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HYPER-CONJUGATE HARMONIC FUNCTION OF CONIC REGULAR FUNCTIONS IN CONIC QUATERNIONS

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Abstract. We give a rth conic regular functions with conic quaternion variables in $\mathbb{C}^2$ and obtain a hyper-conjugate harmonic function of conic regular function in conic quaternions in the sense of Clifford analysis.

1. Introduction

The quaternions in Clifford algebra are a normed division algebra with four dimensions over the real numbers. The quaternions are non-commutative and non-associative, but satisfy a weaker form of associativity. The quaternions were envisioned by Musès to a complete, integrated, connected, and natural number system. Musès [12, 13] sketched certain fundamental types of hypernumbers and arranged them in hyperbolic quaternions and conic quaternions with associated arithmetic and geometry. The conic quaternions have been applied in fields such as special theory and string theory of relativity and quantum theory. Deavours [1] provided a mathematical summary of quaternion algebra such as calculus and properties of several operators in quaternions. Kajiwara *et al.* [2, 3] obtained several regenerations in complex and studied the inhomogeneous Cauchy-Riemann system of quaternions and Clifford analysis in ellipsoid. In 2011, Koriyama *et al.* [8] gave some regularities of quaternionic functions based on holomorphic mappings in a domain in $\mathbb{C}^2$. Naser [14] and Nôno [15, 16, 17] gave some properties of quaternionic hyperholomorphic functions in quaternions. Sudbery [18] gave the line of quaternionic analysis which remedies these deficiencies by using the exterior differential calculus. He was able to clarify the relationship between quaternionic analysis and complex analysis.

For any complex harmonic function f_1 in a domain of holomorphy D in $\mathbb{C}^2$, we [10, 11] investigated the uniqueness and existence of hyper-conjugate harmonic functions of an octonion number system and dual quaternion in Clifford

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analysis. We [4, 5, 6, 7] researched certain properties of a regularity of functions with values in special quaternions on Clifford analysis and corresponding Cauchy-Riemann systems in special quaternions. Also, we gave a regular function with values in dual split quaternions and some analogous conditions of complex Cauchy-Riemann systems and relations between a corresponding Cauchy-Riemann system and a regularity of functions with values in dual split quaternions.

In this paper, we give the conditions of conic Cauchy-Riemann systems and conic harmonicity in $\mathbb{C}^2$. Then for any complex valued function $g_1(z)$ satisfying the condition of harmonicity in a pseudoconvex domain Ω in $\mathbb{C}^2$, we can find a hyper-conjugate harmonic function $g_2(z)$ on Ω such that $g(z) = g_1(z) + g_2(z)e_2$ is a conic regular function on Ω.

2. Preliminaries

Suppose the following base elements

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

These satisfy the following commutative multiplication rules:

$$e_0^2 = e_2^2 = 1, \; e_1^2 = e_3^2 = -1, \; e_1e_2 = e_3, \; e_2e_3 = e_1, \; e_3e_1 = -e_2.$$

Consider the field

$$\mathcal{CQ} = \{Z = x_0 + x_1e_1 + x_2e_2 + x_3e_3 | \; x_l(l = 0, 1, 2, 3) \in \mathbb{R}\}, \tag{1}$$

where the element e_0 is the identity of $\mathcal{CQ}$ and e_1 identifies the imaginary unit $\sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers. A conic quaternion Z is given by (1),

$$Z = z_1 + z_2e_2 \in \mathcal{CQ},$$

where $z_1 = x_0 + x_1e_1$ and $z_2 = x_2 + x_3e_1$ are complex numbers in $\mathbb{C}$. Conic quaternions are built on bases $\{1, e_1, e_2, e_3\}$ and form a commutative, associative, and distributive arithmetic. They contain non-trivial idempotents and zero divisors, but no nilpotents. Conic quaternions are isomorphic to tessarines, and also to bicomplex numbers. Thus, we identify $\mathcal{CQ}$ with $\mathbb{C}^2$.

We write a conic quaternion $Z = z_1 + z_2e_2$, the 1st conic quaternion conjugate number is $Z^{\dagger_1} = z_1 - z_2e_2$ and its modulus is

$$ZZ^{\dagger_1} = z_1^2 + z_2^2 = (x_0 + x_1e_1)^2 + (x_2 + x_3e_1)^2.$$

Analogously, the 2nd conic quaternion conjugate number is $Z^{\dagger_2} = \overline{z_1} + \overline{z_2}e_2$ and its modulus is

$$ZZ^{\dagger_2} = z_1\overline{z_1} + z_2\overline{z_2} + (z_1\overline{z_2} + z_2\overline{z_1})e_2 = (x_0 + x_2e_2)^2 + (x_1 + x_3e_2)^2.$$

Also, the 3rd conic quaternion conjugate number is $Z^{\dagger_3} = \overline{z_1} - \overline{z_2}e_2$ and its modulus is

$$ZZ^{\dagger_3} = z_1\overline{z_1} - z_2\overline{z_2} - (z_1\overline{z_2} - z_2\overline{z_1})e_2 = (x_0 + x_3e_3)^2 + (x_1 - x_2e_3)^2.$$

We use the following differential operators:

$$\begin{aligned}
\frac{\partial}{\partial Z} &:= \frac{\partial}{\partial z_1} + e_2\frac{\partial}{\partial z_2} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3}\Big),\\
\frac{\partial}{\partial Z^{\dagger_1}} &= \frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}\Big),\\
\frac{\partial}{\partial Z^{\dagger_2}} &= \frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}\Big),\\
\frac{\partial}{\partial Z^{\dagger_3}} &= \frac{\partial}{\partial \overline{z_1}} - e_2\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3}\Big),
\end{aligned}$$

where $\frac{\partial}{\partial z_1}$, $\frac{\partial}{\partial \overline{z_1}}$, $\frac{\partial}{\partial z_2}$, $\frac{\partial}{\partial \overline{z_2}}$ are usual differential operators used in complex analysis.

And we use the following differential operators:

$$\begin{aligned}
\Delta_{\dagger_1} &:= \frac{\partial^2}{\partial Z \partial Z^{\dagger_1}} = \frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2^2}\\
&= \frac{1}{4}\Big(\frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1}\Big)^2 + \frac{1}{4}\Big(\frac{\partial}{\partial x_2} - e_1\frac{\partial}{\partial x_3}\Big)^2,\\
\Delta_{\dagger_2} &:= \frac{\partial^2}{\partial Z \partial Z^{\dagger_2}} = \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \frac{\partial^2}{\partial z_2 \partial \overline{z_2}} + e_2\Big(\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} + \frac{\partial^2}{\partial z_2 \partial \overline{z_1}}\Big)\\
&= \frac{1}{4}\Big(\frac{\partial}{\partial x_0} + e_2\frac{\partial}{\partial x_2}\Big)^2 + \frac{1}{4}\Big(\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_3}\Big)^2,\\
\Delta_{\dagger_3} &:= \frac{\partial^2}{\partial Z \partial Z^{\dagger_2}} = \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} - \frac{\partial^2}{\partial z_2 \partial \overline{z_2}} + e_2\Big(-\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} + \frac{\partial^2}{\partial z_2 \partial \overline{z_1}}\Big)\\
&= \frac{1}{4}\Big(\frac{\partial}{\partial x_0} - e_3\frac{\partial}{\partial x_3}\Big)^2 + \frac{1}{4}\Big(\frac{\partial}{\partial x_1} + e_3\frac{\partial}{\partial x_2}\Big)^2
\end{aligned}$$

are the analogous complex Laplacian operators. Specially, $\Delta_{\dagger_1}$ and $\Delta_{\dagger_3}$ are called complex multiplicative moduli.

3. Hyper-conjugate harmonic functions on $\mathcal{CQ}$

Let Ω be a bounded open set in $\mathcal{CQ}$. A function $f(Z) : \Omega \to \mathcal{CQ}$ is defined on Ω with values in $\mathcal{CQ}$ as follows:

$$f(Z) = f(z_1 + z_2e_2) = f_1(z_1, z_2) + f_2(z_1, z_2)e_2,$$

where $f_1(z_1, z_2) = u_0 + u_1e_1$ and $f_2(z_1, z_2) = u_2 + u_3e_1$ are complex valued functions with real valued functions $u_l = u_l(x_0, x_1, x_2, x_3)$, $(l = 0, 1, 2, 3)$ are real valued functions.

Remark 1. From the definition of differential operators, we have the following equations:

$$\begin{aligned}
\frac{\partial f}{\partial Z} &= \Big(\frac{\partial f_1}{\partial z_1}+\frac{\partial f_2}{\partial z_2}\Big)+\Big(\frac{\partial f_2}{\partial z_1}+\frac{\partial f_1}{\partial z_2}\Big)e_2,\\
\frac{\partial f}{\partial Z^{\dagger_1}} &= \Big(\frac{\partial f_1}{\partial z_1}-\frac{\partial f_2}{\partial z_2}\Big)+\Big(\frac{\partial f_2}{\partial z_1}-\frac{\partial f_1}{\partial z_2}\Big)e_2,\\
\frac{\partial f}{\partial Z^{\dagger_2}} &= \Big(\frac{\partial f_1}{\partial \overline{z_1}}+\frac{\partial f_2}{\partial \overline{z_2}}\Big)+\Big(\frac{\partial f_2}{\partial \overline{z_1}}+\frac{\partial f_1}{\partial \overline{z_2}}\Big)e_2,\\
\frac{\partial f}{\partial Z^{\dagger_3}} &= \Big(\frac{\partial f_1}{\partial \overline{z_1}}-\frac{\partial f_2}{\partial \overline{z_2}}\Big)+\Big(\frac{\partial f_2}{\partial \overline{z_1}}-\frac{\partial f_1}{\partial \overline{z_2}}\Big)e_2.
\end{aligned}$$

Definition 1. Let Ω be an open set in $\mathcal{CQ}$. A function $f(Z)$ is the rth conic regular (r=1,2,3) in Ω if and only if :

(i) f_1 and f_2 are continuously differential functions in Ω,
(ii) f satisfies the following equations called the rth conic Cauchy-Riemann system:

$$\frac{\partial f}{\partial Z^{\dagger_r}} = 0.$$

Definition 2. Let Ω be an open set in $\mathcal{CQ}$. A function $f(Z)$ is the rth conic harmonic $(r = 1, 2, 3)$ in Ω if and only if :

(i) f_1 and f_2 are continuously differential functions in Ω.
(ii) f satisfies the following equations called the rth conic Cauchy-Riemann system:

$$\Delta_{\dagger_r} f = 0.$$

Remark 2. We have

$$\begin{aligned}
\Delta_{\dagger_1} f &= \frac{\partial^2 f_1}{\partial z_1^2}-\frac{\partial^2 f_1}{\partial z_2^2}+\Big(\frac{\partial^2 f_2}{\partial z_1^2}-\frac{\partial^2 f_2}{\partial z_2^2}\Big)e_2,\\
\Delta_{\dagger_2} f &= \frac{\partial^2 f_1}{\partial z_1\partial \overline{z_1}}+\frac{\partial^2 f_1}{\partial z_2\partial \overline{z_2}}+\frac{\partial^2 f_2}{\partial z_1\partial \overline{z_2}}+\frac{\partial^2 f_2}{\partial z_2\partial \overline{z_1}}\\
&\quad+\Big(\frac{\partial^2 f_2}{\partial z_1\partial \overline{z_1}}+\frac{\partial^2 f_2}{\partial z_2\partial \overline{z_2}}+\frac{\partial^2 f_1}{\partial z_1\partial \overline{z_2}}+\frac{\partial^2 f_1}{\partial z_2\partial \overline{z_1}}\Big)e_2,\\
\Delta_{\dagger_3} f &= \frac{\partial^2 f_1}{\partial z_1\partial \overline{z_1}}-\frac{\partial^2 f_1}{\partial z_2\partial \overline{z_2}}-\frac{\partial^2 f_2}{\partial z_1\partial \overline{z_2}}+\frac{\partial^2 f_2}{\partial z_2\partial \overline{z_1}}\\
&\quad+\Big(\frac{\partial^2 f_2}{\partial z_1\partial \overline{z_1}}-\frac{\partial^2 f_2}{\partial z_2\partial \overline{z_2}}-\frac{\partial^2 f_1}{\partial z_1\partial \overline{z_2}}+\frac{\partial^2 f_1}{\partial z_2\partial \overline{z_1}}\Big)e_2.
\end{aligned}$$

Proposition 3.1. *Let Ω be an open set in $\mathcal{CQ}$. If a function $f(Z)$ is the rth conic regular $(r = 1, 2, 3)$ in Ω, then $f(Z)$ is the rth conic harmonic $(r = 1, 2, 3)$ in Ω.*

Proof. Since f is the 1st conic regular in Ω, it satisfies the equations:

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_2}{\partial z_1} = \frac{\partial f_1}{\partial z_2}.$$

Hence, we have the following result:

$$\begin{aligned}
\Delta_{\dagger_1} f &= \frac{\partial^2 f_1}{\partial z_1^2} - \frac{\partial^2 f_1}{\partial z_2^2} + \Big(\frac{\partial^2 f_2}{\partial z_1^2} - \frac{\partial^2 f_2}{\partial z_2^2}\Big) e_2 \\
&= \frac{\partial^2 f_2}{\partial z_1 \partial z_2} - \frac{\partial f_2}{\partial z_2 \partial z_1} + \Big(\frac{\partial^2 f_1}{\partial z_1 \partial z_2} - \frac{\partial f_1}{\partial z_2 \partial z_1}\Big) e_2 = 0.
\end{aligned}$$

Also, since f is the 2nd conic regular in Ω, it satisfies the equations:

$$\frac{\partial f_1}{\partial \overline{z_1}} = -\frac{\partial f_2}{\partial \overline{z_2}} \quad \text{and} \quad \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial f_1}{\partial \overline{z_2}}.$$

Hence, we have

$$\begin{aligned}
\Delta_{\dagger_2} f &= -\frac{\partial^2 f_2}{\partial z_1 \partial \overline{z_2}} - \frac{\partial^2 f_2}{\partial z_2 \partial \overline{z_1}} + \frac{\partial f_2}{\partial z_1 \partial \overline{z_2}} + \frac{\partial^2 f_2}{\partial z_2 \partial \overline{z_1}} \\
&\quad + \Big(-\frac{\partial^2 f_1}{\partial z_1 \partial \overline{z_2}} - \frac{\partial^2 f_1}{\partial z_2 \partial \overline{z_1}} + \frac{\partial^2 f_1}{\partial z_1 \partial \overline{z_2}} + \frac{\partial^2 f_1}{\partial z_2 \partial \overline{z_1}}\Big) e_2 = 0.
\end{aligned}$$

Since f is the 3rd conic regular in Ω, it satisfies the equations

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial f_2}{\partial \overline{z_2}} \quad \text{and} \quad \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial f_1}{\partial \overline{z_2}}.$$

Hence, we have

$$\begin{aligned}
\Delta_{\dagger_3} f &= \frac{\partial^2 f_2}{\partial z_1 \partial \overline{z_2}} - \frac{\partial^2 f_2}{\partial z_2 \partial \overline{z_1}} - \frac{\partial^2 f_2}{\partial z_1 \partial \overline{z_2}} + \frac{\partial^2 f_2}{\partial z_2 \partial \overline{z_1}} \\
&\quad + \Big(\frac{\partial^2 f_1}{\partial z_1 \partial \overline{z_2}} - \frac{\partial^2 f_1}{\partial z_2 \partial \overline{z_1}} - \frac{\partial^2 f_1}{\partial z_1 \partial \overline{z_2}} + \frac{\partial^2 f_1}{\partial z_2 \partial \overline{z_1}}\Big) e_2 = 0.
\end{aligned}$$

□

Definition 3. Let Ω in $\mathbb{C}^2$ be an open set with a $\mathcal{C}^2$ boundary. Let $\Omega = \{Z \mid \rho(Z) < 0\}$, where $\rho \in \mathcal{C}^2$ is in a neighborhood of $\overline{\Omega}$ and grad $\rho \neq 0$ on $b\Omega$. If

$$\sum_{j,k=1}^{2} \frac{\partial^2 \rho(Z)}{\partial z_j \partial \overline{z_k}} w_j \overline{w_k} \geq 0,$$

for all $z \in b\Omega$ and $w \in \mathbb{C}^2$ satisfying

$$\sum_{j,k=1}^{2} \frac{\partial \rho(Z)}{\partial z_j} w_j = 0,$$

then Ω is pseudoconvex.

Consider an automorphism γ_1:

$$\gamma_1(z_1, z_2) := (\overline{z_1}, \overline{z_2})$$

of $\mathbb{C}^2$. If $\gamma_1(\Omega)$ is a pseudoconvex domain of the space $\mathbb{C}^2$ of two complex variables $\overline{z_1}, \overline{z_2}$, then a domain $\Omega \in \mathbb{C}^2 \cong \mathcal{CQ}$ is said to be pseudoconvex with respect to the complex variables z_1, z_2.

Also, consider an automorphism γ_2:

$$\gamma_2(z_1, z_2) := (z_1, z_2)$$

of $\mathbb{C}^2$. A domain $\Omega \in \mathbb{C}^2 \cong \mathcal{CQ}$ is said to be pseudoconvex with respect to the complex variables z_1, z_2 if $\gamma_2(\Omega)$ is a pseudoconvex domain of the space $\mathbb{C}^2$ of two complex variables z_1, z_2.

Theorem 3.2. *Let Ω be a domain in $\mathcal{CQ}$, which is a pseudoconvex domain with respect to the complex variables $\overline{z_1}, \overline{z_2}$ and let $g_1(z_1, z_2)$ be complex-valued functions of class $\mathcal{C}^2$ on Ω satisfying the 1st conic Cauchy-Riemann system. Then there exist 1st conic conjugate harmonic functions $g_2(z_1, z_2)$ of class $\mathcal{C}^2$ on Ω such that $g(Z)$ is the 1st conic regular function in Ω.*

Proof. We consider the 1-forms and the differential operator on $\gamma_1(\Omega)$:

$$\psi := \frac{\partial g_1}{\partial z_2} dz_1 + \frac{\partial g_1}{\partial z_1} dz_2$$

and

$$\delta = \frac{\partial}{\partial z_1} dz_1 + \frac{\partial}{\partial z_2} dz_2.$$

We compute the operator δ from the left-hand side of the 1-forms ψ on $\gamma_1(\Omega)$:

$$\delta\psi \quad = \quad (\frac{\partial}{\partial z_1}\frac{\partial g_1}{\partial z_1} - \frac{\partial}{\partial z_2}\frac{\partial g_1}{\partial z_2}) dz_1 \wedge dz_2.$$

By the 1st conic Cauchy-Riemann system, the coefficient vanishes. From Krantz [9], the δ-closed form ψ of z_1 and z_2 are δ-exact form on $\gamma_1(\Omega)$. Since Ω is a pseudoconvex domain, there exists the 1st conic conjugate harmonic function g_2 of class $\mathcal{C}^\infty$ in Ω, where $\overline{\partial}$-closed form $\gamma_1^{-1}\psi = \overline{\partial} g_2$ on Ω of $\overline{z_1}$ and $\overline{z_2}$ are $\overline{\partial}$-exact $(0,1)$-forms on Ω such that $g(Z)$ is the 1st conic regular function in Ω. □

Corollary 3.3. *Let Ω be a domain in $\mathcal{CQ}$, which is a pseudoconvex domain with respect to z_1 and z_2. Let $g_1(z_1, z_2)$ be complex-valued functions of class $\mathcal{C}^2$ on Ω satisfying the rth conic Cauchy-Riemann system $(r = 2, 3)$. Then there exist rth conic conjugate harmonic functions $g_2(z_1, z_2)$ of class $\mathcal{C}^2$ on Ω such that $g(Z)$ is a rth conic regular function in Ω $(r = 2, 3)$.*

Proof. We consider the 1-forms and the differential operator on $\gamma_2(\Omega)$:

$$\psi := \frac{\partial g_1}{\partial \overline{z_2}} d\overline{z_1} + \frac{\partial g_1}{\partial \overline{z_1}} d\overline{z_2}$$

and

$$\delta = \frac{\partial}{\partial \overline{z_1}} d\overline{z_1} + \frac{\partial}{\partial \overline{z_2}} d\overline{z_2}.$$

We operate the operator δ from the left-hand side of the 1-forms ψ on $\gamma_2(\Omega)$:

$$\delta\psi \quad = \quad (\frac{\partial}{\partial \overline{z_1}} \frac{\partial g_1}{\partial \overline{z_1}} - \frac{\partial}{\partial \overline{z_2}} \frac{\partial g_1}{\partial \overline{z_2}}) d\overline{z_1} \wedge d\overline{z_2}.$$

By the rth conic Cauchy-Riemann system $(r = 2, 3)$, the coefficient vanishes. Similarly, from Krantz [9], the δ-closed form ψ of z_1 and z_2 are δ-exact form on $\gamma(\Omega)$. Since Ω is a pseudoconvex domain, there exists the rth conic conjugate harmonic function g_2 of class $\mathcal{C}^\infty$ in Ω, where $\overline{\partial}$-closed form $\gamma_2^{-1}\psi = \overline{\partial} g_2$ on Ω of z_1 and z_2 are $\overline{\partial}$-exact $(0, 1)$-forms on Ω such that $g(Z)$ is the rth conic regular function in Ω. □

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