 | 1009.942 | 1004.334 |
| **LG** | | 0.192 | | 0.026 | | | 0.121 | 1349.523 | 1898.027 | 1903.731 | 1898.123 |

**Table5:** Maximum-likelihood estimates, AIC, BIC and AICC values, and Kolmogorov-Smirnov statistics for the waiting times (in minutes) before service of 100 bank customers.

| Model | MLEs | | | | | | Measures | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | λ | θ | β | p | a | b | K_S | -2logL | AIC | BIC | AICC |
| **KEG** | 0.387 | 0.977 | | | 1.254 | 0.448 | *0.037* | *634.249* | *642.249* | *652.67* | *642.671* |
| **GIW** | 5.023 | 0.661 | 1.759 | | | | 0.436 | 668.892 | 674.892 | 682.707 | 675.142 |
| **IW** | 4.287 | 1.2 | | | | | 0.168 | 671.919 | 675.919 | 681.13 | 676.043 |
| **LG** | | 0.182 | | 0.063 | | | 0.061 | 1074.762 | 1078.762 | 1083.973 | 1078.886 |

In order to compare distributions, we consider the K_S (Kolmogorov-Smirnov) statistic, -2logL, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion). The best distribution corresponds to lower -2logL, AIC, BIC, AICC statistics value.

Table 4 and Table 5 show parameter MLEs, the values of K_S, -2logL, AIC, BIC, AICC statistics for the three data set consecutively. From the above results, it is evident that the KEG distribution is the best distribution for fitting these data sets compared to other distributions considered here. And is a strong competitor to other distributions commonly used in literature for fitting lifetime data.

A CDF plot compares the fitted densities of the models with the empirical curve of the observed data (Figure 4) and (Figure 5) The fitted CDF for the KEG model is closer to the empirical curve.

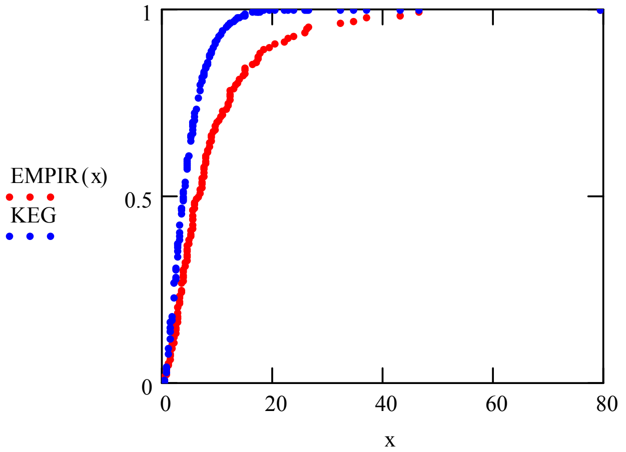


**Figure 4.** Empirical, fitted KEG cdf of the bladder cancer patients data

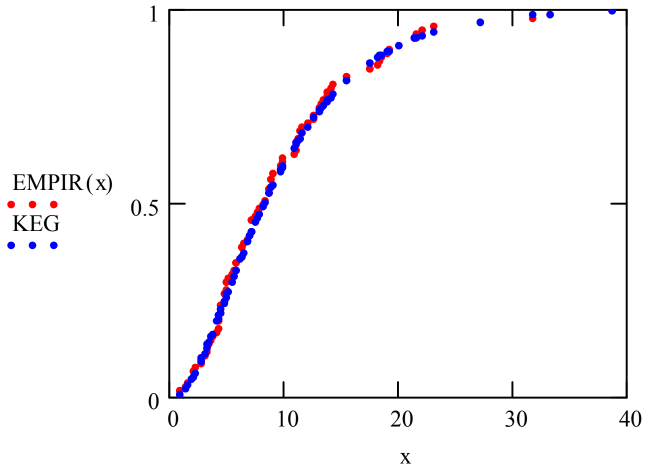


**Figure 5.** Empirical, fitted KEG cdf for the waiting times (in minutes) before service of 100 bank customers

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