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Comparative analysis of Bayesian and maximum likelihood estimators in change point problems with Poisson process

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Abstract

Nowadays the application of change point analysis has been indispensable in a wide range of areas such as quality control, finance, environmetrics, medicine, geographics, and engineering. Identification of times where process changes would help minimize the consequences that might happen afterwards. The main objective of this paper is to compare the change-point detection capabilities of Bayesian estimate and maximum likelihood estimate. We applied Bayesian and maximum likelihood techniques to formulate change points having a step change and multiple number of change points in a Poisson rate. After a signal from *c*-chart and Poisson cumulative sum control charts have been detected, Monte Carlo simulation has been applied to investigate the performance of Bayesian and maximum likelihood estimation. Change point detection capacities of Bayesian and maximum likelihood estimation techniques have been investigated through simulation. It has been found that the Bayesian estimates outperforms standard control charts well specially when there exists a small to medium size of step change. Moreover, it performs convincingly well in comparison with the maximum likelihood estimator and remains good choice specially in confidence interval statistical inference.

Keywords: Bayesian estimate, change point, control chart, maximum likelihood estimate, Monte Carlo simulation.

1. Introduction

Originally change point models have been developed in connection with applications in quality control, where a change from the in-control to the out-of-control state has to be detected based on the available random observations. Various change point models have been suggested for a broad spectrum of applications like quality control, reliability, econometrics or medicine. In change point problems, a random process indexed by time is observed and we want to investigate whether a change in the distribution of the random elements occurs.

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That means we are interested in determining whether the observed stochastic process is homogeneous or not. Generally, in the discrete time case, let $X_1, X_2, \ldots$, denote a sequence of independent random variables, where the elements $X_1, \ldots, X_\theta$, have an identical distribution function F_0 and $X_{\theta+1}, \ldots$, are distributed according to F_1 and the change-point θ is unknown. Several statistical tests of the null hypothesis F_0 is equal to F_1 against the alternative F_0 is not equal to F_1 for some θ have been suggested. Majority change point problem methodologies assume that the number of change points is known and has fixed value. Like threshold models (Chen and Lee, 1995), the methods based on maximum likelihood estimators considered by Hawkins (2001) and many others considered this situation. Other authors have studied the one change point problem using a Bayesian approach (Menzefricke, 1981; Hsu, 1984; Smith, 1975). The model that introduces more flexibility into the analysis of change point problems is product partition model (PPM) developed by Hartigan (1990). The product partition model (PPM) have been applied by Barry and Hartigan (1993) to identify multiple change points in the case of normal means only. Loschi *et al.* (2003) add some features on PPM that help to identify multiple changes in both means and variances of normal data. Moreover, they proposed a Gibbs sampling scheme to compute the posterior distributions of the random partition generated by using change points and the posterior distributions of the number of change points (Kim and Seo, 2002; Lee and Lee, 2007)

A change point model in a Poisson process using a Bayesian framework have been discussed. We analyze and discuss the performance of the Bayesian change point model through posterior estimates and probability based intervals.

Bayesian and maximum likelihood estimation methods are compared via a simulation study. Performances based on goodness of fit of the estimation methods have been assessed based on the output from R software using carefully checked simulated data.

Let a Poisson process $X_t, t = 1, \ldots, T$, be initially in-control with independent observations coming from a Poisson distribution with a known rate λ_0. At an unknown point in time, τ, the Poisson rate parameter changes from its in-control state of λ_0 to λ_1, $\lambda_1 = \lambda_0 + \delta$, $\delta \neq 0$. The Poisson process step change model can thus be parametrized as follows:

$$P(x_t|\lambda_t) = \begin{cases} \frac{e^{-\lambda_0}\lambda_0^{x_t}}{x_t!}, & t = 1, 2, \cdots, \tau, \\ \\ \frac{e^{-\lambda_1}\lambda_1^{x_t}}{x_t!}, & t = \tau + 1, \cdots, T, \end{cases}$$

where δ, t and T are the magnitude of the step change, the change time and the current time respectively. The change from the in-control state may occur due to a non-constant change type scenario which can be explained by a linear trend model $\lambda_t = \lambda_0 + \beta(t - \tau)$ for $t>\tau$. If the magnitude of linear trend disturbance (slope) is positive, an increasing trend in which $\lambda_t>\lambda_0$, while a negative β leads to a linear reduction of the Poisson rate and $\lambda_t<\lambda_0$ for $t = \tau + 1, \ldots T$. The Poisson process linear trend change model can be modeled as follows:

$$P(x_t|\lambda_t) = \begin{cases} \frac{e^{-\lambda_0}\lambda_0^{x_t}}{x_t!}, & t = 1, 2, \cdots, \tau \\ \\ \frac{e^{(-\lambda_0+\beta(t-\tau))}(-\lambda_0+\beta(t-\tau))^{x_t}}{x_t!}, & t = \tau + 1, \cdots, T. \end{cases} \tag{1.1}$$

The Poisson process linear trend change model can be parameterized as follows. Each observation consists of a count from a subgroup formed from the output of the process.

During the formulation of subgroups $i = 1, 2, \cdots, \tau$ the process rate λ_i is equal to its known in-control value λ_0. For subgroups $i = \tau + 1, \cdots, T$ the process rate λ_i is equal to some unknown rate $\lambda_i = \lambda_0 + \beta(i - \tau)$ where T is the most recent subgroup sample. The model assumes two unknowns in τ and β, representing the last subgroup taken from the in-control process and the slope parameter of the linear trend, respectively. This model can be used to derive a maximum likelihood estimator for the process change point.

2. Methodological approach

2.1. Bayesian estimation

Barry and Hartigan (1993) assume that the observations are independent $N(\mu_i, \sigma^2)$, and that the probability of a change point at a position i is p, independently at each i. The prior distribution of μ_{ij} (the mean of the block beginning at position $i+1$ and ending at position j) is chosen as $N(\mu_0, \sigma_0^2/(j-i))$.

The algorithm uses a partition $\rho = (U_1, U_2, ..., U_n)$, where $U_i = 1$ indicates a change point at position $i+1$; Erdman and Emerson (2007) initialize U_i to 0 for all $i < n$, with $U_n \equiv 1$. In each step of the Markov chain, at each position i, a value of Ui is drawn from the conditional distribution of Ui given the data and the current partition. Following Barry and Hartigan, Erdman and Emerson let b denote the number of blocks obtained if $U_i = 0$, conditional on U_j , for $i \neq j$. The transition probability, ρ, for the conditional probability of a change point at the position $i+1$, may be obtained from the simplified ratio presented in Barry and Hartigan:

$$\begin{aligned}\frac{p_i}{1-p_i} &= \frac{P(U_i = 1|X, U_j, j \neq i)}{P(U_i = 0|X, U_j, j \neq i)} \\ &= \frac{\left[\int_0^\gamma p^b(1-p)^{n-b-1}dp\right]\left[\int_0^\lambda \frac{w^{b/2}}{(W_1+B_1w)^{(n-1)/2}}dw\right]}{\left[\int_0^\gamma p^b(1-p)^{n-b-1}dp\right]\left[\int_0^\lambda \frac{w^{(b-1)/2}}{(W_0+B_0w)^{(n-1)/2}}dw\right]}\end{aligned}$$

where W_0, B_0, W_1 and B_1 are the within and between block sums of squares obtained when $U_i = 0$ and $U_i = 1$ respectively, and X is the data. The tuning parameters γ and λ may take values in $[0, 1]$, chosen so that this method After each iteration, the posterior means are updated conditional on the current partition. A direct implementation of the Barry and Hartigan MCMC algorithm is numerically unstable for long sequences because the integrands of

$$\int_0^\lambda \frac{w^{b/2}}{(W_1 + B_1w)^{(n-1)/2}}dw$$

and

$$\int_0^\lambda \frac{w^{(b-1)/2}}{(W_0 + B_0w)^{(n-1)/2}}dw$$

either diverge or go to 0 for long sequences. Fortunately, these integrals can be simplified as incomplete beta integrals. The odds of a change point at a particular position in the partition (given the data and the current partition) may be re-expressed as

$$\frac{p_i}{1-p_i} = \frac{P(U_i = 1|X, U_j, j \neq i)}{P(U_i = 0|X, U_j, j \neq i)}$$

$$= \left(\frac{W_0}{W_1}\right)^{\frac{n-b-2}{2}} \left(\frac{B_0}{B_1}\right)^{\frac{b+2}{2}} \sqrt{\frac{W_1}{B_1}} \cdot \frac{\int_0^{\frac{B_1\lambda/W_1}{1+B_1\lambda/W_1}} p^{(b+2)/2}(1-p)^{(n-b-3)/2}dp \cdot \int_0^{\gamma} p^b(1-p)^{n-b-1}dp}{\int_0^{\frac{B_1\lambda/W_1}{1+B_1\lambda/W_1}} p^{(b+2)/2}(1-p)^{(n-b-2)/2}dp \cdot \int_0^{\gamma} p^{b-1}(1-p)^{n-b}dp}.$$

This expression consists of numerically stable terms, allowing application of the Barry and Hartigan procedure to sequences of any length. The MCMC implementation of Barry and Hartigan estimates the posterior distributions of the change points and the means, θ_{ij}.

However, performing Bayesian inference for multiple change points parameter θ estimation is a challenging problem. Even when θ is assumed known, exact computation of the posterior distribution of the change points is intractable for large data sets. This issue is typically tackled using Markov chain Monte Carlo (MCMC) techniques.

We used Monte Carlo simulation to study the performance of the constructed Bayesian estimates in change estimation following a signal from c-chart, Poisson CUSUM, and Poisson EWMA control charts when a change (step, linear) is simulated to occur at $\tau = 100$. We generated 100 observations of a Poisson process with an in-control rate of $\lambda_0 = 20$. To investigate the behavior of the Bayesian estimators over the population for different change sizes, we replicated this simulation 100 times. Simulated datasets that were obvious outliers were excluded. The number of replication studies is a compromise between excessive computational time, considering MCMC iterations and sufficiency of the achievable distributions even for tails.

In the step of change scenario, we induced step changes of sizes $\delta = 6$ as an example and $\delta = +2, +6$ for a replication study until control chart signaled. In this scenario, the replication study was limited to c-chart, since other control charts mostly signaled prior to the induction of the second change point.

Because we know that the process is in-control, if an out-of-control observation was generated in the simulation of the early 100 in-control observations, it was taken as a false alarm and the simulation was restarted.

2.2. Maximum likelihood estimation

Maximum likelihood, also called the maximum likelihood method, is the procedure of finding the value of one or more parameters for a given statistic which makes the known likelihood distribution a maximum. Maximum likelihood estimation of an unknown change point first begins with obtaining the maximum likelihood estimate (MLE) as a point estimate. Interval estimates of any desired level, which are preferred over point estimates can be constructed around the MLE. Hawkins (1977) studied change point detection in the series following independent uni-variate normal distribution with possible change of mean.

It was shown that more precise estimates were obtained when ML estimators were used in conjunction with Poisson control charts, compared to c-chart signals and CUSUM or EWMA built-in estimators. A confidence interval on the estimated change point was constructed.

We consider a model for step change in the rate of a Poisson process. A step change in the rate parameter occurs when the rate suddenly shifts from its in-control value to some out-of-control value, and remains at that out-of-control value until the appropriate process adjustments are made. From this model, a maximum likelihood estimator of the process change point is derived.

Consider a step change model for the behavior of a Poisson process rate parameter λ. The model assumes that the process is initially in control with independent observations coming from a Poisson distribution with known rate λ_0. After an unknown point in time, the rate parameter changes from an in-control state of $\lambda = \lambda_0$ to an unknown out-of-control state of $\lambda = \lambda_i$, for $i = \tau + 1, \cdots$, and the functional form of λ_i is given as

$$\lambda_i = \lambda_0 + \beta(i - \tau) \tag{2.1}$$

where β is the magnitude (or slope) of the linear trend disturbance. During the formulation of subgroups $t = 1, 2, \cdots, \tau$ the process rate λ_i is equal to its known in-control value λ_0. For subgroups $i = \tau + 1, \cdots, T$ the process rate λ_i is equal to some unknown rate $\lambda_i = \lambda_0 + \beta(i - \tau)$ where T is the most recent subgroup sample. This model can be used to derive a maximum likelihood estimator for the process change point. We denoted the MLE of the proposed change point estimator as τ. Assuming a process change point at τ, the likelihood function is

$$L(\tau, \beta|x) = \prod \frac{e^{(-\lambda_0)}\lambda_0^{x_i}}{x_i!} \prod \frac{e^{(-(\lambda_0+\beta(i-\tau)))}(\lambda_0 + \beta(i-\tau))^{x_i}}{x_i!} \tag{2.2}$$

where x_i is the count corresponding to the ith subgroup. The MLE of τ is the value of τ that maximizes the likelihood in (2.2), or equivalently, its logarithm.

2.3. Assessment of data analysis

In this paper, Monte Carlo simulation have been applied to investigate the performance of the Bayesian estimation in change estimation after a signal from c-chart and Poisson cumulative sum (CUSUM) control charts observed. A multiple change is simulated to occur at $\tau = 100$. Moreover, Poisson exponential weighted moving average (EWMA) chart have been applied for comparing change point detection capacity with others considered. A Poisson process of 100 observations with an in-control rate of $\lambda_0 = 20$ have been generated for this study. Different change sizes have been used to examine the behavior of the Bayesian estimators over the population, the simulation method have been simulated 100 times. To calculate the Poisson CUSUM statistic measures S_i^+, S_i^-, k^+, k^- are defined as follows;

$$S_i^+ = \max\{0, S_{i-1} + x_i - k^+\}, \tag{2.3}$$

$$S_i^- = \max\{0, k^- - x_i + S_{i-1}\}, \tag{2.4}$$

$$k^+ = \frac{\lambda_1 - \lambda_0}{\ln \lambda_1 - \ln \lambda_0}, \tag{2.5}$$

$$k^- = \frac{\lambda_0 - \lambda_1}{\ln \lambda_0 - \ln \lambda_1}, \tag{2.6}$$

where S_{i-1} is the cumulative sum from the previous observations and x_i is the numerical value for $i = 1, \ldots, n$ observation. The k^+ and k^- are appropriate constants that utilizes the average occurences of events λ_i in a specified time interval to calculate their values. If

$S_i^{\pm}$ exceeds the specified decision interval then the control chart signals that an increase (or a decrease) in the Poisson rate occurred. We set the charts to detect a 25% shift in Poisson rates. An alternative time-weighted chart which considered in this paper is the the Poisson EWMA chart. This chart is typically used for two-sided alternatives. The EWMA statistic is;

$$Z_i = \phi X_i + (1-\phi) Z_{i-1}, \tag{2.7}$$

where X_i is i^{th} observation, Z_{i-1} is value of the EWMA statistic from the previous observation and $0 < \phi < 1$ is weighting constant.

The Shewhart (1926) methods have been used for c-chart and Poisson cumulative sum control charts, respectively. A Poisson cumulative sum accumulates the difference between an observed value and a reference value.

The R package had been used for all changes and control. To obtain posterior distributions of the time and the magnitude of the changes, we used the R2WinBUGS interface in R to generate 100,000 samples through MCMC iterations in WinBUGS for all change point scenarios with the first 20,000 samples ignored as burn in. We then analyzed the results using the CODA package in R.

2.4. Change models performance analysis

In Figure 2.1, the posterior distributions for the time and the magnitude of a step change of size +6 are presented. Although there is slight difference in almost all control charts, posterior distributions of the change point concentrate on the 100th sample which is the real change point. As seen in the figure, the posteriors are not symmetric and are skewed specially for the change in time. Therefore, the posterior mode can be an estimator for the change point model parameters time and magnitude of step change (τ, δ).

A confidence interval (CI) is a posterior probability based interval which involves those values of highest probability in the posterior density of the parameter of interest. The 50% and 80% confidence intervals for the estimated time and the magnitude of step changes in all three control charts are depicted in Table 2.1. Under the same probability of 0.8 for the c-chart, the CI for the time of the step change of size $\delta = +2$ covers 53 samples around the 100th sample whereas it decreases to 6 samples for $\delta = +6$ due to the smaller standard deviation.

The comparison of the 50% and 80% CIs for the estimated time of a step change of size $\delta = +6$ in the Poisson EWMA reveals that the posterior distribution of the time is highly left skewed and the increase in the probability contracts the left boundary of the interval, from 96.9 to 88 in comparison with the shift in the right boundary.

Table 2.1 Credible intervals for the estimated time τ and the magnitude of step changes δ

δ		c-chart		Poisson EWMA		Poisson CUSUM	
		50%	80%	50%	80%	50%	80%
+2	τ	(101,105)	(65,118)	(96.6,114)	(71.2,125.8)	(98.2,105)	(65.2,108)
	δ	(2.1,3.2)	(1.9,3.2)	(1.41,2.65)	(0.76,3.05)	(0.12,2.50)	(0.23,4.8)
+6	τ	(97.9,100)	(96,102)	(96.9,101)	(88,103)	(96,101)	(83,106)
	δ	(3.9,5.2)	(3.4,5.5)	(2.2,4.1)	(1.2,4.8)	(1.31,4)	(0.05,4.9)

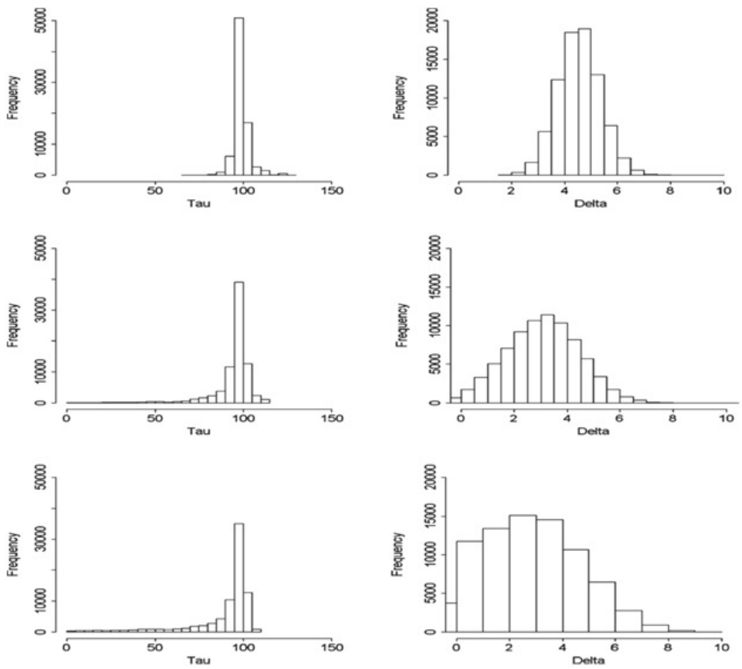

Figure 2.1 Posterior distributions of the time and the magnitude of a step change $\lambda_0 = 20, \delta = +6, \tau = 100$ following signals from (τ, δ) c - chart, (τ, δ) EWMA and (τ, δ) Poisson cumulative sum

The probability of the occurrence of the change point in the last 10, 25, and 50 observed samples prior to signaling in the control charts is depicted in Table 2.2. For a step change of size $\delta = +2$, since the c-chart signals very late, it is unlikely that the change point occurred in the last 10, 25, and even 50 samples. In contrast, in the Poisson EWMA and CUSUM charts, where they both signal earlier than the c-chart, the probabilities of occurrence in the last 10 samples are 0.55 and 0.59, then increase to 0.76 and 0.82, respectively as the next 15 samples are included. In the case of $\delta = +6$, for the c-chart, the 98% probability of occurrences of change point is located between the last 25 and 50 samples for the c - chart.

Table 2.2 Probability of occurrence of change point in last 10, 25, and 50 observed samples

	c-chart			Poisson EWMA			Poisson CUSUM		
δ	10	25	50	10	25	50	10	25	50
+2	0.00	0.00	0.01	0.55	0.76	0.86	0.59	0.82	0.91
+6	0.00	0.01	0.99	0.06	0.86	0.95	0.91	0.97	0.99

3. Comparison of Bayesian and maximum likelihood estimators

To study the performance of the proposed Bayesian estimators in comparison with others, we run the alternatives, built-in estimators of Poisson EWMA, CUSUM charts and ML estimators.

Table 3.1 Average of estimated time of linear trend in Poisson process

	c-chart		Poisson EWMA		Poisson CUSUM	
δ	$E(\tau_{MLE})$	$E(\tau_b)$	$E(\tau_{MLE})$	$E(\tau_b)$	$E(\tau_{MLE})$	$E(\tau_b)$
-2	-	100.83	-	100.75	-	100.92
	-	-1.16	-	-0.93	-	-0.96
-1	-	102.05	-	102.14	-	102.74
	-	-2.36	-	-2.07	-	-2.18
-0.5	103.55	102.96	-	104.6	-	104.7
	-3.48	-2.5	-	-2.91	-	-2.91
0.5	102.7	103.75	102.02	104.45	102.12	104.78
	-3.19	-2.99	-9.23	-2.94	-11.68	-2.78
1	100.23	102.55	101.08	102.75	101.57	102.78
	-3.19	-2.05	-12.42	-2.11	-3.59	-2.36
2	100.23	101.2	100.57	101.18	100.59	101.19
	-2.81	-1.02	-4.07	-1.04	-3.81	-1.04

Table 3.1 shows the mean of the Bayesian estimates and detected change points provided by built-in estimators of Poisson EWMA equation (2.7) and Poisson CUSUM equation (2.3)-(2.6) charts and the ML estimator for a linear trend change in a Poisson process at different magnitude of step change δ. Application of the proposed ML estimator is restricted to trends with a positive slope as Newton method is not tractable for decreasing trends in Poisson mean.

The Bayesian estimator, τ_b, the average time required to detect the change point in Bayesian method almost outperforms the built-in estimator of EWMA, τ_{EWMA} which is the average time required to detect change point in Poisson EWMA using MLE technique as in equation (2.7), where there exists a decreasing trend. This superiority increases when the slope size β raises. The CUSUM estimator, τ_{CUSUM}, the average time taken to detect change point in Poisson CUSUM as in equation (2.3) - (2.6) chart estimates the change point more precisely than τ_{EWMA}, the average time taken to detect change point in Poisson EWMA as in equation (2.7) in both MLE and Bayesian estimation techniques. However the Bayesian estimator, τ_b, still remains the best alternative for detection of linear trends with negative slopes, when the variation of the estimates is taken into account.

Table 3.1 reveals that the Bayesian estimator, τ_b, that is the average time needed to detect change point using Bayesian estimation in Poisson process is slightly outperformed by the ML estimator, τ_{MLE} which is the average time when applying Maximum likelihood estimation across the charts when there exists an increasing linear trend in the process mean. However, the Bayesian estimator can still be a reasonable alternative in light of the obtained standard deviations which are less than those observed from the ML estimator over replications.

Apart from the accuracy and precision criteria used for the comparison study, the posterior distributions for the time and the magnitude of a change enable us to construct probabilistic intervals around estimates and probabilistic inferences about the location of change point. This is a significant advantage of the proposed Bayesian approach. The approach to change point identification described in this paper has the advantage of building on control charts that may be already in place in practice.

4. Conclusions

To identify the special causes of a failure of a given process, recognition of time when the process has changed plays a great role. The technique of identifying the change point drastically reduces the effort to resolve the problems happened. This paper modeled the change point estimation for a Poisson process in a Bayesian framework. A step and multiple change settings have been considered when the number of changes is known. We built posterior distributions for change point estimates using MCMC. Comparisons of Bayesian estimators with c-chart, Poisson EWMA, and CUSUM control charts have been performed. It has been found that the Bayesian estimates outperform standard control charts in change estimation, particularly where there exists a small to medium size of step change. Bayesian estimator performs convincingly well in comparison with the maximum likelihood estimator and remains good specially in conditions like confidence intervals estimation.

In Poisson process, detailed analysis of performance of Bayesian estimates over different change scenario showed that each Bayesian change point model outperforms other models. The importance of such analysis in any process specially in quality control confirmation process is unquestionable. Therefore further in-depth study incorporating data from a given process and using some more other change models is highly recommended.

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