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COMMON COUPLED FIXED POINT FOR HYBRID PAIR OF MAPPINGS UNDER GENERALIZED NONLINEAR CONTRACTION

Bhavana Deshpande and Amrish Handa

Abstract. We establish a coupled coincidence and common coupled fixed point theorem for hybrid pair of mappings under generalized nonlinear contraction. An example supporting to our result has also been cited. We improve, extend and generalize several known results.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X. Let $D(x, A)$ denote the distance from x to $A \subset X$ and H denote the Hausdorff metric induced by d, that is,

$$\begin{aligned} D(x,\ A) &= \inf_{a\in A} d(x,\ a) \\ \text{and } H(A,\ B) &= \max\left\{\sup_{a\in A} D(a,\ B),\ \sup_{b\in B} D(b,\ A)\right\},\ \text{for all } A,\ B \in CB(X). \end{aligned}$$

The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was initiated by Markin [20]. The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to ([2], [7], [8], [9], [10], [11], [13], [14], [15], [16], [21], [25], [26], [27]) and the reference therein. The theory of multivalued mappings has applications in control theory, convex optimization, differential inclusions and economics. Nadler [21] extended the famous Banach Contraction Principle [3] from single-valued mapping to multivalued mapping.

Bhaskar and Lakshmikantham [5] introduced the notion of coupled fixed point and mixed monotone mappings for single valued mappings. Bhaskar and Lakshmikantham [5] established some coupled fixed point theorems and

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applied these results to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [17] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results of Bhaskar and Lakshmikantham [5]. For more details on coupled fixed point theory, we also refer the reader to ([1], [4], [6], [12], [18], [19], [22], [23], [24]).

In [2], Abbas et al. introduced the following for multivalued mappings:

Definition 1. [2]. Let X be a nonempty set, $F : X \times X \to 2^X$ (a collection of all nonempty subsets of X) and g be a self-mapping on X. An element $(x, y) \in X \times X$ is called

(1) a coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.

(2) a coupled coincidence point of hybrid pair $\{F, g\}$ if $g(x) \in F(x, y)$ and $g(y) \in F(y, x)$.

(3) a common coupled fixed point of hybrid pair $\{F, g\}$ if $x = g(x) \in F(x, y)$ and $y = g(y) \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F and g by $C(F, g)$. Note that if $(x, y) \in C(F, g)$, then (y, x) is also in $C(F, g)$.

Definition 2. [2]. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called $w-$compatible if $g(F(x, y)) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.

Definition 3. [2]. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called $F-$weakly commuting at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.

Lemma 1.1. [11]. *Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in CB(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b\in B} d(a, b)$.*

In this paper, we establish a coupled coincidence and common coupled fixed point theorem for hybrid pair of mappings under generalized nonlinear contraction. We improve, extend and generalize the results of Ding et al. [12]. Theorem 14 of Abbas et al. [2] is a special case of our result. An example is also given to validate our result.

2. Main results

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_φ) φ is non-decreasing,

(ii_φ) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, where $\varphi^{n+1}(t) = \varphi^n(\varphi(t))$.

It is clear that $\varphi(t) < t$ for each $t > 0$. In fact, if $\varphi(t_0) \geq t_0$ for some $t_0 > 0$, then, since φ is non-decreasing, $\varphi^n(t_0) \geq t_0$ for all $n \in \mathbb{N}$, which contradicts with $\lim_{n\to\infty} \varphi^n(t_0) = 0$. In addition, it is easy to see that $\varphi(0) = 0$.

Theorem 2.1. *Let (X, d) be a metric space, $F : X \times X \to CB(X)$ and $g : X \to X$ be two mappings. Assume there exists some $\varphi \in \Phi$ such that*

$$\begin{aligned} & H(F(x, y), F(u, v)) \qquad (1) \\ \leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx, gu),\ D(gx, F(x, y)),\ D(gu, F(u, v)), \\ d(gy, gv),\ D(gy, F(y, x)),\ D(gv, F(v, u)), \\ \frac{D(gx, F(u, v))+D(gu, F(x, y))}{2},\ \frac{D(gy, F(v, u))+D(gv, F(y, x))}{2} \end{array}\right\}\right], \end{aligned}$$

for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x, y) \in C(F, g)$ and gx and gy are fixed points of g, that is, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

Proof. Let $x_0, y_0 \in X$ be arbitrary. Then $F(x_0, y_0)$ and $F(y_0, x_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$, because $F(X \times X) \subseteq g(X)$. Since $F : X \times X \to CB(X)$, therefore by Lemma 1.1, there exist $z_1 \in F(x_1, y_1)$ and $z_2 \in F(y_1, x_1)$ such that

$$\begin{aligned} d(gx_1, z_1) &\leq H(F(x_0, y_0), F(x_1, y_1)), \\ d(gy_1, z_2) &\leq H(F(y_0, x_0), F(y_1, x_1)). \end{aligned}$$

Since $F(X \times X) \subseteq g(X)$, therefore there exist $x_2, y_2 \in X$ such that $z_1 = gx_2$ and $z_2 = gy_2$. Thus

$$\begin{aligned} d(gx_1, gx_2) &\leq H(F(x_0, y_0), F(x_1, y_1)), \\ d(gy_1, gy_2) &\leq H(F(y_0, x_0), F(y_1, x_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n,\ y_n)$ and $gy_{n+1} \in F(y_n,\ x_n)$ such that

$$\begin{aligned}
& d(gx_n,\ gx_{n+1}) \\
\leq\ & H(F(x_{n-1},\ y_{n-1}),\ F(x_n,\ y_n)) \\
\leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ D(gx_{n-1},\ F(x_{n-1},\ y_{n-1})), \\ D(gx_n,\ F(x_n,\ y_n)),\ d(gy_{n-1},\ gy_n), \\ D(gy_{n-1},\ F(y_{n-1},\ x_{n-1})),\ D(gy_n,\ F(y_n,\ x_n)), \\ \frac{D(gx_{n-1},\ F(x_n,\ y_n))+D(gx_n,\ F(x_{n-1},\ y_{n-1}))}{2}, \\ \frac{D(gy_{n-1},\ F(y_n,\ x_n))+D(gy_n,\ F(y_{n-1},\ x_{n-1}))}{2} \end{array}\right\}\right] \\
\leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gx_{n-1},\ gx_n),\ d(gx_n,\ gx_{n+1}), \\ d(gy_{n-1},\ gy_n),\ d(gy_{n-1},\ gy_n),\ d(gy_n,\ gy_{n+1}), \\ \frac{d(gx_{n-1},\ gx_{n+1})+d(gx_n,\ gx_n)}{2},\ \frac{d(gy_{n-1},\ gy_{n+1})+d(gy_n,\ gy_n)}{2} \end{array}\right\}\right] \\
\leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}), \\ \frac{d(gx_{n-1},\ gx_{n+1})}{2},\ \frac{d(gy_{n-1},\ gy_{n+1})}{2} \end{array}\right\}\right].
\end{aligned}$$

Thus

$$d(gx_n,\ gx_{n+1}) \leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}), \\ \frac{d(gx_{n-1},\ gx_{n+1})}{2},\ \frac{d(gy_{n-1},\ gy_{n+1})}{2} \end{array}\right\}\right]. \qquad (2)$$

Similarly

$$d(gy_n,\ gy_{n+1}) \leq \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}), \\ \frac{d(gx_{n-1},\ gx_{n+1})}{2},\ \frac{d(gy_{n-1},\ gy_{n+1})}{2} \end{array}\right\}\right]. \qquad (3)$$

Combining (2) and (3), we get

$$\begin{aligned}
& \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} \\
\leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}), \\ \frac{d(gx_{n-1},\ gx_{n+1})}{2},\ \frac{d(gy_{n-1},\ gy_{n+1})}{2} \end{array}\right\}\right] \\
\leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}), \\ \frac{d(gx_{n-1},\ gx_n)+d(gx_n,\ gx_{n+1})}{2}, \\ \frac{d(gy_{n-1},\ gy_n)+d(gy_n,\ gy_{n+1})}{2} \end{array}\right\}\right] \\
\leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}) \end{array}\right\}\right].
\end{aligned}$$

Thus

$$\begin{aligned} & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} \\ \leq\ & \varphi\left[\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}) \end{array}\right\}\right]. \end{aligned} \tag{4}$$

If we suppose that

$$\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}) \end{array}\right\} = \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}.$$

Then, by (4) and by the fact that $\varphi(t) < t$ for all $t > 0$, we have

$$\begin{aligned} & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} \\ \leq\ & \varphi[\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}] \\ <\ & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}, \end{aligned}$$

which is a contradiction. Thus, we must have

$$\max\left\{\begin{array}{c} d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n), \\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}) \end{array}\right\} = \max\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\}.$$

Hence by (4), we have for all $n \in \mathbb{N}$,

$$\begin{aligned} & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} \\ \leq\ & \varphi[\max\{d(gx_{n-1},\ gx_n),\ d(gy_{n-1},\ gy_n)\}] \\ \leq\ & \varphi^n[\max\{d(gx_0,\ gx_1),\ d(gy_0,\ gy_1)\}] \\ \leq\ & \varphi^n(\delta), \end{aligned}$$

where

$$\delta = \max\{d(gx_0,\ gx_1),\ d(gy_0,\ gy_1)\}.$$

Thus

$$\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} \leq \varphi^n(\delta). \tag{5}$$

Without loss of generality, one can assume that $\max\{d(gx_0, gx_1), d(gy_0, gy_1)\} \neq 0$. In fact, if this is not true, then $gx_0 = gx_1 \in F(x_0, y_0)$ and $gy_0 = gy_1 \in F(y_0, x_0)$, that is, (x_0, y_0) is a coupled coincidence point of F and g.

Thus, for $m, n \in \mathbb{N}$ with $m > n$, by triangle inequality and (5), we get

$$\begin{aligned} & d(gx_n,\ gx_{m+n}) \\ \leq\ & d(gx_n,\ gx_{n+1}) + d(gx_{n+1},\ gx_{n+2}) + ... + d(gx_{n+m-1},\ gx_{m+n}) \\ \leq\ & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} \\ & + \max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\} \\ & + ... + \max\{d(gx_{n+m-1},\ gx_{n+m}),\ d(gy_{n+m-1},\ gy_{n+m})\} \\ \leq\ & \varphi^n(\delta) + \varphi^{n+1}(\delta) + ... + \varphi^{n+m-1}(\delta) \\ \leq\ & \sum_{i=n}^{n+m-1} \varphi^i(\delta), \end{aligned}$$

which implies, by (ii_φ), that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Similarly we obtain that $\{gy_n\}$ is also a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, therefore there exist $x,\ y \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \text{ and } \lim_{n\to\infty} gy_n = gy. \tag{6}$$

Now, since $gx_{n+1} \in F(x_n,\ y_n)$ and $gy_{n+1} \in F(y_n,\ x_n)$, therefore by using condition (1), we get

$$D(gx_{n+1},\ F(x,\ y)) \le H(F(x_n,\ y_n),\ F(x,\ y)) \ \le \varphi[\Delta_n], \tag{7}$$

and

$$D(gy_{n+1},\ F(y,\ x)) \ \le H(F(y_n,\ x_n),\ F(y,\ x)) \le \varphi[\Delta_n], \tag{8}$$

where

$$\Delta_n = \max \left\{ \begin{array}{c} d(gx_n,\ gx),\ d(gx_n,\ gx_{n+1}),\ D(gx,\ F(x,\ y)), \\ d(gy_n,\ gy),\ d(gy_n,\ gy_{n+1}),\ D(gy,\ F(y,\ x)), \\ \frac{D(gx_n, F(x,y)) + d(gx, gx_{n+1})}{2},\ \frac{D(gy_n, F(y,x)) + d(gy, gy_{n+1})}{2} \end{array} \right\}.$$

Since $\lim_{n\to\infty} gx_n = gx$ and $\lim_{n\to\infty} gy_n = gy$, therefore there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\Delta_n = \max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\}.$$

Combining this with (7) and (8), we get for all $n > n_0$,

$$\begin{aligned} & \max \{D(gx_{n+1},\ F(x,\ y)),\ D(gy_{n+1},\ F(y,\ x))\} \\ \le\ & \varphi [\max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\}]. \end{aligned} \tag{9}$$

Now, we claim that

$$\max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\} = 0. \tag{10}$$

If this is not true, then

$$\max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\} > 0. \tag{11}$$

Thus, by (9), (11) and by the fact that $\varphi(t) < t$ for all $t > 0$, we get for all $n > n_0$,

$$\begin{aligned} & \max\{D(gx_{n+1},\ F(x,\ y)),\ D(gy_{n+1},\ F(y,\ x))\} \\ \le\ & \varphi [\max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\}] \\ <\ & \max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\}. \end{aligned}$$

Thus, we get for all $n > n_0$,

$$\begin{aligned} & \max \{D(gx_{n+1},\ F(x,\ y)),\ D(gy_{n+1},\ F(y,\ x))\} \\ <\ & \max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\}. \end{aligned} \tag{12}$$

Letting $n \to \infty$ in (12), by using (6), we obtain

$$\begin{aligned} & \max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\} \\ <\ & \max \{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\}, \end{aligned}$$

which is a contradiction. So (10) holds. Thus, it follows that

$$gx \in F(x,\ y) \text{ and } gy \in F(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of F and g. Hence $C(F,\ g)$ is nonempty.

Suppose now that (a) holds. Assume that for some $(x,\ y) \in C(F,\ g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v, \tag{13}$$

where $u,\ v \in X$. Since g is continuous at u and v. We have, by (13), that u and v are fixed points of g, that is,

$$gu = u \text{ and } gv = v. \tag{14}$$

As F and g are $w-$compatible, so

$$(g^n x,\ g^n y) \in C(F,\ g),\ \text{ for all } n \geq 1, \tag{15}$$

that is, for all $n \geq 1$,

$$g^n x \in F(g^{n-1}x,\ g^{n-1}y) \text{ and } g^n y \in F(g^{n-1}y,\ g^{n-1}x). \tag{16}$$

Now, by using (1) and (16), we obtain

$$D(g^n x,\ F(u,\ v))\ \leq H(F(g^{n-1}x,\ g^{n-1}y),\ F(u,\ v)) \leq \varphi[\nabla_n], \tag{17}$$

and

$$D(g^n y,\ F(v,\ u))\ \leq H(F(g^{n-1}y,\ g^{n-1}x),\ F(v,\ u)) \leq \varphi[\nabla_n], \tag{18}$$

where

$$\nabla_n = \max\left\{ \begin{array}{ll} d(g^n x,\ gu),\ D(gu,\ F(u,\ v)), & \frac{D(g^n x,\ F(u,\ v))+d(gu,\ g^n x)}{2}, \\ d(g^n y,\ gv),\ D(gv,\ F(v,\ u)), & \frac{D(g^n y,\ F(v,\ u))+d(gv,\ g^n y)}{2} \end{array} \right\}.$$

By (13) and (14), there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\nabla_n = \max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}.$$

Combining this with (17) and (18), we get for all $n > n_0$,

$$\begin{aligned} & \max\left\{D(g^n x,\ F(u,\ v)),\ D(g^n y,\ F(v,\ u))\right\} \\ \leq\ & \varphi\left[\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right]. \end{aligned} \tag{19}$$

Now, we claim that

$$\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\} = 0. \tag{20}$$

If this is not true, then

$$\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\} > 0. \tag{21}$$

Thus, by (19), (21) and by the fact that $\varphi(t) < t$ for all $t > 0$, we get for all $n > n_0$,

$$\begin{aligned} & \max\{D(g^n x,\ F(u,\ v)),\ D(g^n y,\ F(v,\ u))\} \\ \leq\ & \varphi\left[\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right] \\ <\ & \max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}. \end{aligned}$$

Thus, we get for all $n > n_0$,

$$\begin{aligned} & \max\{D(g^n x,\ F(u,\ v)),\ D(g^n y,\ F(v,\ u))\} \\ < \ & \max\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\}. \end{aligned} \tag{22}$$

On taking limit as $n \to \infty$ in (22), by using (13) and (14), we get

$$\begin{aligned} & \max\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\} \\ < \ & \max\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\}, \end{aligned}$$

which is a contradiction. So (20) holds. Thus, it follows that

$$gu \in F(u,\ v) \text{ and } gv \in F(v,\ u). \tag{23}$$

Now, from (14) and (23), we have

$$u = gu \in F(u,\ v) \text{ and } v = gv \in F(v,\ u),$$

that is, $(u,\ v)$ is a common coupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x,\ y) \in C(F,\ g)$, g is $F-$weakly commuting, that is, $g^2x \in F(gx,\ gy)$ and $g^2y \in F(gy,\ gx)$ and $g^2x = gx$ and $g^2y = gy$. Thus $gx = g^2x \in F(gx,\ gy)$ and $gy = g^2y \in F(gy,\ gx)$, that is, $(gx,\ gy)$ is a common coupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x,\ y) \in C(F,\ g)$ and for some $u,\ v \in X$,

$$\lim_{n\to\infty} g^n u = x \text{ and } \lim_{n\to\infty} g^n v = y. \tag{24}$$

Since g is continuous at x and y. Therefore, by (24), x and y are fixed points of g, that is,

$$gx = x \text{ and } gy = y. \tag{25}$$

Since $(x,\ y) \in C(F,\ g)$. Therefore, by (25), we obtain

$$x = gx \in F(x,\ y) \text{ and } y = gy \in F(y,\ x),$$

that is, $(x,\ y)$ is a common coupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C(F,\ g)) = \{(x,\ x)\}$. Then $\{x\} = \{gx\} = F(x,\ x)$. Hence $(x,\ x)$ is a common coupled fixed point of F and g. ∎

Example 1. Suppose that $X = [0,\ 1]$, equipped with the metric $d : X \times X \to [0,\ +\infty)$ defined as $d(x,\ y) = \max\{x,\ y\}$ and $d(x,\ x) = 0$ for all $x,\ y \in X$. Let $F : X \times X \to CB(X)$ be defined as

$$F(x,\ y) = \begin{cases} \{0\}, & \text{for } x,\ y = 1, \\ \left[0,\ \frac{x^2+y^2}{4}\right], & \text{for } x,\ y \in [0,\ 1), \end{cases}$$

and $g : X \to X$ be defined as

$$g(x) = x^2, \text{ for all } x \in X.$$

Define $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{2}, & \text{for } t \neq 1, \\ \frac{3}{4}, & \text{for } t = 1. \end{cases}$$

Now, for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v \in [0,\ 1)$, we have

Case (a). If $x^2 + y^2 = u^2 + v^2$, then

$$\begin{aligned}
& H(F(x,\ y),\ F(u,\ v)) \\
= \ & \frac{u^2+v^2}{4} \\
\leq \ & \frac{1}{4}\max\{x^2,\ u^2\} + \frac{1}{4}\max\{y^2,\ v^2\} \\
\leq \ & \frac{1}{4}d(gx,\ gu) + \frac{1}{4}d(gy,\ gv) \\
\leq \ & \frac{1}{2}\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\}\right] \\
\leq \ & \varphi\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\}\right].
\end{aligned}$$

Case (b). If $x^2 + y^2 \neq u^2 + v^2$ with $x^2 + y^2 < u^2 + v^2$, then

$$\begin{aligned}
& H(F(x,\ y),\ F(u,\ v)) \\
= \ & \frac{u^2+v^2}{4} \\
\leq \ & \frac{1}{4}\max\{x^2,\ u^2\} + \frac{1}{4}\max\{y^2,\ v^2\} \\
\leq \ & \frac{1}{4}d(gx,\ gu) + \frac{1}{4}d(gy,\ gv) \\
\leq \ & \frac{1}{2}\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\}\right] \\
\leq \ & \varphi\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\}\right].
\end{aligned}$$

Similarly, we obtain the same result for $u^2+v^2 < x^2+y^2$. Thus the contractive condition (1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v \in [0,\ 1)$. Again, for all $x,\ y,\ u,\ v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$, we have

$$\begin{aligned}
& H(F(x,\ y),\ F(u,\ v)) \\
= \ & \frac{x^2 + y^2}{4} \\
\leq \ & \frac{1}{4}\max\{x^2,\ u^2\} + \frac{1}{4}\max\{y^2,\ v^2\} \\
\leq \ & \frac{1}{4}d(gx,\ gu) + \frac{1}{4}d(gy,\ gv) \\
\leq \ & \frac{1}{2}\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v)) + D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(v,\ u)) + D(gv,\ F(y,\ x))}{2} \end{array}\right\}\right] \\
\leq \ & \varphi\left[\max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v)) + D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(v,\ u)) + D(gv,\ F(y,\ x))}{2} \end{array}\right\}\right].
\end{aligned}$$

Thus the contractive condition (1) is satisfied for all $x,\ y,\ u,\ v \in X$ with x, $y \in [0,\ 1)$ and $u,\ v = 1$. Similarly, we can see that the contractive condition (1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v = 1$. Hence, the hybrid pair $\{F,\ g\}$ satisfies the contractive condition (1), for all $x,\ y,\ u,\ v \in X$. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = (0,\ 0)$ is a common coupled fixed point of hybrid pair $\{F,\ g\}$. The function $F : X \times X \to CB(X)$ involved in this example is not continuous at the point $(1,\ 1) \in X \times X$.

Remark 1. We improve, extend and generalize the result of Ding et al. [12] in the following sense:

(i) We prove our result in the settings of multivalued mapping and for hybrid pair of mappings while Ding et al. [12] proved result for single valued mappings.

(ii) To prove the result we consider non complete metric space and the space is also not partially ordered.

(iii) The mapping $F : X \times X \to CB(X)$ is discontinuous and not satisfying mixed g-monotone property.

(iv) The function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ involved in our theorem and example is discontinuous.

(v) Our proof is simple and different from the other results in the existing literature.

If we put $g = I$ (the identity mapping) in the Theorem 2.1, we get the following result:

Corollary 2.2. *Let (X, d) be a complete metric space, $F : X \times X \to CB(X)$ be a mapping. Assume there exists some $\varphi \in \Phi$ such that*

$$\begin{aligned} &H(F(x,\ y),\ F(u,\ v)) \\ \leq\ &\varphi\left[\max\left\{\begin{array}{c} d(x,\ u),\ D(x,\ F(x,\ y)),\ D(u,\ F(u,\ v)), \\ d(y,\ v),\ D(y,\ F(y,\ x)),\ D(v,\ F(v,\ u)), \\ \frac{D(x,F(u,v))+D(u,F(x,y))}{2},\ \frac{D(y,F(v,u))+D(v,F(y,x))}{2} \end{array}\right\}\right], \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$. Then F has a coupled fixed point.

If we put $\varphi(t) = kt$ in Theorem 2.1 where $0 < k < 1$, then we obtain the following result of Abbas et al. [2]:

Corollary 2.3. *Let (X, d) be a metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} &H(F(x,\ y),\ F(u,\ v)) \\ \leq\ &k\max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,F(u,v))+D(gu,F(x,y))}{2},\ \frac{D(gy,F(v,u))+D(gv,F(y,x))}{2} \end{array}\right\}, \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, where $0 < k < 1$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x,\ y) \in C(F,\ g)$ and for some $u,\ v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x,\ y) \in C(F,\ g)$ and gx and gy are fixed points of g, that is, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x,\ y) \in C(F,\ g)$ and for some $u,\ v \in X$.

(d) $g(C(F,\ g))$ is a singleton subset of $C(F,\ g)$.

If we put $g = I$ (the identity mapping) in the Corollary 2.3, we get the following result:

Corollary 2.4. *Let (X, d) be a complete metric space, $F : X \times X \to CB(X)$ be a mapping satisfying*

$$\begin{aligned} &H(F(x,\ y),\ F(u,\ v)) \\ \leq\ &k\max\left\{\begin{array}{c} d(x,\ u),\ D(x,\ F(x,\ y)),\ D(u,\ F(u,\ v)), \\ d(y,\ v),\ D(y,\ F(y,\ x)),\ D(v,\ F(v,\ u)), \\ \frac{D(x,F(u,v))+D(u,F(x,y))}{2},\ \frac{D(y,F(v,u))+D(v,F(y,x))}{2} \end{array}\right\}, \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, where $0 < k < 1$. Then F has a coupled fixed point.

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Bhavana Deshpande
Department of Mathematics, Govt. P. G. Arts and Science College, Ratlam (M.P.), India
E-mail address: bhavnadeshpande@yahoo.com

Amrish Handa
Department of Mathematics, Govt. P. G. Arts and Science College, Ratlam (M.P.), India
E-mail address: amrishhanda83@gmail.com