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Optimal three step-stress accelerated life tests for Type-I hybrid censored data

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Abstract

In this paper, the maximum likelihood estimators for parameters are derived under three step-stress accelerated life tests for Type-I hybrid censored data. The exponential distribution and the cumulative exposure model are considered based on the assumption that a log quadratic relationship exits between stress and the mean lifetime θ. The test plan to search optimal stress change times minimizing the asymptotic variance of maximum likelihood estimators are presented. A numerical example to illustrate the proposed inferential procedures and some simulation results to investigate the sensitivity of the optimal stress change times by the guessed parameters are given.

Keywords: Accelerated life tests, cumulative exposure model, hybrid Type-I censoring, maximum likelihood estimator, optimal test plan, step-stress.

1. Introduction

It takes too much time to get a lifetime of a reliable item under typical conditions because of greatly improved units. The accelerated life tests (ALTs) are used in order to reduce the long testing time, where test units are put on higher stress than usual stress to yield information on test unit quickly. A model is fitted using data from ALTs and then extrapolated to make inferences on the lifetimes under usual stress. The step-stress accelerated life test (SSALT) is a kind of ALTs, where the stress are increased until a pre-specified time or upon occurrence of a fixed number of failures.

The Type-I hybrid censoring scheme (HCS) is considered in this paper. The experiment is completed at the time t_r if the rth failure occurs before preassigned time τ_c. on the other hand, the experiment is completed at the time τ_c in HCS. Thus, the experiment is completed at the time $\tau^* = \min(\tau_c, t_r)$, where τ_c and r are predetermined. The total testing time of HCS is at most τ_c, compared with the conventional Type-I censoring scheme. That is why this is referred to as the Type-I HCS. If the experiment is completed at the time $\tau^* = \max(\tau_c, t_r)$, it is referred to as Type-II HCS, because it assures at least r failures.

Some authors presented the results on the Type-I HCS. Bai *et al.* (1989) presented the optimal plan to search the stress change time which minimizes the asymptotic variance of MLE of the log scale parameter at the use stress. Balakrishnan and Han (2009) considered

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the C-optimality, D-optimality and A-optimality criteria for k-step-stress ALT with an equal step duration under progressively Type-I censored data from exponential distribution. Balakrishnan and Xie (2007) obtained the exact distribution of MLEs for a simple step-stress model with Type-II hybrid censored data from the exponential distribution and a cumulative exposure (CE) model. Childs *et al*. (2003) derived exact distribution of MLE of the mean of the exponential distribution and an exact lower confidence bound based on Type-I and Type-II hybrid censored sample. Chandrasekar *et al*. (2004) proposed generalized Type-I and Type-II hybrid censoring scheme and derived the exact distribution of MLE as well as exact confidence intervals for the mean of the exponential distribution. Gouno *et al*. (2004) studied optimal k-step-stress ALT with an equal step duration for progressively Type-I censored data from exponential distribution. Ling *et al*. (2009) considered a simple step-stress ALT model under progressive Type-I censoring scheme for the exponential distribution and a cumulative exposure model. Ling *et al*. (2011) proposed the optimum plan for a step-stress ALT with two stress variables under Type-I hybrid censoring scheme. Moon (2008) studied the optimal plan with a grouped and censored data obtained from the three step-stress ALT under the exponential distribution with a log-quadratic failure rate function of stress and a tampered failure rate (TFR) model. Moon and Park (2009) presented the optimal plan using data obtained from periodic inspection under type I censored three step stress ALT with a log-linear failure rate function of stress and a TFR model. Moon (2012) obtained MLEs and their confidence regions of parameters on the Type II censored three step-stress ALT for two-parameter exponential distribution.

In this paper, the MLEs of model parameters for three step-stress ALT under the assumption that a quadratic relationship exits between a stress and log(mean lifetime) for an exponential distribution are derived and one of models that have been commonly used on analysis of step-stress ALTs, the CE model proposed by Nelson (1980) is considered. In section 2, the model and some assumptions are given and the MLEs of parameters are obtained under three step-stress ALT in section 3. In section 4, the optimal plan to search optimal stress change times which minimize the asymptotic variance of the MLE of logarithm of the mean lifetime at the use stress. An example and some simulation results for the proposed procedures are presented in section 5 and some conclusions are given in section 6.

2. Model and assumptions

Suppose that there are three step-stress with $s_0 < s_1 < s_2 < s_3$, where s_0 is the use stress. We use the notation in our results without loss of generality as follows.

$$x_i = \frac{s_i - s_0}{s_3 - s_0}, \qquad i = 1, 2, 3.$$

For the step-stress ALTs, all units are simultaneously put on stress x_1 and examined until a preassigned time τ_1, but if all units do not fail before τ_1, the unfailed units are put on a stronger stress x_2 and observed until time τ_2. The remaining units at τ_2 are also put on a much stronger stress x_3 and examined. The experiment is continued until time $\tau^* = \min(\tau_c, t_r)$ in the Type-I HCS, where τ_1, τ_2, τ_c and r are preassigned beforehand.

Some assumptions and useful notations are introduced as follows.

(1) For any stress level x_i, $i = 1, 2, 3$, the lifetime of test unit T_{ij}, $j = 1, 2, \cdots, n_i$ distributes as an exponential distribution with the mean lifetime θ_i.

(2) Let n_1 be the number of failures before the time τ_1 at the stress x_1, n_2 be the number of failures before the time τ_2 at the stress x_2, n_3 be the number of failures before the time τ_c at the stress x_3. Let n_t be the number of units that fail before the test terminates. Then

$$n_t = \begin{cases} n_1 = r, & t_r \leq \tau_1 \\ n_1 + n_2 = r, & \tau_1 < t_r \leq \tau_2 \\ n_1 + n_2 + n_3 = r, & \tau_2 < t_r \leq \tau_c \\ n_1 + n_2 + n_3 < r, & \tau_c < t_r < \infty. \end{cases}$$

(3) The mean lifetime θ_i and each stress level x_i are assumed to be a log-quadratic function. That is,

$$\log \theta_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2, \qquad i = 1, 2, 3,$$

where β_0, β_1 and β_2 are unknown model parameters.

(4) A CE model is assumed that the remaining lifetime of test unit depends on the present CE.

By the assumption, the probability density function (PDF) for exponential distribution and cumulative distribution function (CDF) at each stress level x_i are given by

$$\begin{aligned} f_i(t, \theta_i) &= \frac{1}{\theta_i} \exp\left(- \left(\frac{t}{\theta_i} \right) \right), \\ F_i(t, \theta_i) &= 1 - \exp\left(- \left(\frac{t}{\theta_i} \right) \right), \qquad i = 1, 2, 3, \end{aligned} \tag{2.1}$$

where $t \geq 0$ and $\theta_i > 0$. Then, under three SSALT, the cumulative exposure distribution (CED), $G(t)$ is given by

$$G(t) = \begin{cases} F_1(t,\ \theta_1), & 0 \leq t < \tau_1 \\ F_2(t - \tau_1 + \nu_1,\ \theta_2), & \tau_1 \leq t < \tau_2 \\ F_3(t - \tau_2 + \nu_2,\ \theta_3), & \tau_2 \leq t < \infty, \end{cases} \tag{2.2}$$

where $F_i(\cdot, \theta_i)$ is given by (2.1), ν_i is the solution of $F_{i+1}(\nu_i, \theta_{i+1}) = F_i((\tau_i - \tau_{i-1}) + \nu_{i-1}, \theta_i)$ with $\nu_1 = \frac{\theta_2}{\theta_1}\tau_1$ and $\nu_2 = \frac{\theta_3}{\theta_2}(\tau_2 - \tau_1) + \frac{\theta_3}{\theta_1}\tau_1$.
Thus, the CED, $G(t)$ and the corresponding PDF, $g(t)$ are given by

$$G(t) = \begin{cases} 1 - \exp\left(-\frac{t}{\theta_1} \right), & 0 \leq t < \tau_1 \\ 1 - \exp\left(-\frac{t - \tau_1}{\theta_2} - \frac{\tau_1}{\theta_1} \right), & \tau_1 \leq t < \tau_2 \\ 1 - \exp\left(-\frac{t - \tau_2}{\theta_3} - \frac{\tau_2 - \tau_1}{\theta_2} - \frac{\tau_1}{\theta_1} \right), & \tau_2 \leq t < \infty, \end{cases} \tag{2.3}$$

$$g(t) = \begin{cases} \frac{1}{\theta_1} \exp\left(-\frac{t}{\theta_1} \right), & 0 \leq t < \tau_1 \\ \frac{1}{\theta_2} \exp\left(-\frac{t - \tau_1}{\theta_2} - \frac{\tau_1}{\theta_1} \right), & \tau_1 \leq t < \tau_2 \\ \frac{1}{\theta_3} \exp\left(-\frac{t - \tau_2}{\theta_3} - \frac{\tau_2 - \tau_1}{\theta_2} - \frac{\tau_1}{\theta_1} \right), & \tau_2 \leq t < \infty, \end{cases} \tag{2.4}$$

respectively.

3. Maximum likelihood estimators and Fisher information matrix

In this section, the MLEs of the model parameters β_0, β_1, β_2 by Newton-Rapshon method are derived. From the CED in (2.3) and the PDF in (2.4), the likelihood function based on observations t_{ij}, $i = 1, 2, 3$, $j = 1, 2, \cdots, n_i$ at stress $x_i, i = 1, 2, 3$ for the three cases under the Type-I HCS are given as follows.

Case 1 : If $t_r \le \tau_1$,

$$L(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{r} g_1(t_{1j}) \left[1 - G_1(t_r)\right]^{n-r}.$$

Case 2 : If $\tau_1 \le t_r < \tau_2$,

$$L(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{n_1} g_1(t_{1j}) \prod_{j=1}^{n_2} g_2(t_{2j}) \left[1 - G_2(t_r)\right]^{n-r}.$$

Case 3 : If $\tau_2 < t_r < \infty$,

(i) $t_r \le \tau_c$,

$$L(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{n_1} g_1(t_{1j}) \prod_{j=1}^{n_2} g_2(t_{2j}) \prod_{j=1}^{n_3} g_3(t_{3j}) \left[1 - G_3(t_r)\right]^{n-r}.$$

(ii) $\tau_c < t_r$,

$$L(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{n_1} g_1(t_{1j}) \prod_{j=1}^{n_2} g_2(t_{2j}) \prod_{j=1}^{n_3} g_3(t_{3j}) \left[1 - G_3(\tau_c)\right]^{n-(n_1+n_2+n_3)}.$$

Then, the likelihood function in Case 3 is given by

$$L(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{n_1} g_1(t_{1j}) \prod_{j=1}^{n_2} g_2(t_{2j}) \prod_{j=1}^{n_3} g_3(t_{3j}) \left[1 - G_3(\tau^*)\right]^{n-(n_1+n_2+n_3)}.$$

From the likelihood function in above, we can observe the followings.

(1) The likelihood function in Case 1 is obtained as

$$L(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{r} \frac{1}{\theta_1} \exp\left(-\frac{t_{1j}}{\theta_1}\right) \left(\exp\left(-\frac{t_r}{\theta_1}\right)\right)^{n-r}.$$

The MLEs of θ_2 and θ_3 do not exist in Case 1, because the likelihood function is independent of θ_2 and θ_3.

(2) The likelihood function in Case 2 is obtained as

$$L(\theta_1, \theta_2, \theta_3) = \prod_{j=1}^{n_1} \frac{1}{\theta_1} \exp\left(-\frac{t_{1j}}{\theta_1}\right) \prod_{j=1}^{n_2} \frac{1}{\theta_2} \exp\left(-\frac{t_{2j} - \tau_1}{\theta_2} - \frac{\tau_1}{\theta_1}\right) \times \left(\exp\left(-\frac{t_r - \tau_1}{\theta_2} - \frac{\tau_1}{\theta_1}\right)\right)^{n-r}.$$

The MLE of θ_3 does not exist in Case 2, because the likelihood function is independent of θ_3. Hence, the situation that no failures are observed at stress x_i, $i = 1, 2, 3$ is not considered, because the MLEs of parameters do not always exist.

Therefore, for $\tau_2 < t_r < \infty$, the likelihood function is obtained as

$$\begin{aligned} L(\theta_1, \theta_2, \theta_3) = &\prod_{j=1}^{n_1} \frac{1}{\theta_1} \exp\left(-\frac{t_{1j}}{\theta_1}\right) \prod_{j=1}^{n_2} \frac{1}{\theta_2} \exp\left(-\frac{t_{2j}-\tau_1}{\theta_2} - \frac{\tau_1}{\theta_1}\right) \\ &\times \prod_{j=1}^{n_3} \frac{1}{\theta_3} \exp\left(-\frac{t_{3j}-\tau_2}{\theta_3} - \frac{\tau_2-\tau_1}{\theta_2} - \frac{\tau_1}{\theta_1}\right) \\ &\times \left(\exp\left(-\frac{\tau^*-\tau_2}{\theta_3} - \frac{\tau_2-\tau_1}{\theta_2} - \frac{\tau_1}{\theta_1}\right)\right)^{n-(n_1+n_2+n_3)}. \end{aligned}$$

Then, the log-likelihood function of β_0, β_1 and β_2 is given by

$$\log L(\beta_0, \beta_1, \beta_2) = -\sum_{i=1}^{3} \left[n_i \left(\beta_0 + \beta_1 x_i + \beta_2 x_i^2\right) + U_i \exp\left(-\beta_0 - \beta_1 x_i - \beta_2 x_i^2\right)\right], \tag{3.1}$$

where

$$\begin{aligned} U_1 &= \sum_{j=1}^{n_1} t_{1j} + (n - n_1)\tau_1, \\ U_2 &= \sum_{j=1}^{n_2} (t_{2j} - \tau_1) + (n - (n_1 + n_2))(\tau_2 - \tau_1), \\ U_3 &= \sum_{j=1}^{n_3} (t_{3j} - \tau_2) + (n - (n_1 + n_2 + n_3))(\tau^* - \tau_2). \end{aligned}$$

The MLEs for model parameters β_0, β_1 and β_2 can be obtained by solving the following equation in (3.2) using the Newton-Raphson method.

$$\frac{\partial \log L(\beta_0, \beta_1, \beta_2)}{\partial \beta_k} = -\sum_{i=1}^{3} \left[n_i x_i^k - x_i^k D_i \exp\left(-\beta_0 - \beta_1 x_i - \beta_2 x_i^2\right)\right] = 0, \ \ k = 0, 1, 2. \tag{3.2}$$

The second partial and mixed derivatives of $\log L(\beta_0, \beta_1, \beta_2)$ in (3.1) with respect to β_0, β_1 and β_2 are given as follows.

$$\frac{\partial^2 \log L(\beta_0, \beta_1, \beta_2)}{\partial \beta_k \partial \beta_l} = -\sum_{i=1}^{3} x_i^{k+l} D_i \exp\left(-\beta_0 - \beta_1 x_i - \beta_2 x_i^2\right), \quad k, l = 0, 1, 2. \tag{3.3}$$

The Fisher information matrix $F = (f_{kl}), k, l = 0, 1, 2$ is obtained by taking the means of the second partial and mixed derivatives of $\log L(\beta_0, \beta_1, \beta_2)$ in (3.3). Therefore, the Fisher information matrix is given by

$$F = n \begin{pmatrix} f_{00} & f_{01} & f_{02} \\ f_{10} & f_{11} & f_{12} \\ f_{20} & f_{21} & f_{22} \end{pmatrix},$$

where, for $k, l = 0, 1, 2$,

$$
\begin{aligned}
f_{kl} =& - E\left(\frac{\partial^2 \log L(\beta_0, \beta_1, \beta_2)}{\partial \beta_k \partial \beta_l}\right) = \sum_{i=1}^{3} x_i^{k+l}(1 - A_{i-2})(1 - A_{i-1})A_i, \\
A_i =& 1 - \exp\left(-\frac{\tau_i - \tau_{i-1}}{\theta_i}\right), \ i = 1, 2, 3 \quad \text{and} \quad \tau_3 = \tau^*, \ \ \tau_0 = 0.
\end{aligned}
\tag{3.4}
$$

4. Optimal plan for three SSALT

In this section, the optimal plan of the three SSALT is presented, which is very important to improve the precision of MLEs of model parameters and the quality of the statistical inference at the use stress.

Let $V = \left(v_{kl}^2\right)$ be the covariance matrix, F^{-1} of $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$. Then the asymptotic variance (AsVar) for $\log \hat{\theta}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{\beta}_2 x_i^2$ is given by

$$
\begin{aligned}
\sigma_i^2 =& (1, x_i, x_i^2) V (1, x_i, x_i^2)' \\
=& v_{00}^2 + x_i^2 v_{11}^2 + x_i^4 v_{22}^2 + 2x_i v_{01}^2 + 2x_i^2 v_{02}^2 + 2x_i^3 v_{12}^2, \quad i = 1, 2, 3.
\end{aligned}
$$

Then,

$$
\sigma_0^2 = n\text{AsVar}(\log \hat{\theta}_0) = \frac{d_1^2}{A_1} + \frac{d_2^2}{(1 - A_1)A_2} + \frac{d_3^2}{(1 - A_1)(1 - A_2)A_3}, \tag{4.1}
$$

where A_i, $i = 1, 2, 3$ are given in (3.4) and

$$
d_1 = \frac{x_2 x_3}{(x_2 - x_1)(x_3 - x_1)}, \ \ d_2 = \frac{x_1 x_3}{(x_2 - x_1)(x_3 - x_2)}, \ \ d_3 = \frac{x_1 x_2}{(x_3 - x_1)(x_3 - x_2)}.
$$

The optimal plan under the three SSALT is to find the optimal stress change times based on the minimization of the asymptotic variance, $n\text{AsVar}(\log \hat{\theta}_0)$ in (4.1). Hence, the optimal stress change times τ_1^* and τ_2^* are obtained by solving the following equations given by

$$
\frac{\partial \sigma_0^2}{\partial \tau_1} = 0 \qquad \text{and} \qquad \frac{\partial \sigma_0^2}{\partial \tau_2} = 0,
$$

where

$$
\begin{aligned}
\frac{\partial \sigma_0^2}{\partial \tau_1} =& - \frac{d_1^2(1 - A_1)}{A_1^2 \theta_1} + \frac{d_2^2}{(1 - A_1)A_2^2}\left(\frac{A_2}{\theta_1} + \frac{1 - A_2}{\theta_2}\right) \\
&+ \frac{d_3^2}{(1 - A_1)(1 - A_2)A_3}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right), \\
\frac{\partial \sigma_0^2}{\partial \tau_2} =& - \frac{d_2^2(1 - A_2)}{(1 - A_1)A_2^2 \theta_2} + \frac{d_3^2}{(1 - A_1)(1 - A_2)A_3^2}\left(\frac{A_3}{\theta_2} + \frac{1 - A_3}{\theta_3}\right).
\end{aligned}
\tag{4.2}
$$

Theorem 4.1 Optimal stress change times, τ_1^* and τ_2^* minimizing σ_0^2, the asymptotic variance of $\log \hat{\theta}_0$ are the unique solutions of (4.2).

Proof: (4.2) can be obtained by equating the first derivative of σ_0^2 with respect to τ_1 and τ_2 to zero. To show that σ_0^2 has the minimum value at τ_1^* and τ_2^*, it is sufficient to prove that the determinant of the second order partial derivatives matrix H is positive definite by showing that two conditions are satisfied, which are given by

$$\frac{\partial^2\sigma_0^2}{\partial\tau_1^2} > 0 \quad \text{and} \quad |H| > 0,$$

where

$$H = \begin{pmatrix} \frac{\partial^2\sigma_0^2}{\partial\tau_1^2} & \frac{\partial^2\sigma_0^2}{\partial\tau_1\partial\tau_2} \\ & \frac{\partial^2\sigma_0^2}{\partial\tau_2^2} \end{pmatrix}.$$

The second order partial derivative of σ_0^2 with respect to τ_1 is given by

$$\begin{aligned}\frac{\partial^2\sigma_0^2}{\partial\tau_1^2} =& \frac{d_1^2(1-A_1)(2-A_1)}{\theta_1^2A_1^3} + \frac{d_2^2}{\theta_1(1-A_1)A_2^2}\left(\frac{A_2}{\theta_1} + \frac{1-A_2}{\theta_2}\right) \\ &+ \frac{d_2^2(1-A_2)}{\theta_2(1-A_1)A_2^3}\left(\frac{A_2}{\theta_1} + \frac{2-A_2}{\theta_2}\right) + \frac{d_3^2}{(1-A_1)(1-A_2)A_3}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)^2,\end{aligned}$$

which is positive by $1-A_i > 0, 2-A_i > 0$ for $i = 1, 2$ and the determinant of the second order partial derivatives matrix H is given by

$$\begin{aligned}|H| =& \frac{d_1^2d_2^2(1-A_2)^2(2-A_2)}{\theta_1^2\theta_2^2A_1^3A_2^3} + \frac{d_1^2d_3^2}{\theta_1^2\theta_2A_1^3A_3^2}\left(\frac{A_3}{\theta_2} + \frac{1-A_3}{\theta_3}\right) + \frac{d_1^2d_3^2(1-A_3)}{\theta_2^2\theta_3A_1^3A_3^3}\left(\frac{A_3}{\theta_2} + \frac{2-A_3}{\theta_3}\right) \\ &+ \frac{d_2^2d_3^2}{\theta_1(1-A_1)^2(1-A_2)A_2^3A_3^2}\left(\frac{A_3}{\theta_2} + \frac{1-A_3}{\theta_3}\right)\left(\frac{A_2(2-A_2)}{\theta_1^2} + \frac{(1-A_2)(4-A_2)}{\theta_1\theta_2}\right) \\ &+ \frac{d_2^2d_3^2(1-A_3)}{\theta_3(1-A_1)^2(1-A_2)A_2^3A_3^3}\left(\frac{A_3}{\theta_2} + \frac{2-A_3}{\theta_3}\right)\left(\frac{A_2^2}{\theta_1^2} + \frac{2A_2(1-A_2)}{\theta_1\theta_2} + \frac{(1-A_2)(2-A_2)}{\theta_2^2}\right) \\ &+ \frac{d_2^4A_2^2(1-A_2)}{\theta_1^2\theta_2^2(1-A_1)^2A_2^6} + \frac{d_3^4(1-A_3)}{\theta_3^2(1-A_1)^2(1-A_2)^2A_3^4}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)^2,\end{aligned}$$

which is also positive. Therefore, we can see that the unique solution satisfying (4.2) exists and minimize the asymptotic variance given in (4.1). □

But, the solutions of (4.2) are not in closed forms.

5. Numerical example and simulation studies

In this section, a numerical example is first presented to illustrate the procedure of proposed optimal plan and some simulation results are presented to compare the performances for different values of θ_1, θ_2 and θ_3 .

5.1. Example

The $n = 30$ simulated failure times from three SSALT are generated based on model parameters $\beta_0 = 2.0$, $\beta_1 = -2.0$, $\beta_2 = -2.0$, three stress level $x_0 = 0$, $x_1 = 0.35$, $x_2 = 0.65$, $x_3 = 1.0$ and $\tau_1 = 1.237$, $\tau_2 = 1.430$. Then $\theta_1 = 2.872$, $\theta_2 = 0.865$ and $\theta_3 = 0.135$ are obtained by the relationship $\log\theta_i = \beta_0 + \beta_1x_i + \beta_2x_i^2$, $i = 1, 2, 3$.

A sample Y is first simulated from the exponential distribution with mean 1. Then the sample from GED $G(t)$ in (2.3) is obtained from the transformation given by

$$T = \begin{cases} \theta_1 \cdot Y, & 0 \le Y < \frac{\tau_1}{\theta_1} \\ \tau_1 + \left(Y - \frac{\tau_1}{\theta_1}\right) \cdot \theta_2, & \frac{\tau_1}{\theta_1} \le Y < \frac{\tau_2}{\theta_1} \\ \tau_2 + \left(Y - \frac{\tau_1}{\theta_1} - \frac{\tau_2 - \tau_1}{\theta_2}\right) \cdot \theta_3, & \frac{\tau_2}{\theta_1} \le Y < \infty. \end{cases} \tag{5.1}$$

The sample generated from model (2.3) using (5.1) is presented in Table 5.1.

Table 5.1 Simulated failure times

stress level	failure times									
x_1	0.037	0.099	0.307	0.308	0.314	0.344	0.389	0.484	0.765	0.986
x_2	1.304	1.325	1.352	1.357						
x_3	1.439	1.467	1.472	1.485	1.510	1.524	1.543	1.551	1.562	1.582
	1.606	1.668	1.758	1.764	1.805	1.902				

Table 5.2 MLEs and optimal stress change times

(τ_c , r)	τ^*	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	τ_1^*	τ_2^*
(1.57,24)	1.570	2.878	0.869	0.184	0.860	1.446
(1.80,24)	1.582	2.878	0.869	0.175	0.866	1.460

The optimal stress change times τ_1^* and τ_2^* for three SSALT can be obtained from (3.2), (3.4) and (4.1) based on the Type-I HCS. They are given for different censoring times in Table 5.2.

5.2. Simulation studies

In order to use this optimal plan, unknown parameters θ_i, $i = 1, 2, 3$ must be approximated by the past data set. However, inaccurate pre-estimated values may not lead to optimal stress change times and result in poor estimators of parameters at the use stress. Thus the effects of changes in values of the pre-estimated θ_1, θ_2 and θ_3 for two different censoring times are examined to investigate the performances of the optimal stress change times, τ_1^* and τ_2^* and the results are presented in Table 5.3-Table 5.5, where $\theta_1 = 2.872$, $\theta_2 = 0.865$ and $\theta_3 = 0.135$ and τ_{ti}^*, $i = 1, 2$ means the true optimal stress change times for true values of θ_i under the various Type-I HCS.

Table 5.3 Optimal stress change times for different θ_1 and θ_2 values when $\theta_3 = 0.13$

			θ_2	0.8	0.85	0.9	0.95
(τ_c , r)	τ^*	(τ_{t1}^*, τ_{t2}^*)	θ_1	(τ_1^*, τ_2^*)	(τ_1^*, τ_2^*)	(τ_1^*, τ_2^*)	(τ_1^*, τ_2^*)
(1.57,21)	1.543	(0.855,1.438)	2.7	(0.851,1.432)	(0.840,1.436)	(0.830,1.438)	(0.821,1.441)
			2.8	(0.862,1.435)	(0.852,1.438)	(0.841,1.441)	(0.832,1.443)
			2.9	(0.873,1.437)	(0.862,1.440)	(0.852,1.443)	(0.842,1.445)
			3.0	(0.883,1.440)	(0.872,1.442)	(0.862,1.445)	(0.852,1.447)
(1.57,24)	1.570	(0.868,1.462)	2.7	(0.864,1.456)	(0.853,1.459)	(0.843,1.462)	(0.833,1.465)
			2.8	(0.876,1.459)	(0.865,1.462)	(0.854,1.465)	(0.844,1.467)
			2.9	(0.886,1.461)	(0.875,1.464)	(0.865,1.467)	(0.855,1.469)
			3.0	(0.897,1.464)	(0.886,1.466)	(0.875,1.469)	(0.866,1.471)
(1.57,27)	1.570	(0.868,1.462)	2.7	(0.864,1.456)	(0.853,1.459)	(0.843,1.462)	(0.833,1.465)
			2.8	(0.876,1.459)	(0.865,1.462)	(0.854,1.465)	(0.844,1.467)
			2.9	(0.886,1.461)	(0.875,1.464)	(0.865,1.467)	(0.855,1.469)
			3.0	(0.897,1.464)	(0.886,1.466)	(0.875,1.469)	(0.866,1.471)
(1.80,24)	1.582	(0.874,1.473)	2.7	(0.870,1.467)	(0.859,1.470)	(0.848,1.473)	(0.838,1.476)
			2.8	(0.882,1.470)	(0.870,1.473)	(0.860,1.476)	(0.850,1.478)
			2.9	(0.893,1.472)	(0.881,1.475)	(0.871,1.478)	(0.861,1.480)
			3.0	(0.903,1.474)	(0.892,1.477)	(0.881,1.480)	(0.872,1.482)
(1.80,27)	1.758	(0.958,1.629)	2.7	(0.953,1.621)	(0.940,1.625)	(0.928,1.629)	(0.917,1.632)
			2.8	(0.966,1.624)	(0.954,1.628)	(0.942,1.632)	(0.931,1.635)
			2.9	(0.979,1.627)	(0.967,1.631)	(0.955,1.635)	(0.944,1.638)
			3.0	(0.991,1.630)	(0.979,1.634)	(0.967,1.637)	(0.956,1.640)
(1.80,30)	1.800	(0.978,1.665)	2.7	(0.972,1.657)	(0.959,1.661)	(0.947,1.665)	(0.936,1.669)
			2.8	(0.986,1.660)	(0.973,1.665)	(0.961,1.668)	(0.950,1.672)
			2.9	(0.999,1.664)	(0.986,1.668)	(0.974,1.671)	(0.963,1.675)
			3.0	(1.012,1.667)	(0.999,1.671)	(0.987,1.674)	(0.975,1.678)

Table 5.4 Optimal stress change times for different θ_1 and θ_2 values when $\theta_3 = 0.16$

			θ_2	0.8	0.85	0.9	0.95
$(\tau_c$, r)	τ^*	$(\tau^*_{t1}, \tau^*_{t2})$	θ_1	(τ^*_1, τ^*_2)	(τ^*_1, τ^*_2)	(τ^*_1, τ^*_2)	(τ^*_1, τ^*_2)
(1.57,21)	1.543	(0.855,1.438)	2.7	(0.846,1.421)	(0.835,1.424)	(0.825,1.427)	(0.816,1.430)
			2.8	(0.857,1.424)	(0.847,1.427)	(0.837,1.430)	(0.827,1.433)
			2.9	(0.868,1.426)	(0.857,1.429)	(0.847,1.432)	(0.838,1.435)
			3.0	(0.878,1.429)	(0.867,1.432)	(0.857,1.434)	(0.848,1.437)
(1.57,24)	1.570	(0.868,1.462)	2.7	(0.859,1.444)	(0.848,1.448)	(0.838,1.451)	(0.828,1.454)
			2.8	(0.870,1.447)	(0.859,1.451)	(0.849,1.454)	(0.840,1.457)
			2.9	(0.881,1.450)	(0.870,1.453)	(0.860,1.456)	(0.851,1.459)
			3.0	(0.892,1.452)	(0.881,1.456)	(0.871,1.458)	(0.861,1.461)
(1.57,27)	1.570	(0.868,1.462)	2.7	(0.859,1.444)	(0.848,1.448)	(0.838,1.451)	(0.828,1.454)
			2.8	(0.870,1.447)	(0.859,1.451)	(0.849,1.454)	(0.840,1.457)
			2.9	(0.881,1.450)	(0.870,1.453)	(0.860,1.456)	(0.851,1.459)
			3.0	(0.892,1.452)	(0.881,1.456)	(0.871,1.458)	(0.861,1.461)
(1.80,24)	1.582	(0.874,1.473)	2.7	(0.865,1.455)	(0.854,1.459)	(0.843,1.462)	(0.834,1.465)
			2.8	(0.876,1.458)	(0.865,1.461)	(0.855,1.465)	(0.845,1.467)
			2.9	(0.887,1.461)	(0.876,1.464)	(0.866,1.467)	(0.856,1.470)
			3.0	(0.898,1.463)	(0.887,1.466)	(0.877,1.469)	(0.867,1.472)
(1.80,27)	1.758	(0.958,1.629)	2.7	(0.946,1.606)	(0.934,1.611)	(0.923,1.615)	(0.912,1.619)
			2.8	(0.960,1.610)	(0.948,1.614)	(0.936,1.618)	(0.925,1.622)
			2.9	(0.973,1.613)	(0.961,1.618)	(0.949,1.622)	(0.938,1.625)
			3.0	(0.985,1.617)	(0.973,1.621)	(0.961,1.624)	(0.950,1.628)
(1.80,30)	1.800	(0.978,1.665)	2.7	(0.965,1.642)	(0.953,1.647)	(0.941,1.651)	(0.930,1.655)
			2.8	(0.980,1.646)	(0.967,1.650)	(0.955,1.655)	(0.944,1.659)
			2.9	(0.993,1.649)	(0.980,1.654)	(0.968,1.658)	(0.957,1.662)
			3.0	(1.006,1.653)	(0.993,1.657)	(0.981,1.661)	(0.970,1.665)

Table 5.5 Optimal stress change times for different θ_1 and θ_2 values when $\theta_3 = 0.19$

			θ_2	0.8	0.85	0.9	0.95
$(\tau_c$, r)	τ^*	$(\tau^*_{t1}, \tau^*_{t2})$	θ_1	(τ^*_1, τ^*_2)	(τ^*_1, τ^*_2)	(τ^*_1, τ^*_2)	(τ^*_1, τ^*_2)
(1.57,21)	1.543	(0.855,1.438)	2.7	(0.841,1.411)	(0.831,1.414)	(0.821,1.418)	(0.812,1.421)
			2.8	(0.853,1.414)	(0.842,1.417)	(0.832,1.420)	(0.823,1.423)
			2.9	(0.863,1.416)	(0.853,1.420)	(0.843,1.423)	(0.834,1.426)
			3.0	(0.873,1.419)	(0.863,1.422)	(0.853,1.425)	(0.844,1.428)
(1.57,24)	1.570	(0.868,1.462)	2.7	(0.854,1.434)	(0.843,1.438)	(0.833,1.441)	(0.824,1.444)
			2.8	(0.866,1.437)	(0.855,1.441)	(0.845,1.444)	(0.835,1.447)
			2.9	(0.876,1.440)	(0.866,1.443)	(0.856,1.447)	(0.846,1.449)
			3.0	(0.887,1.443)	(0.876,1.446)	(0.866,1.449)	(0.857,1.452)
(1.57,27)	1.570	(0.868,1.462)	2.7	(0.854,1.434)	(0.843,1.438)	(0.833,1.441)	(0.824,1.444)
			2.8	(0.866,1.437)	(0.855,1.441)	(0.845,1.444)	(0.835,1.447)
			2.9	(0.876,1.440)	(0.866,1.443)	(0.856,1.447)	(0.846,1.449)
			3.0	(0.887,1.443)	(0.876,1.446)	(0.866,1.449)	(0.857,1.452)
(1.80,24)	1.582	(0.874,1.473)	2.7	(0.860,1.444)	(0.849,1.448)	(0.839,1.452)	(0.829,1.455)
			2.8	(0.871,1.448)	(0.861,1.451)	(0.851,1.455)	(0.841,1.458)
			2.9	(0.883,1.450)	(0.872,1.454)	(0.862,1.457)	(0.852,1.460)
			3.0	(0.893,1.453)	(0.882,1.457)	(0.872,1.460)	(0.863,1.463)
(1.80,27)	1.758	(0.958,1.629)	2.7	(0.941,1.593)	(0.929,1.598)	(0.917,1.603)	(0.907,1.607)
			2.8	(0.954,1.597)	(0.942,1.602)	(0.931,1.607)	(0.920,1.611)
			2.9	(0.967,1.601)	(0.955,1.606)	(0.944,1.610)	(0.933,1.614)
			3.0	(0.980,1.605)	(0.968,1.609)	(0.956,1.613)	(0.945,1.617)
(1.80,30)	1.800	(0.978,1.665)	2.7	(0.960,1.628)	(0.947,1.634)	(0.936,1.639)	(0.925,1.643)
			2.8	(0.974,1.633)	(0.961,1.638)	(0.950,1.642)	(0.939,1.647)
			2.9	(0.987,1.637)	(0.975,1.641)	(0.963,1.646)	(0.952,1.650)
			3.0	(1.000,1.640)	(0.987,1.645)	(0.976,1.649)	(0.965,1.653)

From Table 5.3-Table 5.5, we can see that the optimal stress change time τ^*_1 and τ^*_2 increase as τ^* increases. The optimal stress change time τ^*_1 increases as θ_1 increases and decrease as θ_2 increases, and τ^*_2 increase as θ_1 or θ_2 increases. But, τ^*_1 and τ^*_2 decrease as θ_3 increases. If pre-estimated values of θ_1 and θ_2 are less than 2% far from true values as a relative error, they have less than 1% effect on τ^*_1 and less than 1.5% on τ^*_2 regardless of θ_3. If pre-estimated values of θ_1 and θ_2 are less than 5% and θ_3 is less than 20% far from true values, they have less than 2.5% effect on τ^*_1, but less than 1% on τ^*_2. If pre-estimated values of θ_1 and θ_2 are about 10% far from true values, they have less than 5% effect on τ^*_1 and less than 1% on τ^*_2 when the pre-estimated θ_3 is less than 20% far from true values, but τ^*_1 has about 5.5% effect on τ^*_1 and less than 1.5% on τ^*_2 when the pre-estimated value of θ_3 is about 40% far from true values. On the whole, τ^*_2 is not sensitive to the pre-estimated values of θ_1, θ_2 and θ_3. On the other hand, τ^*_1 is affected slightly if the pre-estimated values of θ_1 and θ_2 are larger than 10% far from true values. τ^*_1 and τ^*_2 are not nearly affected by the pre-estimated value of θ_3.

6. Conclusion

In this paper, the MLEs of model parameters θ_i, $i = 1, 2, 3$ are derived, when the lifetime, T_{ij}, $j = 1, 2, \cdots, n_i$ of failures on stress x_i follows an exponential distribution with mean lifetime θ_i for the three SSALT based on Type-I HCS. The optimal plan to search the optimal stress change times τ_1^* and τ_2^* minimizing the asymptotic variance of $\log\hat{\theta}_0$ on the use stress is presented based on the MLEs of model parameters and the Fisher information matrix. By simulation results, we can see that the pre-estimated values of θ_1, θ_2 and θ_3 have a small effect on the optimal stress change times.

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