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SOME RELATIONSHIPS BETWEEN THE INTEGRAL TRANSFORM AND THE CONVOLUTION PRODUCT ON ABSTRACT WIENER SPACE

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Abstract. In this paper we establish several formulas for multiple integral transform of functionals defined on abstract Wiener space. We then use the these results to establish several basic formulas involving multiple convolution products.

1. Introduction

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|_H = \sqrt{\langle\cdot,\cdot\rangle}$. Let $\|\cdot\|_0$ be a measurable norm on H with respect to the Gauss measure μ [9]. Let B denote the completion of H with respect to $\|\cdot\|_0$. Let i denote the natural injection form H into B. The adjoint operator i^* of i is one-to-one and maps B^* continuously onto a dense subset of H^*, where H^* and B^* are topological duals of H and B, respectively. We then have a triple (B^*, H, B) such that $B^* \subset H^* \equiv H \subset B$ and $\langle h, x\rangle = (h, x)$ for all x in B^* and h in H, where $(\cdot,\cdot)$ denotes the natural dual pairing between B^* and B. By the results of Gross in [6], $\mu \circ i^{-1}$ has a unique countably additive extension m to the Borel σ-algebra $\mathcal{B}(B)$ on B. The triple (B, H, m) is called an abstract Wiener space. For more details see [3, 5, 7, 9, 13]. The classical Wiener space is an example of abstract Wiener space.

In 1981, Lee introduced an integral transform which is called the Fourier-Gauss transform, on abstract Wiener space in his unifying paper [13]. He then applied the integral transform to investigate the existence of solutions of the system of the differential equations (1.1) and (1.2) (Cauchy problems) associated with an appropriate operator N_c;

$$\begin{cases} u_t(x,t) = P(N_c)u(x,t), & x \in B, t > 0 \\ u(x,0) = f(x) \end{cases} \tag{1.1}$$

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and

$$\begin{cases} u_{tt}(x,t) = P(N_c)u(x,t), & x \in B, t > 0 \\ P(N_c)u(x) = f(x) \end{cases} \tag{1.2}$$

where $P(\eta) = a_0 + a_1\eta + \cdots + a_m\eta^m$, f is an appropriate function and c is a fixed nonzero complex number. He also showed that the solutions of differential equations (1.1) and (1.2) above can be represented as integrals with respect to the Wiener measure.

Since the concept of the integral transform was introduced by Lee, many mathematicians studied the integral transform and related topics for functionals in several classes [2, 4, 10, 11]. In particular, Chang, Chung and Skoug established several basic formulas for the integral transform and the convolution product [2]. They also established a Fubini theorem for the integral transform and the convolution product of functionals on classical Wiener space [4].

In this paper, we extend the results in [2, 4] to functionals on the abstract Wiener space B. The most results and formulas in [2, 4] follow immediately from the results and the formulas in this paper.

2. Definitions and preliminaries

In this section we list some definitions and results from [13]. First, we denote the abstract Wiener integral of a functional F by

$$\int_B F(x)m(dx).$$

A subset E of B is said to be scale-invariant measurable [8] provided ρE is measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho E) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.).

Throughout this paper we will assume that each functional $F : B \to \mathbb{C}$ we consider is scale-invariant measurable and that

$$\int_B |F(\rho x)|m(dx) < \infty$$

for each $\rho > 0$.

Let $[B]$ be the space of all complex-valued continuous functions defined on $[0, T]$ which vanish at $t = 0$ and whose real and imaginary parts are elements of B.

First we state the definition of the integral transform $\mathcal{F}_{\gamma,\beta}$ introduced in [13].

Definition 1. Let F be a functional defined on $[B]$. For each pair of nonzero complex numbers γ and β, the integral transform $\mathcal{F}_{\gamma,\beta}F$ of F is defined by

$$\mathcal{F}_{\gamma,\beta}F(y) \equiv \mathcal{F}_{\gamma,\beta}(F)(y) \equiv \int_B F(\gamma x + \beta y)m(dx), \quad y \in [B], \tag{2.1}$$

if it exists.

Remark 1. When $\gamma = \sqrt{c}(c > 0)$ and $\beta = i$, $\mathcal{F}_{\gamma,\beta}$ is the Fourier-Wiener c-transform introduced in [12] and is denoted by $\mathcal{F}_c$. When $\gamma = (-iq)^{-\frac{1}{2}}(q > 0)$ and $\beta = 1$, $\mathcal{F}_{\gamma,\beta}$ is the Fourier-Feynman transform in [1] and is denoted by T_q.

Next we state the definition of the convolution product $(F * G)_\gamma$.

Definition 2. Let F and G be functionals defined on $[B]$. Then the convolution product $(F * G)_\gamma$ of F and G is defined by

$$(F * G)_\gamma(y) = \int_B F\left(\frac{y + \gamma x}{\sqrt{2}}\right) G\left(\frac{y - \gamma x}{\sqrt{2}}\right) m(dx), \quad y \in [B],$$

if it exists.

Now we are ready to describe the class of functionals that we work with in this paper. Let $\mathcal{E}_a$ be the class of functionals F defined on $[B]$ with the following properties;

(1) $|F(z)| \leq c\exp(c'\|z\|_{[B]})$ for some positive real numbers c and c' depending only on F where $\|z\|_{[B]} = (\|x\|_0^2 + \|y\|_0^2)^{\frac{1}{2}}$ for $z = x + iy$ with $x, y \in B$.

(2) $F(x + \lambda y)$ is an entire function of $\lambda \in \mathbb{C}$.

We call $\mathcal{E}_a$ the space of exponential type analytic functionals. By Fernique's theorem [9], there exists some constant a such that $\int_B \exp(a\|x\|_{[B]}^2)m(dx) < \infty$ and so the integral transform $\mathcal{F}_{\gamma,\beta}$ is well-defined on $\mathcal{E}_a$ for each pair of γ and β on $\mathbb{C}$.

The following lemma is due to Lee in [13].

Lemma 2.1. *Let F be an element of $\mathcal{E}_a$ and let γ and β be nonzero complex numbers. Then we have*

$$\mathcal{F}_{\gamma,\beta}(\mathcal{E}_a) \subset \mathcal{E}_a; \tag{2.2}$$

$$\int_B \int_B F(\gamma x + \beta y)m(dx)m(dy) = \int_B F(\sqrt{\gamma^2 + \beta^2}z)m(dz). \tag{2.3}$$

Remark 2. (1) Equation (2.2) tells us that the integral transform $\mathcal{F}_{\gamma,\beta}F$ of a functional F in $\mathcal{E}_a$ is an element of $\mathcal{E}_a$ again.

(2) For nonzero complex numbers γ and β, we note that $\sqrt{\gamma^2 + \beta^2} = \delta$ for some $\delta \in \mathbb{C}$. Since the Wiener measure m is even, the integral on the right-hand side of (2.3) remains of single value no matter, $+\delta$ or $-\delta$, which value we choose for $\sqrt{\gamma^2 + \beta^2}$.

Definition 3. Let

$$A = \{(\gamma, \beta) \in \mathbb{C} \times \mathbb{C} : \gamma^2 + \beta^2 = 1, \gamma \neq 0, \beta \neq 0\}.$$

Remark 3. For all $(\gamma,\beta)\in A$ and $F\in\mathcal{E}_a$, the integral transform always exists and belongs to $\mathcal{E}_a$. Furthermore the equation (2.3) becomes

$$\int_B\int_B F(\gamma x+\beta y)m(dx)m(dy)=\int_B F(z)m(dz). \tag{2.4}$$

3. Multiple integral transforms

In this section we establish the existence of multiple integral transforms of functionals in $\mathcal{E}_a$.

In our next theorem, we establish the Fubini theorem for the integral transform of functionals in $\mathcal{E}_a$.

Theorem 3.1. *Let F be an element of $\mathcal{E}_a$. Then for all (γ_1,β_1) and (γ_2,β_2) in A,*

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y)=\mathcal{F}_{\gamma',\beta'}F(y)=\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y) \tag{3.1}$$

for $y\in[B]$, where $\gamma'=\sqrt{\gamma_1^2+\beta_1^2\gamma_2^2}$ and $\beta'=\beta_1\beta_2$. Furthermore (γ',β') is an element of A.

Proof. Using (2.1) and (2.3) it follows that for $y\in[B]$,

$$\begin{aligned}\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y)&=\int_B\int_B F(\gamma_1 z+\beta_1\gamma_2 x+\beta_1\beta_2 y)m(dz)m(dx)\\&=\int_B F(\sqrt{\gamma_1^2+\beta_1^2\gamma_2^2}w+\beta_1\beta_2 y)m(dw)\\&=\mathcal{F}_{\gamma',\beta'}F(y).\end{aligned}$$

On the other hand, using (2.1) and (2.3) it follows that for $y\in[B]$,

$$\begin{aligned}\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y)&=\int_B\int_B F(\gamma_2 z+\beta_2\gamma_1 x+\beta_1\beta_2 y)m(dz)m(dx)\\&=\int_B F(\sqrt{\gamma_2^2+\beta_2^2\gamma_1^2}w+\beta_1\beta_2 y)m(dw).\end{aligned}$$

Since $\gamma_1^2+\beta_1^2\gamma_2^2=\gamma_2^2+\beta_2^2\gamma_1^2$, we can establish the equation (3.1) as desired. Furthermore (γ',β') is an element of A since (γ_1,β_1) and (γ_2,β_2) are elements of A. ■

Putting $\gamma_1=\gamma_2=\gamma$ and $\beta_1=\beta_2=\beta$, the following corollary follows immediately form equation (3.1).

Corollary 3.2. *Let F be as in Theorem 3.1. Then for all $(\gamma,\beta)\in A$,*

$$\mathcal{F}_{\gamma,\beta}(\mathcal{F}_{\gamma,\beta}F)(y)=\mathcal{F}_{\gamma'',\beta''}F(y)$$

for $y\in[B]$, where $\gamma''=\sqrt{\gamma^2(1+\beta^2)}$ and $\beta''=\beta^2$.

Corollary 3.3. *Let $\mathcal{F}_c$ be the Fourier-Wiener c-transform used in* [12]. *Then*

$$\mathcal{F}_{\sqrt{2}}(\mathcal{F}_{\sqrt{2}}F)(y)=\mathcal{F}_{0,-1}F(y)=F(-y)=\mathcal{F}_{\sqrt{2}}(\mathcal{F}_{\sqrt{2}}F)(y)$$

for $y\in[B]$.

Note that if (γ,β) is an element of A, then $(i\frac{\beta}{\gamma},\frac{1}{\beta})$ also is an element of A, and so using equation (3.1), we can establish a basic formula for the inverse integral transform.

Theorem 3.4. *Let F be as in Theorem 3.1. For all $(\gamma,\beta)\in A$,*

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}F)(y)=\mathcal{F}_{\gamma',\beta'}F(y)=\mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}F)(y)$$

for $y\in B$. But $\gamma'=\sqrt{\gamma^2+\beta^2(i\frac{\gamma}{\beta})^2}=0$ and $\beta'=\beta\frac{1}{\beta}=1$ and so

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}F)(y)=F(y)=\mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}F)(y) \tag{3.2}$$

for $y\in[B]$.

To obtain an n-dimensional version of Theorem 3.1, we use the following notations. Let $\{(\gamma_n,\beta_n)\}$ be a sequence in A. For all $n=1,2,\cdots$, let

$$\tilde{\gamma}_n=\sqrt{\sum_{k=1}^{n}\gamma_k^2(\prod_{i=1}^{k}\beta_{i-1}^2)} \tag{3.3}$$

and

$$\tilde{\beta}_n=\prod_{k=1}^{n}\beta_k \tag{3.4}$$

where $\beta_0=1$. Note that $(\tilde{\gamma}_1,\tilde{\beta}_1)=(\gamma_1,\beta_1)$, $(\tilde{\gamma}_2,\tilde{\beta}_2)=(\gamma',\beta')$ and $(\tilde{\gamma}_n,\tilde{\beta}_n)$ are also elements of A for all $n=1,2,\cdots$.

In our next theorem, we give an n-dimensional version of Theorem 3.1. The following theorem is one of our main results in this paper.

Theorem 3.5. *Let F be as in Theorem 3.1. For all $(\gamma_1,\beta_1),\cdots,(\gamma_n,\beta_n)\in A$,*

$$\mathcal{F}_{\gamma_n,\beta_n}\cdots\mathcal{F}_{\gamma_1,\beta_1}F(y)=\mathcal{F}_{\tilde{\gamma}_n,\tilde{\beta}_n}F(y)=\mathcal{F}_{\gamma_1,\beta_1}\cdots\mathcal{F}_{\gamma_n,\beta_n}F(y) \tag{3.5}$$

for $y\in[B]$, where $\tilde{\gamma}_n$ and $\tilde{\beta}_n$ are given by (3.3) *and* (3.4) *above.*

Proof. To prove this theorem, we use the mathematical induction for $n\geq 2$. First note that for $n=2$, using (3.1) it follows that for $y\in[B]$,

$$\mathcal{F}_{\gamma_2,\beta_2}\mathcal{F}_{\gamma_1,\beta_1}F(y)=\mathcal{F}_{\gamma',\beta'}F(y)=\mathcal{F}_{\tilde{\gamma}_2,\tilde{\beta}_2}F(y).$$

Now we assume that for n,

$$\mathcal{F}_{\gamma_n,\beta_n}\cdots\mathcal{F}_{\gamma_1,\beta_1}F(y)=\mathcal{F}_{\tilde{\gamma}_n,\tilde{\beta}_n}F(y). \tag{3.6}$$

Then using (2.3) and (3.6) it follow that for $y \in [B]$,

$$\begin{aligned}
&\mathcal{F}_{\gamma_{n+1},\beta_{n+1}}\mathcal{F}_{\gamma_n,\beta_n}\cdots\mathcal{F}_{\gamma_1,\beta_1}F(y) = \mathcal{F}_{\gamma_{n+1},\beta_{n+1}}\mathcal{F}_{\tilde{\gamma}_n,\tilde{\beta}_n}F(y)\\
&= \int_B \mathcal{F}_{\tilde{\gamma}_n,\tilde{\beta}_n}F(\gamma_{n+1}x+\beta_{n+1}y)m(dx)\\
&= \int_B\int_B F(\tilde{\gamma}_n z+\tilde{\beta}_n\gamma_{n+1}x+\tilde{\beta}_n\beta_{n+1}y)m(dz)m(dx)\\
&= \int_B F(\sqrt{\tilde{\gamma}_n^2+\tilde{\beta}_n^2\gamma_{n+1}^2}w+\tilde{\beta}_n\beta_{n+1}y)m(dw).
\end{aligned}$$

Now using (3.3) and (3.4), we can easily check that

$$\begin{aligned}
\sqrt{\tilde{\gamma}_n^2+\tilde{\beta}_n^2\gamma_{n+1}^2} &= \sqrt{\sum_{k=1}^{n}\gamma_k^2\prod_{i=1}^{k}\beta_{i-1}^2+\prod_{k=1}^{n}\beta_k^2\gamma_{n+1}^2}\\
&= \sqrt{\sum_{k=1}^{n+1}\gamma_k^2\prod_{i=1}^{k}\beta_{i-1}^2}\\
&= \tilde{\gamma}_{n+1}
\end{aligned}$$

and

$$\tilde{\beta}_n\beta_{n+1} = \beta_1\cdots\beta_n\beta_{n+1} = \tilde{\beta}_{n+1}.$$

Hence

$$\mathcal{F}_{\gamma_{n+1},\beta_{n+1}}\mathcal{F}_{\gamma_n,\beta_n}\cdots\mathcal{F}_{\gamma_1,\beta_1}F(y) = \mathcal{F}_{\tilde{\gamma}_{n+1},\tilde{\beta}_{n+1}}F(y),$$

and so using (3.1) we can establish equation (3.5) as desired. ■

Corollary 3.6. *Let F be as in Theorem 3.5 and let (γ,β) be an element of A. Then*

$$\mathcal{F}_{\gamma,\beta}\overbrace{\cdots}^{n\text{-times}}\mathcal{F}_{\gamma,\beta}F(y) = \mathcal{F}_{\sqrt{1-\beta^{2n}},\beta^n}F(y)$$

for $y \in [B]$. Furthermore

$$\mathcal{F}_{\sqrt{2}}\overbrace{\cdots}^{n\text{-times}}\mathcal{F}_{\sqrt{2}}F(y) = \mathcal{F}_{\sqrt{1-(-1)^n},i^n}F(y)$$

for $y \in [B]$.

Remark 4. In fact, there are $n!$-formulas corresponding for the equation (3.5). For example, if $n=3$, then there are six-formulas ;

$$\begin{aligned}
&\mathcal{F}_{\gamma_3,\beta_3}(\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)))(y)\\
&= \mathcal{F}_{\gamma_3,\beta_3}(\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)))(y)\\
&= \mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_3,\beta_3}(\mathcal{F}_{\gamma_1,\beta_1}F)))(y)\\
&= \mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_3,\beta_3}F)))(y)\\
&= \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_3,\beta_3}F)))(y)\\
&= \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_3,\beta_3}(\mathcal{F}_{\gamma_2,\beta_2}F)))(y)
\end{aligned}$$

for $y \in [B]$.

4. Application

In this section we use the result of the multiple integral transform to establish basic formulas involving convolution product.

In our next lemma, we give a basic formula for integral transform for convolution product.

Lemma 4.1. *Let F and G be elements of $\mathcal{E}_a$ and let (γ,β) be an element of A. Then*

$$\begin{aligned}\mathcal{F}_{\gamma,\beta}(F*G)_\gamma(y) &= \mathcal{F}_{\gamma,\beta}F(y/\sqrt{2})\mathcal{F}_{\gamma,\beta}G(y/\sqrt{2})\\ &= \mathcal{F}_{\gamma,\beta/\sqrt{2}}F(y)\mathcal{F}_{\gamma,\beta/\sqrt{2}}G(y)\end{aligned} \tag{4.1}$$

for $y \in [B]$.

Next we give several corollaries of Lemma 4.1.

Corollary 4.2. *Let $\mathcal{F}_c$ be the Fourier-Wiener c-transform used in* [12]. *Then*

$$\mathcal{F}_{\sqrt{2}}(F*G)_{\sqrt{2}}(y) = \mathcal{F}_{\sqrt{2}}F(y/\sqrt{2})\mathcal{F}_{\sqrt{2}}G(y/\sqrt{2}).$$

Corollary 4.3. *For $F \in \mathcal{E}_a$ and $(\gamma,\beta) \in A$,*

$$\mathcal{F}_{\gamma,\beta}(F*F)_\gamma(y) = [\mathcal{F}_{\gamma,\beta}F(y/\sqrt{2})]^2$$

and

$$\mathcal{F}_{\gamma,\beta}(F*1)_\gamma(y) = \mathcal{F}_{\gamma,\beta}F(y/\sqrt{2})$$

for $y \in [B]$.

The following theorem is also one of our main results in this paper.

Theorem 4.4. *Let F and G be as in Lemma 4.1. For all (γ,β) in A,*

$$(F*G)_\gamma(y) = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}F(\cdot/\sqrt{2})\mathcal{F}_{\gamma,\beta}G(\cdot/\sqrt{2}))(y) \tag{4.2}$$

for $y \in [B]$.

Proof. Using (4.1) and (3.2) with F replaced with $(F*G)_\gamma$, we can obtain the equation (4.2) as desired. ∎

Remark 5. Interchanging (γ,β) and $(i\frac{\gamma}{\beta},\frac{1}{\beta})$ in equation (4.2) we obtain the formula

$$(F*G)_{i\frac{\gamma}{\beta}}(y) = \mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}F(\cdot/\sqrt{2})\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}G(\cdot/\sqrt{2}))(y)$$

for $y \in [B]$.

The following theorem is the last main theorem in this paper.

Theorem 4.5. *Let F and G be as in Lemma 4.1. For all (γ_1,β_1) and (γ_2,β_2) in A,*

$$\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F*\mathcal{F}_{\gamma_2,\beta_2}G)_{\gamma_1}(y) = \mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F*\mathcal{F}_{\gamma_1,\beta_1}G)_{\gamma_2}(y)$$

for $y \in [B]$.

Proof. Using equations (4.1) and (3.1) it follows that for $y \in [B]$,

$$\begin{aligned}
&\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F * \mathcal{F}_{\gamma_2,\beta_2}G)_{\gamma_1}(y)\\
&= \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y/\sqrt{2})\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}G)(y/\sqrt{2})\\
&= \mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y/\sqrt{2})\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}G)(y/\sqrt{2})\\
&= \mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F * \mathcal{F}_{\gamma_1,\beta_1}G)_{\gamma_2}(y),
\end{aligned}$$

which completes the proof of Theorem 4.5. ■

We obtain the following corollary by letting $G(y) = F(y)$ or by letting $G(y)$ be identically one on $[B]$.

Corollary 4.6. *Let F be as in Theorem 3.1. Then for all (γ_1, β_1) and (γ_2, β_2) in A,*

$$\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F * \mathcal{F}_{\gamma_2,\beta_2}F)_{\gamma_1}(y) = [\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y/\sqrt{2})]^2$$

and

$$\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F * 1)_{\gamma_1}(y) = \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y/\sqrt{2}) = \mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F * 1)_{\gamma_2}(y)$$

for $y \in [B]$.

Corollary 4.7. *Let F and G be as in Theorem 4.5. For all $(\gamma, \beta) \in A$,*

$$\begin{aligned}
&\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}F * \mathcal{F}_{\gamma,\beta}G)_{i\frac{\gamma}{\beta}}(y)\\
&= \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}F)(y/\sqrt{2})\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}G)(y/\sqrt{2}) \qquad (4.3)\\
&= F(y/\sqrt{2})G(y/\sqrt{2})
\end{aligned}$$

for $y \in [B]$. Now taking the integral transform $\mathcal{F}_{\gamma,\beta}$ of each side of equation (4.3)*, we can obtain a basic formula for convolution product*

$$(\mathcal{F}_{\gamma,\beta}F * \mathcal{F}_{\gamma,\beta}G)_{i\frac{\gamma}{\beta}}(y) = \mathcal{F}_{\gamma,\beta}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2}))(y) \qquad (4.4)$$

$y \in [B]$. Furthermore, interchanging (γ, β) and $(i\frac{\gamma}{\beta}, \frac{1}{\beta})$ in equation (4.4) *we obtain the formula*

$$(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}F * \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}G)_{\gamma}(y) = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2}))(y)$$

$y \in [B]$.

Remark 6. Clearly there is an n-dimensional version of the Theorem 4.5. For example, if $n = 3$, then there are six formulas as follows;

$$\begin{aligned}
&\mathcal{F}_{\gamma_3,\beta_3}(\mathcal{F}_{\gamma_2,\beta_2}\mathcal{F}_{\gamma_1,\beta_1}F * \mathcal{F}_{\gamma_2,\beta_2}\mathcal{F}_{\gamma_1,\beta_1}G)_{\gamma_3}(y)\\
&= \mathcal{F}_{\gamma_3,\beta_3}(\mathcal{F}_{\gamma_1,\beta_1}\mathcal{F}_{\gamma_2,\beta_2}F * \mathcal{F}_{\gamma_1,\beta_1}\mathcal{F}_{\gamma_2,\beta_2}G)_{\gamma_3}(y)\\
&= \mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_3,\beta_3}\mathcal{F}_{\gamma_1,\beta_1}F * \mathcal{F}_{\gamma_3,\beta_3}\mathcal{F}_{\gamma_1,\beta_1}G)_{\gamma_2}(y)\\
&= \mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}\mathcal{F}_{\gamma_3,\beta_3}F * \mathcal{F}_{\gamma_1,\beta_1}\mathcal{F}_{\gamma_3,\beta_3}G)_{\gamma_2}(y)\\
&= \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_3,\beta_3}\mathcal{F}_{\gamma_2,\beta_2}F * \mathcal{F}_{\gamma_3,\beta_3}\mathcal{F}_{\gamma_2,\beta_2}G)_{\gamma_1}(y)\\
&= \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}\mathcal{F}_{\gamma_3,\beta_3}F * \mathcal{F}_{\gamma_2,\beta_2}\mathcal{F}_{\gamma_3,\beta_3}G)_{\gamma_1}(y)
\end{aligned}$$

for $y \in [B]$. Hence we know that there are $n!$ formulas for the case of n-dimensional.

5. Conclusions

For certain values of the parameters γ and β and for certain classes of functionals, the Fourier-Wiener transform, the modified Fourier-Wiener transform, the Fourier-Feynman transform and the Gauss transform are special cases of Lee's integral transform $\mathcal{F}_{\gamma,\beta}$. These transforms play an important role in the studies of stochastic processes and functional integrals on infinite dimensional spaces. In this paper, we have extended various results and formulas in previous papers. That is to say, all results and formulas in previous papers are special cases in this paper.

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