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CERTAIN INTEGRAL REPRESENTATIONS OF GENERALIZED STIELTJES CONSTANTS $\gamma_k(a)$

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Abstract. A large number of series and integral representations for the Stieltjes constants (or generalized Euler-Mascheroni constants) γ_k and the generalized Stieltjes constants $\gamma_k(a)$ have been investigated. Here we aim at presenting certain integral representations for the generalized Stieltjes constants $\gamma_k(a)$ by choosing to use four known integral representations for the generalized zeta function $\zeta(s,a)$. As a by-product, our main results are easily seen to specialize to yield those corresponding integral representations for the Stieltjes constants γ_k. Some relevant connections of certain special cases of our results presented here with those in earlier works are also pointed out.

1. Introduction and Preliminaries

Throughout this paper let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of real numbers, complex numbers, positive integers and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For simplicity, we also denote $(\log z)^\alpha$ by $\log^\alpha z$.

The Hurwitz (or generalized) zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \quad \left(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \tag{1}$$

whose special case when $a = 1$ is the Riemann zeta function $\zeta(s,1) := \zeta(s)$ defined by (see, *e.g.*, [19, Section 2.3])

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1). \end{cases} \tag{2}$$

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It is known (see, *e.g.*, [19, Section 2.2]) that both the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s,a)$ can be continued meromorphically to the whole complex s-plane, except for a simple pole only at $s=1$ with their respective residue 1, in many different ways, for example, by means of the contour integral representation (see, *e.g.*, [19, p. 156, Eq. (3)]):

$$\zeta(s,a) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}\, e^{-az}}{1-e^{-z}}\, dz, \tag{3}$$

where the contour C is the Hankel loop (*cf.*, *e.g.*, Whittaker and Watson [20, p. 245]), which starts from ∞ along the upper side of the positive real axis, encircles the origin once in the positive (counter-clockwise) direction, and then returns to ∞ along the lower side of the positive real axis.

The Laurent series expansion of $\zeta(s,a)$ centered at its simple pole $s=1$ is given by

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, \gamma_k(a)\, (s-1)^k \quad (s \in \mathbb{C} \setminus \{1\};\ \Re(a) > 0)\,, \tag{4}$$

where $\{\gamma_k(a)\}_{k\in\mathbb{N}_0}$ are known as *generalized Stieltjes constants* (see, *e.g.*, [1, p. 1356]). The series representation (4) is equivalently written in the following form:

$$(s-1)\,\zeta(s,a) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, \gamma_k(a)\, (s-1)^{k+1} \quad (s \in \mathbb{C};\ \Re(a) > 0)\,. \tag{5}$$

We find from (4) and (5), respectively, that

$$\lim_{s\to 1} \left(\frac{\partial^k}{\partial s^k} Z(s,a) \right) = (-1)^k\, \gamma_k(a) \quad (k \in \mathbb{N}_0;\ \Re(a) > 0) \tag{6}$$

and

$$\lim_{s\to 1} \left(\frac{\partial^{k+1}}{\partial s^{k+1}} (s-1)\,\zeta(s,a) \right) = (-1)^k\, (k+1)\, \gamma_k(a) \quad (k \in \mathbb{N}_0;\ \Re(a) > 0)\,, \tag{7}$$

where, for simplicity,

$$Z(s,a) := \zeta(s,a) - \frac{1}{s-1}. \tag{8}$$

Berndt [3, Theorem 1] (see also [1, Eq. (1.2)]) showed that

$$\gamma_k(a) = \lim_{n\to\infty} \left\{ \sum_{j=0}^{n} \frac{\log^k(j+a)}{j+a} - \frac{\log^{k+1}(n+a)}{k+1} \right\} \quad (k \in \mathbb{N}_0;\ 0 < a \leq 1)\,. \tag{9}$$

The Stieltjes constants γ_k ($k \in \mathbb{N}_0$) arise from the following Laurent expansion of the Riemann zeta function $\zeta(s)$ about $s=1$ (see, *e.g.*, [14, pp. 166-169], [15, p. 255] and [19, p. 165]):

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, \gamma_k\, (s-1)^k, \tag{10}$$

where

$$\begin{aligned}\gamma_k &= \lim_{n\to\infty}\left\{\sum_{j=1}^{n}\frac{\log^k j}{j}-\int_1^n\frac{\log^k x}{x}\,dx\right\}\\ &= \lim_{n\to\infty}\left\{\sum_{j=1}^{n}\frac{\log^k j}{j}-\frac{\log^{k+1} n}{k+1}\right\}\quad (k\in\mathbb{N}_0)\end{aligned}\tag{11}$$

and, in particular, γ_0 (denoted by γ) is the Euler-Mascheroni constant (see, for details, [14, Section 1.5] and [19, Section 1.2]):

$$\gamma := \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j}-\log n\right)\cong 0.57721\,56649\,\cdots\,. \tag{12}$$

The Stieltjes constants γ_k are named after Thomas Jan Stieltjes and often referred to as generalized Euler-Mascheroni constants. It is easy to see that

$$\gamma_k(1)=\gamma_k \quad (k\in\mathbb{N}_0)\,. \tag{13}$$

Adell [1] approximated each generalized Stieltjes constants $\gamma_k(a)$ by means of a finite sum involving Bernoulli numbers. Kreminski [17] presented a new approach to high-precision approximation of $\gamma_k(a)$. A remarkably large number of integral formulas for the Euler-Mascheroni constant γ have been presented (see, *e.g.*, [7], [16], and [19, Section 1.2]). The Stieltjes and generalized Stieltjes constants γ_k and $\gamma_k(a)$ $(k\in\mathbb{N}_0)$ have been investigated in various ways, especially for their series and integral representations (see, *e.g.*, [2, 5, 8, 9, 11, 12, 13, 18] and the references cited therein; see also [14, Section 2.21]).

In 1985, using contour integration, Ainsworth and Howell [2] showed that

$$\gamma_k = 2\,\Re\left\{\int_0^\infty\frac{(x-i)\,\log^k(1-ix)}{(1+x^2)\,(e^{2\pi x}-1)}dx\right\}\quad (k\in\mathbb{N}) \tag{14}$$

and

$$\begin{aligned}\gamma=\gamma_0 &= \frac{1}{2}+2\,\Re\left\{\int_0^\infty\frac{(x-i)}{(1+x^2)\,(e^{2\pi x}-1)}dx\right\}\\ &= \frac{1}{2}+2\int_0^\infty\frac{x}{(1+x^2)\,(e^{2\pi x}-1)}dx.\end{aligned}\tag{15}$$

Coffey [9, Proposition 3] (see also Coffey [10, Eq. (2.17)]) found several integral representations for the generalized Stieltjes constants $\gamma_k(a)$, one of which is recalled here:

$$\gamma_k(a)=\frac{1}{2a}\,\log^k a-\frac{\log^{k+1} a}{k+1}+\frac{2}{a}\,\Re\left\{\int_0^\infty\frac{(y/a-i)\,\log^k(a-iy)}{(1+y^2/a^2)\,(e^{2\pi y}-1)}\,dy\right\} \tag{16}$$

$$(\Re(a) > 0;\ k \in \mathbb{N}),$$

which, upon setting $a = 1$, yields (14). By using binomial theorem, we have

$$\begin{aligned}\log^{2k}(a - iy) &= \left\{\frac{1}{2}\ \ln\left(a^2 + y^2\right) - i\ \arctan\left(\frac{y}{a}\right)\right\}^{2k} \\ &= \mathcal{A}_k(a, y) + i\,\mathcal{B}_k(a, y) \quad (k \in \mathbb{N}),\end{aligned} \tag{17}$$

where, for convenience and simplicity,

$$\mathcal{A}_k(a, y) := \sum_{j=0}^{k} \frac{(-1)^j}{2^{2k-2j}} \binom{2k}{2j} \arctan^{2j}\left(\frac{y}{a}\right) \cdot \ln^{2k-2j}\left(a^2 + y^2\right)$$

and

$$\mathcal{B}_k(a, y) := \sum_{j=1}^{k} \frac{(-1)^j}{2^{2k+1-2j}} \binom{2k}{2j-1} \arctan^{2j-1}\left(\frac{y}{a}\right) \cdot \ln^{2k+1-2j}\left(a^2 + y^2\right).$$

From (16) and (17), we obtain a more explicit integral representation for the generalized Stieltjes constants $\gamma_{2k}(a)$:

$$\gamma_{2k}(a) = \frac{1}{2a}\ \ln^{2k} a - \frac{\ln^{2k+1} a}{2k+1} + \frac{2}{a} \int_0^{\infty} \frac{\frac{y}{a}\,\mathcal{A}_k(a, y) + \mathcal{B}_k(a, y)}{\left(1 + \frac{y^2}{a^2}\right)(e^{2\pi y} - 1)}\, dy \quad (a > 0;\ k \in \mathbb{N}). \tag{18}$$

where $\mathcal{A}_k(a, y)$ and $\mathcal{B}_k(a, y)$ are given in (17). Similarly, we have

$$\begin{aligned}\log^{2k+1}(a - iy)) &= \left\{\frac{1}{2}\ \ln\left(a^2 + y^2\right) - i\ \arctan\left(\frac{y}{a}\right)\right\}^{2k+1} \\ &= \mathcal{C}_k(a, y) + i\,\mathcal{D}_k(a, y) \quad (k \in \mathbb{N}_0),\end{aligned} \tag{19}$$

where, for convenience and simplicity,

$$\mathcal{C}_k(a, y) := \sum_{j=0}^{k} \frac{(-1)^j}{2^{2k+1-2j}} \binom{2k+1}{2j} \arctan^{2j}\left(\frac{y}{a}\right) \cdot \ln^{2k+1-2j}\left(a^2 + y^2\right)$$

and

$$\mathcal{D}_k(a, y) := \sum_{j=0}^{k} \frac{(-1)^{j+1}}{2^{2k-2j}} \binom{2k+1}{2j+1} \arctan^{2j+1}\left(\frac{y}{a}\right) \cdot \ln^{2k-2j}\left(a^2 + y^2\right).$$

From (16) and (19), we get a more explicit integral representation for the generalized Stieltjes constants $\gamma_{2k+1}(a)$:

$$\begin{aligned}\gamma_{2k+1}(a) =& \frac{1}{2a}\ \ln^{2k+1} a - \frac{\ln^{2k+2} a}{2k+2} \\ &+ \frac{2}{a} \int_0^{\infty} \frac{\frac{y}{a}\,\mathcal{C}_k(a, y) + \mathcal{D}_k(a, y)}{\left(1 + \frac{y^2}{a^2}\right)(e^{2\pi y} - 1)}\, dy \quad (a > 0;\ k \in \mathbb{N}_0).\end{aligned} \tag{20}$$

where $\mathcal{C}_k(a, y)$ and $\mathcal{D}_k(a, y)$ are given in (19).

Here we aim at presenting certain interesting integral representations of the generalized Stieltjes constants $\gamma_k(a)$ of a similar nature as those in (18) and (20) by mainly using four known integral representations of the generalized zeta function $\zeta(s,a)$. As a by-product, our main results are easily seen to specialize to yield those corresponding integral representations for the Stieltjes constants γ_k. Some relevant connections of certain special cases of our results presented here with those in earlier works are also pointed out.

To do this, we recall the Polygamma functions $\psi^{(n)}(s)$ $(n \in \mathbb{N})$ defined by

$$\psi^{(n)}(s) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(s) = \frac{d^n}{ds^n} \psi(s) \quad \left(n \in \mathbb{N}_0;\ s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \tag{21}$$

where $\psi(s)$ denotes the Psi (or Digamma) function defined by

$$\psi(s) := \frac{d}{ds} \log \Gamma(s) \quad \text{and} \quad \psi^{(0)}(s) = \psi(s) \quad \left(s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{22}$$

A well-known (and potentially useful) relationship between the Polygamma functions $\psi^{(n)}(s)$ and the generalized zeta function $\zeta(s,a)$ is also given by

$$\psi^{(n)}(s) = (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(k+s)^{n+1}} = (-1)^{n+1}\, n!\, \zeta(n+1, s) \tag{23}$$

$$\left(n \in \mathbb{N};\ s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

In particular, we have

$$\psi^{(n)}(1) = (-1)^{n+1}\, n!\, \zeta(n+1) \quad (n \in \mathbb{N}). \tag{24}$$

2. Integral representations for $\gamma_k(a)$

We begin by presenting two formulas asserted by Lemma 1 below.

Lemma 1. *Each of the following formulas holds true:*

$$\lim_{s \to 1} \frac{d^j}{ds^j}\, t^{s-1} = \log^j t \quad (t \in \mathbb{C} \setminus \{0\};\ j \in \mathbb{N}_0). \tag{25}$$

If we define α_j $(j \in \mathbb{N}_0)$ *by*

$$\alpha_j := \lim_{s \to 1} \frac{d^j}{ds^j}\, \Gamma(2-s),$$

then we have a recurrence formula for α_j *as follows:*

$$\alpha_{\ell+1} = \gamma\, \alpha_\ell + \sum_{j=0}^{\ell-1} \frac{\ell!}{j!}\, \alpha_j\, \zeta(\ell - j + 1) \quad (\ell \in \mathbb{N}_0), \tag{26}$$

where an empty sum is understood to be nil throughout this paper, ζ *denotes the Riemann zeta function given in* (2), γ *is the Euler-Mascheroni constant defined by* (12), *and*

$$\alpha_0 = 1 \quad \text{and} \quad \alpha_1 = \gamma. \tag{27}$$

Proof. The formula (25) is straightforward. To prove (26), let $f(s) := \Gamma(2-s)$. The logarithmic derivative of $f(s)$ gives

$$f'(s) = -\Gamma(2-s)\,\psi(2-s), \tag{28}$$

where ψ is the Psi function given in (22). Differentiating each side of (28) ℓ times (Leibniz's generalization of the product rule for differentiation is used for its right-hand side) and taking the limit on each side of the resulting identity as $s \to 1$, we obtain

$$\lim_{s\to 1} f^{(\ell+1)}(s) = -\lim_{s\to 1} \sum_{j=0}^{\ell} \binom{\ell}{j} \left\{ \frac{d^j}{ds^j}\, \Gamma(2-s) \right\} \left\{ \frac{d^{\ell-j}}{ds^{\ell-j}}\, \psi(2-s) \right\},$$

which, upon using (24), yields the desired result (26). Also we have

$$\alpha_0 = \Gamma(1) = 1 \quad \text{and} \quad \alpha_1 = -\Gamma(1)\,\psi(1) = -\psi(1) = \gamma, \tag{29}$$

where γ is the Euler-Mascheroni constant given in (12). □

In addition to the formulas in (29), the next several α_j are given as follows:

$$\begin{aligned} &\alpha_2 = \gamma^2 + \zeta(2); \quad \alpha_3 = \gamma^3 + 3\gamma\,\zeta(2) + 2\,\zeta(3); \\ &\alpha_4 = \gamma^4 + 6\,\gamma^2\,\zeta(2) + 8\,\gamma\,\zeta(3) + \frac{27}{2}\,\zeta(4), \end{aligned} \tag{30}$$

where, for α_4, the following known recurrence formula for $\zeta(2n)$ (see, *e.g.*, [19, p. 167, Eq.(20)]):

$$\zeta(2n) = \frac{2}{2n+1} \sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k) \quad (n \in \mathbb{N} \setminus \{1\}), \tag{31}$$

which can also be used to evaluate $\zeta(2n)$ $(n \in \mathbb{N} \setminus \{1\})$ by recalling the Basler problem $\zeta(2) = \pi^2/6$ (see, *e.g.*, [6] and the references cited therein).

Recall a known contour integral representation of the generalized zeta function $\zeta(s,a)$ (see, *e.g.*, [19, p. 156, Eq.(3)]):

$$\zeta(s,a) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}\, e^{-az}}{1-e^{-z}}\, dz, \tag{32}$$

where the contour C is essentially a Hankel's loop (*cf.*, *e.g.*, Whittaker and Watson [20, p. 245]), which starts from ∞ along the upper side of the positive real axis, encircles the origin once in the positive (counter-clockwise) direction, and then returns to ∞ along the lower side of the positive real axis. Multiplying each side of (32) by $s-1$ and using the fundamental functional relation for the gamma function Γ:

$$\Gamma(s+1) = s\,\Gamma(s), \tag{33}$$

we have

$$(s-1)\,\zeta(s,a) = \frac{\Gamma(2-s)}{2\pi i} \int_C \frac{(-z)^{s-1}\, e^{-az}}{1-e^{-z}}\, dz \quad \left(s \in \mathbb{C};\; a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{34}$$

Differentiating each side of (34), $k+1$ times, with respect to s (Leibniz's generalization of the product rule for differentiation is used for its right-hand side) and taking the limit on both sides of the resulting identity as $s \to 1$, in view of (7), we obtain a contour integral representation of the generalized Stieltjes constants $\gamma_k(a)$ asserted by Theorem 1 below.

Theorem 1. *The following contour integral representation of* $\gamma_k(a)$ *holds true*:

$$\gamma_k(a) = \frac{(-1)^k}{2\pi i\,(k+1)} \sum_{j=0}^{k+1} \binom{k+1}{j} \alpha_j \int_C \frac{\log^{k+1-j}(-z)\, e^{-az}}{1-e^{-z}}\, dz \qquad \left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \tag{35}$$

where α_j *are given in* (26) *and* C *is the Hankel's loop.*

We give two formulas for later use as in the following lemma.

Lemma 2. *The following identities holds true*:

$$\lim_{s\to 1} \left\{ \frac{\partial^j}{\partial s^j} \sin\left(s \arctan \frac{y}{a}\right) \right\} = \arctan^j \left(\frac{y}{a}\right) \sin\left(\arctan \frac{y}{a} + \frac{\pi}{2} j\right) \qquad (j \in \mathbb{N}_0) \tag{36}$$

and

$$\lim_{s\to 1} \left(\frac{\partial^j}{\partial s^j} \frac{a^{1-s}-1}{s-1} \right) = (-1)^{j+1} \frac{\log^{j+1} a}{j+1} \qquad (j \in \mathbb{N}_0)\,. \tag{37}$$

Proof. Here we prove only (37). The other one is easier and direct. Indeed, we have

$$\begin{aligned}
\frac{a^{1-s}-1}{s-1} &= \frac{\exp[(1-s)\log a] - 1}{s-1} \\
&= \frac{1}{s-1} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k\,(s-1)^k\, \log^k a}{k!} - 1 \right\} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\, \log^{k+1} a}{(k+1)!}\, (s-1)^k.
\end{aligned}$$

Then we find that

$$\frac{\partial^j}{\partial s^j} \frac{a^{1-s}-1}{s-1} = \sum_{k=j}^{\infty} \frac{k!}{(k+1)!} (-1)^{k+1} \log^{k+1} a \cdot (s-1)^{k-j},$$

which, upon taking the limit as $s \to 1$, yields (37). □

We find from Hermite's formula for $\zeta(s,a)$ (see, *e.g.*, [19, p. 158, Eq.(12)]) that

$$\begin{aligned} Z(s,a) :=&\zeta(s,a) - \frac{1}{s-1} = \frac{1}{2}a^{-s} + \frac{a^{1-s}-1}{s-1} \\ &+ 2\int_0^\infty \left(a^2+y^2\right)^{-\frac{1}{2}s} \left\{\sin\left(s \arctan\frac{y}{a}\right)\right\} \frac{dy}{e^{2\pi y}-1} \end{aligned} \tag{38}$$

$$\left(s \in \mathbb{C}\setminus\{1\};\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

It is noted that the integral in (38) is an entire function of s.

Using (38) and (6) with the aid of the identities in Lemma 2, as in getting (35), we obtain an integral representation of the generalized Stieltjes constants $\gamma_k(a)$ asserted by Theorem 2 below.

Theorem 2. *The following integral representation for $\gamma_k(a)$ holds true:*

$$\begin{aligned} \gamma_k(a) = &\frac{\log^k a}{2a} - \frac{\log^{k+1} a}{k+1} + 2\,(-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{2^j} \\ &\cdot \int_0^\infty \frac{\log^j\left(a^2+y^2\right)}{\sqrt{a^2+y^2}} \arctan^{k-j}\left(\frac{y}{a}\right) \sin\left(\arctan\frac{y}{a} + \frac{\pi}{2}(k-j)\right) \frac{dy}{e^{2\pi y}-1} \end{aligned} \tag{39}$$

$$\left(k \in \mathbb{N};\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-\right)$$

and

$$\gamma_0(a) = \frac{1}{2a} - \log a + 2\int_0^\infty \frac{\sin\left(\arctan\frac{y}{a}\right)}{\sqrt{a^2+y^2}\,\left(e^{2\pi y}-1\right)}\,dy \quad \left(a \in \mathbb{C}\setminus\mathbb{Z}_0^-\right). \tag{40}$$

If we define β_j by

$$\beta_j := \lim_{s\to 1}\left(\frac{1}{\Gamma(s)}\right)^{(j)} \quad (j \in \mathbb{N}_0), \tag{41}$$

as in getting (26), we have a recurrence formula for β_j (for details, see [5]):

$$\beta_{k+1} = \sum_{j=0}^{k-1} (-1)^{k-j}\,\frac{k!}{j!}\,\zeta(k+1-j)\,\beta_j + \gamma\,\beta_k \quad (k \in \mathbb{N}_0), \tag{42}$$

where $\beta_0 = 1$ and $\beta_1 = \gamma$, and

$$\begin{aligned} &\beta_2 = \gamma^2 - \zeta(2), \quad \beta_3 = \gamma^3 - 3\,\gamma\,\zeta(2) + 2\,\zeta(3), \\ &\beta_4 = \gamma^4 - 6\,\gamma^2\,\zeta(2) + 8\,\gamma\,\zeta(3) + \frac{3}{2}\,\zeta(4). \end{aligned} \tag{43}$$

From a known integral representation for $\zeta(s,a)$ (see, *e.g.*, [19, p. 160, Eq.(22)]), we obtain

$$
\begin{aligned}
Z(s,a) =& \zeta(s,a) - \frac{1}{s-1} = \frac{1}{2}a^{-s} + \frac{a^{1-s}-1}{s-1} \\
&+ \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-at}\, t^{s-1}\, dt \\
& (\Re(s) > -1;\ \Re(a) > 0).
\end{aligned} \tag{44}
$$

Considering (44) and similarly as in getting (35) and (39), we get an integral representation for $\gamma_k(a)$ given in Theorem 3 below.

Theorem 3. *The following integral representation for* $\gamma_k(a)$ *holds true*:

$$
\begin{aligned}
\gamma_k(a) =& \frac{\log^k a}{2a} - \frac{\log^{k+1} a}{k+1} \\
&+ \sum_{j=0}^{k} \binom{k}{j} \beta_j \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-at}\, \log^{k-j} t\, dt
\end{aligned} \tag{45}
$$

$$(k \in \mathbb{N};\ \Re(a) > 0)$$

and

$$
\gamma_0(a) = \frac{1}{2a} - \log a + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-at}\, dt \quad (\Re(a) > 0)\,. \tag{46}
$$

Recall a further generalization of Leibniz's generalization of the product rule for differentiation (see, *e.g.*, [4, p. 1, Entry 1.11.-5]) given in Lemma 3 below.

Lemma 3. *The following derivative formulas hold true*:

$$
\begin{aligned}
\frac{d^n}{dz^n}\, [f_1(z)\, f_2(z) \cdots f_m(z)] &= \sum_{k_1=0}^{n} \binom{n}{k_1} f_1^{(k_1)}(z) \sum_{k_2=0}^{n-k_1} \binom{n-k_1}{k_2} f_2^{(k_2)}(z) \\
\cdots \sum_{k_{m-1}=0}^{n-k_1-\cdots-k_{m-2}} & \binom{n-k_1-\cdots-k_{m-2}}{k_{m-1}} f_{m-1}^{(k_{m-1})}(z)\, f_m^{(n-k_1-\cdots-k_{m-1})}(z),
\end{aligned} \tag{47}
$$

where $m,\ n \in \mathbb{N}$. The special cases of (47) when $m = 2$ and 3 are

$$
\frac{d^n}{dz^n}\, [f_1(z)\, f_2(z)] = \sum_{k_1=0}^{n} \binom{n}{k_1} f_1^{(k_1)}(z)\, f_2^{(n-k_1)}(z) \quad (n \in \mathbb{N}), \tag{48}
$$

which is the Leibniz's generalization of the product rule, and

$$\begin{aligned}&\frac{d^n}{dz^n}\,[f_1(z)\,f_2(z)\,f_3(z)]\\&\quad=\sum_{k_1=0}^{n}\binom{n}{k_1}\sum_{k_2=0}^{n-k_1}\binom{n-k_1}{k_2}\,f_2^{(k_2)}(z)\,f_3^{(n-k_1-k_2)}(z)\quad(n\in\mathbb{N}).\end{aligned}\tag{49}$$

We also have

$$\begin{aligned}\lim_{s\to1}\left\{\frac{\partial^j}{\partial s^j}\cos\left((s-1)\arctan t\right)\right\}&=\arctan^j t\cdot\cos\left(\frac{\pi}{2}j\right)\\&=\frac{1+(-1)^j}{2}\,(-1)^{[j/2]}\,\arctan^j t\quad(j\in\mathbb{N}_0)\,,\end{aligned}\tag{50}$$

where $[x]$ *denotes the greatest integer less than or equal to* $x\in\mathbb{R}$.

From a known integral representation for $\zeta(s,a)$ (see, *e.g.*, [19, p. 160, Eq.(23)]), we get

$$\begin{aligned}&(s-1)\zeta(s,a)=\pi\,2^{s-2}\\&\quad\cdot\int_0^\infty\left[t^2+(2a-1)^2\right]^{\frac{1}{2}(1-s)}\,\frac{\cos\left[(s-1)\arctan\left(\frac{t}{2a-1}\right)\right]}{\cosh^2\left(\frac{1}{2}\pi t\right)}\,dt\end{aligned}\tag{51}$$

$$\left(s\in\mathbb{C};\ \Re(a)>\frac{1}{2}\right).$$

Applying the formula (49) to differentiate each side of (51), $k+1$ times, with respect to s, and taking the limit on both sides of the resulting identity as $s\to1$, and using (50), we obtain an integral representation for $\gamma_k(a)$ given in Theorem 4 below.

Theorem 4. *The following integral representation for* $\gamma_k(a)$ *holds true*:

$$\begin{aligned}\gamma_k(a)=&\frac{\pi}{2(k+1)}\sum_{\ell=0}^{k+1}\binom{k+1}{\ell}\log^\ell 2\int_0^\infty\sum_{j=0}^{k+1-\ell}\binom{k+1-\ell}{j}\frac{1+(-1)^j}{2}\,(-1)^{[j/2]}\\&\cdot\arctan^j\left(\frac{t}{2a-1}\right)\frac{(-1)^{1+\ell+j}}{2^{k+1-\ell-j}}\,\log^{k+1-\ell-j}\left(t^2+(2a-1)^2\right)\\&\cdot\frac{1}{\cosh^2\left(\frac{1}{2}\pi t\right)}\,dt\quad\left(k\in\mathbb{N}_0;\ \Re(a)>\frac{1}{2}\right),\end{aligned}\tag{52}$$

where $[x]$ *denotes the greatest integer less than or equal to* $x\in\mathbb{R}$.

3. Special cases and remarks

In view of the relationship between the Stieltjes constants γ_k and the generalized Stieltjes constants $\gamma_k(a)$ (13), the special cases of (35), (39), (45) and (52) when $a = 1$ yield certain integral representations for the Stieltjes constants γ_k given in Corollary 1 below.

Corollary 1. *Each of the following integral representations for the Stieltjes constants γ_k holds true:*

$$\gamma_k = \frac{(-1)^k}{2\pi i\,(k+1)} \sum_{j=0}^{k+1} \binom{k+1}{j} \alpha_j \int_C \frac{e^{-z}\, \log^{k+1-j}(-z)}{1-e^{-z}}\, dz, \tag{53}$$

where α_j are given in (26) *and C is the Hankel's loop.*

$$\gamma_k = 2\,(-1)^k \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{2^j} \cdot \int_0^\infty \frac{\log^j\left(1+y^2\right)}{\sqrt{1+y^2}} \arctan^{k-j} y\, \sin\left(\arctan y + \frac{\pi}{2}(k-j)\right) \frac{dy}{e^{2\pi y}-1} \quad (k \in \mathbb{N})\,. \tag{54}$$

$$\gamma_k = \sum_{j=0}^{k} \binom{k}{j} \beta_j \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) e^{-t}\, \log^{k-j} t\, dt \quad (k \in \mathbb{N})\,, \tag{55}$$

where β_j are given in (42).

$$\gamma_k = \frac{\pi}{2(k+1)} \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} \log^\ell 2 \int_0^\infty \sum_{j=0}^{k+1-\ell} \binom{k+1-\ell}{j} \frac{1+(-1)^j}{2} (-1)^{[j/2]} \cdot \arctan^j t\, \frac{(-1)^{1+\ell+j}}{2^{k+1-\ell-j}} \log^{k+1-\ell-j}\left(1+t^2\right) \frac{1}{\cosh^2\left(\frac{1}{2}\pi t\right)}\, dt \quad (k \in \mathbb{N}_0)\,, \tag{56}$$

where $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$.

It is noted that the Stieltjes constants in (55) and (54) are seen to be equal to those, respectively, in [5, Eq.(2.14)] and [5, Eq.(2.22)], whose the latter one should be multiplied by 2. A remarkably large number of integral formulas for the Euler-Mascheroni constant γ have been presented (see,*e.g.*, [19, Section 1.2]; see also [7] and references cited therein). Further special cases of (40), (46) and (56) yield some integral representations for the Euler-Mascheroni constant γ given in Corollary 2 below.

Corollary 2. *Each of the following integral representations for the Euler-Mascheroni constant* γ *holds true*:

$$\gamma = \frac{1}{2} + 2 \int_0^\infty \frac{\sin(\arctan y)}{\sqrt{1+y^2}\,(e^{2\pi y}-1)}\, dy. \tag{57}$$

$$\gamma = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + 1 \right) e^{-t}\, dt. \tag{58}$$

$$\gamma = \frac{\pi}{2} \int_0^\infty \log\left(\frac{2}{\sqrt{1+t^2}} \right) \frac{dt}{\cosh^2\left(\frac{1}{2}\pi t\right)}. \tag{59}$$

It is noted that (57) and (58) are known formulas (see, *cf.*, *e.g.*, [19, p.17, Eq.(35) and p. 16, Eq.(9)], respectively).

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