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OPTIMAL CONTROL FOR BELOUSOV-ZHABOTINSKII REACTION MODEL

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Abstract. This paper is concerned with the optimal control problem for Belousov-Zhabotinskii reaction model. That is, we show the existence of the global weak solution. We also show that the existence of the optimal control.

1. Introduction

In this paper we are concerned with the following optimal control problem:

$$(\mathbf{P}) \qquad \text{minimize} \quad J(u)$$

with the cost functional $J(u)$ of the form

$$J(u) = \int_0^T \|y(u) - y_d\|^2_{H^1(I)} dt + \int_0^T \|\rho(u) - \rho_d\|^2_{H^1(I)} dt + \gamma \|u\|^2_{H^1(0,T)}, \quad u \in H^1(0,T),$$

where $y = y(u)$ and $\rho = \rho(u)$ is governed by Belousov-Zhabotinskii reaction model

$$\begin{aligned}
&\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} + \frac{1}{\epsilon^2}\Big[y(1-y) - c(\rho+u)\Big(\frac{y-q}{y+q}\Big)\Big] && \text{in } I\times(0,T],\\
&\frac{\partial \rho}{\partial t} = b\frac{\partial^2 \rho}{\partial x^2} + \frac{1}{\epsilon}(y-\rho) \qquad \text{in } I\times(0,T], && (1.1)\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = \frac{\partial \rho}{\partial x}(0,t) = \frac{\partial \rho}{\partial x}(L,t) = 0 && \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \qquad \text{in } I.
\end{aligned}$$

Here, $I = (0, L)$ is a bounded interval in R. $y(x,t)$ and $\rho(x,t)$ denote the concentrations in a vessel of $HBrO_2$ and Ce^{4+} at $x \in I$ and a time $t \in [0,T]$, respectively. $a > 0$ and $b > 0$ represent the diffusion rate of each species.

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Finally, ϵ, q and c are positive constants where $0 < q < 1$ and $0 < \epsilon \leq 1$. The control term $u(t)$ denotes a light induced bromide production rate to intensity of illumination at a time $t \in [0, T]$([10], [11], [12], [13]).

Belousov-Zhabotinskii reaction models is known as a typical phenomenon of self-organization in the chemical reactions([8]). The model (1.1) was introduced by Keener and Tyson [6] for investigating the mechanics of the Belousov-Zhabotinskii reaction models which is considered to consists of more than ten elementary chemical reactions.

The optimal control problem for the reaction diffusion model is studied in many papers([2], [4], [5], [9]). In particular, Ryu and Yagi [9] studied the optimal control problem for the chemotaxis model of non-monotone type. In this paper, we show the existence of the global weak solution of (1.1). We also show that the existence of the optimal control.

The paper is organized as follows. Section 2 is a preliminary section. In Section 3, we show the existence of the golbal weak solutions. Section 4 show the existence of the optimal control.

2. Preliminaries

Let I be an interval in the real line $\mathbf{R}$. $L^p(I; \mathcal{H})$, $1 \leq p \leq \infty$, denotes the L^p space of measurable functions in I with values in a Hilbert space $\mathcal{H}$. $\mathcal{C}(I; \mathcal{H})$ denotes the space of continuous functions in I with values in $\mathcal{H}$. For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by $a, b, c, \epsilon, \gamma$, m, l and I. In a case when C depends also on some parameter, say θ, it will be denoted by C_θ.

We shall state some inequalities on the Sobolev spaces ([1]). When $s > \frac{1}{2}$, $H^s(I) \subset \mathcal{C}(\bar{I})$ with the estimate

$$\|\cdot\|_{\mathcal{C}} \leq C_s \|\cdot\|_{H^s}.$$

In particular, $H^1(I) \subset L^q(I)$ with

$$\|\cdot\|_{L^q} \leq C_{p,q} \|\cdot\|_{H^1}^r \|\cdot\|_{L^p}^{1-r}, \tag{2.1}$$

where $1 \leq p < q \leq \infty$, $r = \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{p} + \frac{1}{2}}$.

We take the identification of $L^2(I)$ and $(L^2(I))'$ and consider that $H^1(I) \subset L^2(I) \subset (H^1(I))'$. Then, $L^{q'}(I) \subset (H^1(I))'$ for every $q' \in (1, \infty]$ with

$$\|y\|_{(H^1)'} \leq C_{q'} \|y\|_{L^{q'}}, \quad y \in L^{q'}(I). \tag{2.2}$$

3. Global solutions

We set two product Hilbert spaces $\mathcal{V} \subset \mathcal{H}$ as

$$\mathcal{V} = H^1(I) \times H^1(I), \quad \mathcal{H} = L^2(I) \times L^2(I).$$

By identifying $\mathcal{H}$ with its dual space, we consider $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$. It is then seen that

$$\mathcal{V}' = (H^1(I))' \times (H^1(I))'.$$

We set also a symmetric bilinear form on $\mathcal{V} \times \mathcal{V}$:

$$a(Y, \widetilde{Y}) = \left(A_1^{1/2} y, A_1^{1/2} \tilde{y}\right)_{L^2} + \left(A_2^{1/2} y, A_2^{1/2} \tilde{y}\right)_{L^2}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix}, \widetilde{Y} = \begin{pmatrix} \tilde{y} \\ \tilde{\rho} \end{pmatrix} \in \mathcal{V},$$

where $A_1 = -a\frac{\partial^2}{\partial x^2} + 1$ and $A_2 = -b\frac{\partial^2}{\partial x^2} + 1$ with the same domain $\mathcal{D}(A_i) = H_n^2(I) = \{z \in H^2(I); \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(L) = 0\}$ $(i = 1, 2)$. Obviously, the form satisfies

$$|a(Y, \widetilde{Y})| \le M\|Y\|_{\mathcal{V}}\|\widetilde{Y}\|_{\mathcal{V}}, \quad Y, \widetilde{Y} \in \mathcal{V}, \tag{a.i}$$

$$a(Y, Y) \ge \delta\|Y\|_{\mathcal{V}}^2, \quad Y \in \mathcal{V} \tag{a.ii}$$

with some δ and $M > 0$. This form then defines a linear isomorphism $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$.

We consider the following problem

$$\begin{aligned} &\frac{dY}{dt} + AY = F_u(Y), \quad 0 < t \le T, \\ &Y(0) = Y_0 \end{aligned} \tag{3.1}$$

in the space $\mathcal{V}'$. Here, $F_u(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F_u(Y) = \begin{pmatrix} y + \epsilon^{-2}\left[y(1-y) - c(\rho + u)\left(\frac{y-q}{|y|+q}\right)\right] \\ \epsilon^{-1} y + (1 - \epsilon^{-1})\rho \end{pmatrix}. \tag{3.2}$$

Here, Y_0 is defined by $Y_0 = \binom{y_0}{\rho_0}$. $U_{ad} = \{u \in H^1(0, T); \|u\|_{H^1(0,T)} \le m,\ 0 \le u(t) \le l\}$.

For $u \in U_{ad}$, $F_u(\cdot)$ satisfies the following conditions:

Lemma 3.1. (f.i) *For each $\eta > 0$, there exists an increasing continuous function $\phi_\eta : [0, \infty) \to [0, \infty)$ such that*

$$\|F_u(Y)\|_{\mathcal{V}'} \le \eta\|Y\|_{\mathcal{V}} + \phi_\eta(\|Y\|_{\mathcal{H}}), \quad Y \in \mathcal{V},\ a.e.\ (0, T);$$

(f.ii) *For each $\eta > 0$, there exists an increasing continuous function $\psi_\eta : [0, \infty) \to [0, \infty)$ such that*

$$\begin{aligned} &\|F_u(\widetilde{Y}) - F_u(Y)\|_{\mathcal{V}'} \le \eta\|\widetilde{Y} - Y\|_{\mathcal{V}} \\ &+ (\|\widetilde{Y}\|_{\mathcal{V}} + \|Y\|_{\mathcal{V}} + 1)\psi_\eta(\|\widetilde{Y}\|_{\mathcal{H}} + \|Y\|_{\mathcal{H}})\|\widetilde{Y} - Y\|_{\mathcal{H}}, \quad \widetilde{Y}, Y \in \mathcal{V},\ a.e.\ (0, T). \end{aligned}$$

Proof. Indeed, by (2.1), (2.2), it is seen that

$$\begin{aligned}\|y^2\|_{(H^1(I))'} &\le C\|y^2\|_{L^{\frac{3}{2}}(I)} \le C\|y\|_{L^2(I)}^{\frac{2}{3}}\|y\|_{L^4(I)}^{\frac{4}{3}} \\ &\le C\|y\|_{L^2(I)}^{\frac{2}{3}}(\|y\|_{L^2(I)}^{\frac{1}{2}}\|y\|_{H^1(I)}^{\frac{1}{2}})^{\frac{4}{3}} \le \eta\|y\|_{H^1(I)} + C_\eta\|y\|_{L^2(I)}^4, \quad y \in H^1(I)\end{aligned}$$

and

$$\Big\|\frac{y\rho}{|y|+q}\Big\|_{(H^1(I))'} \le C\|\rho\|_{L^2(I)}, \quad y,\ \rho \in H^1(I).$$

Hence, the condition (f.i) is fulfilled.

On the other hand, since

$$\Big|\frac{y\rho}{|y|+q} - \frac{\bar{y}\bar{\rho}}{|\bar{y}|+q}\Big| \le C\Big[|\rho-\bar{\rho}| + (|\rho|+|\bar{\rho}|)|y-\bar{y}|\Big],$$

it is seen that for $y, \bar{y}, \rho, \bar{\rho} \in H^1(I)$,

$$\begin{aligned}&\Big\|\frac{y\rho}{|y|+q} - \frac{\bar{y}\bar{\rho}}{|\bar{y}|+q}\Big\|_{(H^1(I))'} \\ &\le C\Big\|\frac{y\rho}{|y|+q} - \frac{\bar{y}\bar{\rho}}{|\bar{y}|+q}\Big\|_{L^{\frac{3}{2}}(I)} \\ &\le C\Big(\|\rho-\bar{\rho}\|_{L^2(I)} + \|(|\rho|+|\bar{\rho}|)|y-\bar{y}|\|_{L^{\frac{3}{2}}(I)}\Big) \\ &\le C\Big(\|\rho-\bar{\rho}\|_{L^2(I)} + (\|\rho\|_{L^6(I)} + \|\bar{\rho}\|_{L^6(I)})\|y-\bar{y}\|_{L^2(I)}\Big) \\ &\le C\Big(\|\rho-\bar{\rho}\|_{L^2(I)} + (\|\rho\|_{H^1(I)} + \|\bar{\rho}\|_{H^1(I)})\|y-\bar{y}\|_{L^2(I)}\Big)\end{aligned}$$

with the use of (2.1) and (2.2). Hence, the condition (f.ii) is fulfilled. □

In view of above fact, we are led the space of initial values as

$$\mathcal{K} = \Big\{\begin{pmatrix} y_0 \\ \rho_0 \end{pmatrix} \in \mathcal{H}; 0 \le y_0 \in L^2(I) \text{ and } 0 \le \rho_0 \in L^2(I)\Big\}.$$

We then obtain the local existence of the weak solution (for the proof, see Ryu and Yagi [9]).

Theorem 3.2. *Let* (a.i), (a.ii), (f.i), *and* (f.ii) *be satisfied. Then, for any* $Y_0 \in \mathcal{K}$ *and* $u \in U_{ad}$, (3.1) *has a unique weak solution*

$$Y \in H^1(0, T(Y_0); \mathcal{V}') \cap \mathcal{C}([0, T(Y_0)]; \mathcal{H}) \cap L^2(0, T(Y_0); \mathcal{V}),$$

equivalently,

$$y, \rho \in H^1(0, T(Y_0); (H^1(I))') \cap \mathcal{C}([0, T(Y_0)]; L^2(I)) \cap L^2(0, T(Y_0); H^1(I)).$$

Here, the number $T(Y_0) > 0$ *is determined by the norm* $\|Y_0\|_{\mathcal{H}}$.

Theorem 3.3. *For any* $Y_0 \in \mathcal{K}$ *and* $u \in U_{ad}$, *the weak solution* Y *of* (3.1) *is nonnegative. Therefore* Y *is a weak solution of* (1.1).

Proof. We show nonnegative of solutions, which is proved by the same method in Yagi([12]). We introduce the modified nonlinear operator

$$\overline{F}_u(Y) = \begin{pmatrix} y + \epsilon^{-2}\Big[|y|(1-y) - c(|\rho|+u)\Big(\frac{y-q}{|y|+q}\Big)\Big] \\ \epsilon^{-1}|y| + (1-\epsilon^{-1})\rho \end{pmatrix}.$$

to (3.2). And we consider an auxiliary problem

$$\begin{aligned} &\frac{d\overline{Y}}{dt} + A\overline{Y} = \overline{F}_u(\overline{Y}), \quad 0 < t \le T, \\ &\overline{Y}(0) = Y_0. \end{aligned} \tag{3.3}$$

Then, we also know that $\overline{Y} = \binom{\bar{y}}{\bar{\rho}} \in H^1(0,\overline{T}(Y_0);\mathcal{V}') \cap L^2(0,\overline{T}(Y_0);\mathcal{V})$. Let us verify first that $\bar{y} \ge 0$ by the truncation method. Consider $H(\bar{y})$ is $\mathcal{C}^{1.1}$ cutoff function for $-\infty < \bar{y} < \infty$ given by $H(\bar{y}) = \frac{\bar{y}^2}{2}$ for $-\infty \le \bar{y} < 0$ and $H(\bar{y}) = 0$ for $0 \le \bar{y} < \infty$.
Since $\bar{y} \in L^2(0,\overline{T}(Y_0);H^1(I))$, we see that $H'(\bar{y}) \in L^2(0,\overline{T}(Y_0);H^1(I))$.

Therefore, if we take $H'(\bar{y})$ as the test function for the first equation in (3.3), we obtain

$$\begin{aligned} &\langle \bar{y}'(t), H'(\bar{y}(t))\rangle_{(H^1(I))',H^1(I)} \\ =&\Big\langle a\frac{\partial^2\bar{y}}{\partial x^2} + \epsilon^{-2}\Big[|\bar{y}|(1-\bar{y}) - c(|\bar{\rho}|+u)\Big(\frac{\bar{y}-q}{|\bar{y}|+q}\Big)\Big], H'(\bar{y}(t))\Big\rangle_{(H^1(I))',H^1(I)} \\ =&a\Big\langle \frac{\partial^2\bar{y}}{\partial x^2}, H'(\bar{y}(t))\Big\rangle_{(H^1(I))',H^1(I)} + \epsilon^{-2}\Big[\int_0^L |\bar{y}|(1-\bar{y})H'(\bar{y}(t))dx \\ &- c\int_0^L (|\bar{\rho}|+u)\Big(\frac{\bar{y}-q}{|\bar{y}|+q}\Big)H'(\bar{y}(t))dx\Big] \\ =&I_1 + I_2. \end{aligned}$$

Since $I_1 = -a\int_0^L |\frac{\partial H'(\bar{y}(t))}{\partial x}|^2 dx$, we see that $I_1 \le 0$. Since $H'(\bar{y}) \le 0$, $H'(\bar{y})\bar{y} \ge 0$ and $u \ge 0$, it follows that $I_2 \le 0$. Therefore, we obtain

$$\langle \bar{y}'(t), H'(\bar{y}(t))\rangle_{(H^1(I))',H^1(I)} \le 0.$$

If we put

$$\psi(t) = \int_0^L H(\bar{y}(t))dx, \quad 0 \le t \le \overline{T}(Y_0),$$

then we see that

$$\frac{d}{dt}\psi(t) = \langle \bar{y}'(t), H'(\bar{y}(t))\rangle_{(H^1(I))',H^1(I)} \le 0.$$

Therefore, $\psi(t) \le \psi(0)$ for $0 \le t \le \overline{T}(Y_0)$. Thus, $\psi(0) = 0$ implies $\psi(t) = 0$, that is, $\bar{y}(t) \ge 0$ for $0 \le t \le \overline{T}(Y_0)$. Similarily, we obtain that $\rho(t) \ge 0$ for $0 \le t \le \overline{T}(Y_0)$. We conclude that $\overline{F}_u(\overline{Y}) = F_u(\overline{Y})$. Thus we see that $\overline{Y}$ is a local solution of (3.1). By the uniqueness, we see that $\overline{Y} = Y$ for $0 \le t \le \overline{T}(Y_0)$.

Therefore, $Y(t) \geq 0$ for $0 \leq t \leq \overline{T}(Y_0)$. We can repeat the similar argument until we obtain that $Y(t) \geq 0$ for $0 \leq t \leq T(Y_0)$. Hence, since $|y(t)| = y(t)$, the solution Y of (3.1) is a local solution of the problem (1.1). □

Theorem 3.4. *For any $Y_0 \in \mathcal{K}$ and $u \in U_{ad}$,* (3.1) *has a unique global weak solution*

$$0 \leq Y \in H^1(0,T;\mathcal{V}') \cap \mathcal{C}([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V}).$$

Proof. Let y, ρ be any weak solution as in Theorem 3.2 on an interval $[0,S]$. Then, if we use the method as in [12, pp.383-384], we obtain the following estimates

$$\frac{d}{dt}\int_0^L (y^2+\rho^2)dx + \int_0^L (y^2+\rho^2)dx + \int_0^L \Big(a\Big|\frac{\partial y}{\partial x}\Big|^2 + b\Big|\frac{\partial \rho}{\partial x}\Big|^2\Big)dx \leq C\epsilon^{-3}. \quad (3.4)$$

If we solve the differential inequality

$$\frac{d}{dt}\int_0^L (y^2+\rho^2)dx + \int_0^L (y^2+\rho^2)dx \leq C\epsilon^{-3},$$

we have

$$\|y(t)\|^2_{L^2(I)} + \|\rho(t)\|^2_{L^2(I)} \leq \epsilon^{-t}(\|y_0\|^2_{L^2(I)} + \|\rho_1\|^2_{L^2(I)}) + C\epsilon^{-3}, \quad 0 \leq t \leq S. \quad (3.5)$$

If we use (3.4), we obtain

$$\int_0^t (\|y(s)\|^2_{H^1(I)} + \|\rho(s)\|^2_{H^1(I)})ds \leq (\|y_0\|^2_{L^2(I)} + \|\rho_0\|^2_{L^2(I)}) + Ct\epsilon^{-3}, \quad 0 \leq t \leq S. \quad (3.6)$$

Thus, we take $t_1 \in (0,S)$ so that $y(t_1), \rho(t_1) \in L^2(I)$. By (3.5) and (3.6), we see that $\|y\|_{L^2(t_1,S;H^1(I))\cap L^\infty(t_1,S;L^2(I))}$ and $\|\rho\|_{L^2(t_1,S;H^1(I))\cap L^\infty(t_1,S;L^2(I))}$ do not depend on S. As a consequence, $\|y\|_{H^1(t_1,S;(H^1(I))')}$ and $\|\rho\|_{H^1(t_1,S;(H^1(I))')}$, and $\|y\|_{C([t_1,S];L^2(I))}$ and $\|\rho\|_{C([t_1,S];L^2(I))}$ do not depend on S. This shows that y, ρ can be extended as a weak solution beyond the S. By the standard argument on the extension of the weak solutions, we can then prove the desired result. □

4. Optimal controls

Now, let $T > 0$ be such that for each $u \in U_{ad}$, (3.1) has a unique weak solution $Y(u) \in H^1(0,T;\mathcal{V}') \cap \mathcal{C}([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$. Thus, the problem **(P)** is obviously formulated as follows:

$$(\overline{\mathbf{P}}) \qquad \text{minimize } J(u),$$

where

$$J(u) = \int_0^T \|Y(u) - Y_d\|_{\mathcal{V}}^2 dt + \gamma \|u\|_{H^1(0,T)}^2, \quad u \in U_{ad}.$$

Here, $Y_d = \binom{y_d}{\rho_d}$ is a fixed element of $L^2(0,T;\mathcal{V})$ with $y_d \in L^2(0,T;H^1(I))$ and $\rho_d \in L^2(0,T;H^1(I))$. γ is a positive constant.

Theorem 4.1. *There exists an optimal control $\bar{u} \in U_{ad}$ for $(\overline{\mathbf{P}})$ such that $J(\bar{u}) = \min_{u \in U_{ad}} J(u)$.*

Proof. Let $\{u_n\} \subset U_{ad}$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(u_n) = \min_{u \in U_{ad}} J(u).$$

Since $\{u_n\}$ is bounded in $H^1(0,T)$, we can assume that $u_n \to \bar{u}$ weakly in $H^1(0,T)$. By the compactness of $H^1(0,T) \hookrightarrow L^2(0,T)$, we see that

$$u_n \to \bar{u} \text{ strongly in } L^2(0,T). \tag{4.1}$$

For simplicity, we will write Y_n instead of the solution $Y(u_n)$ of (3.1) corresponding to u_n. Using the similar estimates of Y_n, we see as in the proof of Theorem 3.4 that $Y_n \to \overline{Y}$ weakly in $L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}')$. Since $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, it is shown by [7, Chap. 1, Theorem 5.1] that

$$Y_n \to \overline{Y} \text{ strongly in } L^2(0,T;\mathcal{H}). \tag{4.2}$$

Let us verify that $\overline{Y} = \binom{\bar{y}}{\rho}$ is a solution to (3.1) with the control $\bar{u}$. For any $\Phi = \binom{\phi_1}{\phi_2} \in L^2(0,T;\mathcal{V})$,

$$\int_0^T \langle Y_n'(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt + \int_0^T \langle AY_n(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt = \int_0^T \langle F_{u_n}(Y_n(t)), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt.$$

We first observed that for any $\phi_1 \in L^2([0,T];H^1(I))$,

$$\begin{aligned}
&\langle y_n^2 - \bar{y}^2, \phi_1\rangle_{(H^1(I))',H^1(I)} \\
&\le \Big(\int_0^L |y_n^2 - \bar{y}^2| dx\Big) \|\phi\|_{L^\infty(I)} \\
&\le C\Big(\|y_n\|_{L^2(I)} + \|y\|_{L^2(I)}\Big)\|y_n - y\|_{L^2(I)} \|\phi_1\|_{L^\infty(I)} \\
&\le C\Big(\|y_n\|_{L^2(I)} + \|y\|_{L^2(I)}\Big)\|y_n - y\|_{L^2(I)} \|\phi_1\|_{H^1(I)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_0^T \langle y_n^2 - \bar{y}^2, \phi_1\rangle_{(H^1(I))',H^1(I)} dt \\
&\le C\Big(\|y_n\|_{L^\infty(0,T;L^2(I))} + \|y\|_{L^\infty(0,T;L^2(I))}\Big)\|y_n - y\|_{L^2(0,T;L^2(I))} \|\phi_1\|_{L^2(0,T;H^1(I))}.
\end{aligned}$$

From (4.2), we have

$$y_n^2 \to \bar{y}^2 \text{ weakly in } L^2(0,T;(H^1(I))').$$

On the other hand, we observed that

$$\begin{aligned}
&\Big\langle \rho_n \frac{y_n - q}{y_n + q} - \bar{\rho}\frac{\bar{y} - q}{\bar{y} + q}, \phi_1 \Big\rangle_{(H^1(I))', H^1(I)} \\
&= \Big\langle (\rho_n - \bar{\rho})\frac{y_n - q}{y_n + q}, \phi_1 \Big\rangle_{(H^1(I))', H^1(I)} + \Big\langle \bar{\rho}\frac{2q(y_n - \bar{y})}{(y_n + q)(\bar{y} + q)}, \phi_1 \Big\rangle_{(H^1(I))', H^1(I)} \\
&\leq C\Big(\|\rho_n - \bar{\rho}\|_{L^2(I)} + \|y_n - \bar{y}\|_{L^2(I)}\|\bar{\rho}\|_{L^2(I)}\Big)\|\phi_1\|_{H^1(I)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \rho_n \frac{y_n - q}{y_n + q} - \bar{\rho}\frac{\bar{y} - q}{\bar{y} + q}, \phi_1 \Big\rangle_{(H^1(I))', H^1(I)} dt \\
&\leq C \int_0^T \Big(\|\rho_n - \bar{\rho}\|_{L^2(I)} + \|y_n - \bar{y}\|_{L^2(I)}\|\bar{\rho}\|_{L^2(I)}\Big)\|\phi_1\|_{H^1(I)} dt \\
&\leq C\Big(1 + \|\bar{\rho}\|_{L^\infty(0,T;L^2(I))}\Big)\Big(\|y_n - \bar{y}\|_{L^2(0,T;L^2(I))} \\
&\quad + \|\rho_n - \bar{\rho}\|_{L^2(0,T;L^2(I))}\Big)\|\phi_1\|_{L^2(0,T;H^1(I))}.
\end{aligned}$$

From (4.2), we have

$$\rho_n \frac{y_n - q}{y_n + q} \to \bar{\rho}\frac{\bar{y} - q}{\bar{y} + q} \text{ weakly in } L^2(0,T;(H^1(I))').$$

Similarily, we obtain

$$\begin{aligned}
&\int_0^T \Big\langle u_n \frac{y_n - q}{y_n + q} - \bar{u}\frac{\bar{y} - q}{\bar{y} + q}, \phi_1 \Big\rangle_{(H^1(I))', H^1(I)} dt \\
&= \int_0^T \Big\langle (u_n - \bar{u})\frac{y_n - q}{y_n + q} + \bar{u}\frac{2q(y_n - \bar{y})}{(y_n + q)(\bar{y} + q)}, \phi_1 \Big\rangle_{(H^1(I))', H^1(I)} dt \\
&= \int_0^T \Big\langle \frac{y_n - q}{y_n + q}, \phi_1 \Big\rangle_{(H^1(I))', H^1(I)} (u_n - \bar{u}) dt \\
&\quad + \int_0^T \Big\langle \frac{2q(y_n - \bar{y})}{(y_n + q)(\bar{y} + q)}, \phi_1 \Big\rangle_{(H^1(I))', H^1(I)} \bar{u} dt \\
&\leq C\Big(\|u_n - \bar{u}\|_{L^2(0,T)} + \|\bar{u}\|_{H^1(0,T)}\|y_n - \bar{y}\|_{L^2(0,T;L^2(I))}\Big)\|\phi_1\|_{L^2(0,T;H^1(I))}.
\end{aligned}$$

From (4.1) and (4.2), we have

$$u_n \frac{y_n - q}{y_n + q} \to \bar{u}\frac{\bar{y} - q}{\bar{y} + q} \text{ weakly in } L^2(0,T;(H^1(I))').$$

Therefore, we obtain that

$$\int_0^T \langle \overline{Y}'(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt + \int_0^T \langle A\overline{Y}(t), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt = \int_0^T \langle F_{\bar{u}}(\overline{Y}(t)), \Phi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt.$$

This then shows that $\overline{Y}(t)$ satisfies the equation of (3.1) for almost all $t \in (0,T)$. In a similar way it is also shown that $\overline{Y}(0) = Y_0$, note from [3, Chap. XVIII, Theorem 1] that $\overline{Y} \in \mathcal{C}([0,T];\mathcal{H})$. Hence, $\overline{Y}$ is the unique solution to (3.1) with the control $\bar{u}$; that is, $\overline{Y} = Y(\bar{u})$.

Since $Y_n - Y_d$ is weakly convergent to $\overline{Y} - Y_d$ in $L^2(0,T;\mathcal{V})$, we have:

$$\min_{u\in U_{ad}} J(u) \le J(\bar{u}) \le \liminf_{n\to\infty} J(u_n) = \min_{u\in U_{ad}} J(u).$$

Hence, $J(\bar{u}) = \min_{u\in U_{ad}} J(u)$. □

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