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LIGHTLIKE HYPERSURFACES OF A LORENTZ MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study lightlike hypersurfaces M of a Lorentz manifold $\bar{M}$ with a semi-symmetric non-metric connection subject to the conditions; (1) the screen distribution $S(TM)$ is totally geodesic in M, and (2) the second fundamental form B of M is parallel.

1. Introduction

The notion of semi-symmetric non-metric connection on Riemannian manifolds was introduced by Ageshe and Chafle. In [1], they studied some properties of the curvature tensor of a Riemannian manifold endowed with a semi-symmetric non-metric connection. In [2], they gave basic properties of submanifolds of a Riemannian manifold endowed with a semi-symmetric non-metric connection. Yasar, Cöken and Yücesan [6] studied lightlike hypersurfaces in a semi-Riemannian manifold endowed with a semi-symmetric non-metric connection. They found the condition that the Ricci type tensor of a lightlike hypersurface of such a semi-Riemannian manifold be symmetric.

In this paper, we study lightlike hypersurfaces M of a Lorentz manifold $\bar{M}$ endowed with a semi-symmetric non-metric connection subject to the conditions; (1) the screen distribution $S(TM)$ is totally geodesic in M, and (2) the second fundamental form B of M is parallel. We prove the following result:

Theorem 1.1. *Let M be a lightlike hypersurface of a Lorentz manifold $\bar{M}$ admitting a semi-symmetric non-metric connection. If the screen distribution $S(TM)$ is totally geodesic in M and the second fundamental form B of M is parallel, then M is locally a product manifold $L \times M_o \times M_\lambda$, where L is a null curve tangent to the radical distribution $Rad(TM)$, and M_o and M_λ are leaves of some integrable distributions of M.*

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2. Semi-symmetric non-metric connection

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. A connection $\bar{\nabla}$ on $\bar{M}$ is called a *semi-symmetric non-metric connection* [1] if $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_X \bar{g})(Y, Z) = -\pi(Y)\bar{g}(X, Z) - \pi(Z)\bar{g}(X, Y), \tag{2.1}$$

$$\bar{T}(X, Y) = \pi(Y)X - \pi(X)Y, \tag{2.2}$$

for any vector fields X, Y and Z on $\bar{M}$, where π is a 1-form associated with a non-zero vector field ζ by $\pi(X) = \bar{g}(X, \zeta)$.

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with a semi-symmetric non-metric connection. Then the normal bundle $TM^\perp$ of M is a vector subbundle of TM of rank 1 and coincides the radical distribution $Rad(TM) = TM \cap TM^\perp$ of M. Hence the degenerate metric g on M induced by the semi-Riemannian metric $\bar{g}$ has constant rank $dim\, M - 1$. A complementary vector bundle $S(TM)$ of $Rad(TM)$ in TM is non-degenerate distribution on M, which is called a *screen distribution* on M [4], such that

$$TM = Rad(TM) \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. It is well-known [4] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* with respect to $S(TM)$ respectively. Then $T\bar{M}$ is decomposed as

$$T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM).$$

In the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulas of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.4}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.5}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{2.6}$$

where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, and τ and σ are 1-forms on TM.

Using (2.1), (2.2) and (2.3), we show that

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y) - \pi(Y)g(X,Z) - \pi(Z)g(X,Y), \tag{2.7}$$

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2.8}$$

and B is symmetric on TM, where T is the torsion tensor with respect to the induced connection ∇ and η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N).$$

From the fact $B(X,Y) = \bar{g}(\bar{\nabla}_X Y,\, \xi)$, we know that B is independent of the choice of a screen distribution. Taking $Y = \xi$ to this and using (2.1), we get

$$B(X,\xi) = 0. \tag{2.9}$$

The local second fundamental forms are related to their shape operators by

$$g(A^*_\xi X, Y) = B(X,Y) - \lambda g(X,Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.10}$$

$$g(A_N X, PY) = C(X,PY) - \mu g(X,PY) - \pi(PY)\eta(X), \tag{2.11}$$

$$\bar{g}(A_N X, N) = -\mu\eta(X), \qquad \sigma(X) = \tau(X) - \lambda\eta(X),$$

where $\lambda = \pi(\xi)$ and $\mu = \pi(N)$ are smooth functions. By (2.10), we show that A^*_ξ is $S(TM)$-valued self-adjoint shape operators related to B and satisfies

$$A^*_\xi \xi = 0. \tag{2.12}$$

Remark 1. We say that $S(TM)$ is *totally geodesic* [4] in M if $C = 0$. In this case, from (2.5), (2.6) and (2.12), we show that $Rad(TM)$ and $S(TM)$ are parallel distributions on M. Thus, by the decomposition theorem of de Rham [3], M is locally a product manifold $L \times M^*$ where L is a null curve tangent to $Rad(TM)$ and M^* is a leaf of $S(TM)$.

3. Proof of Theorem 1.1

Under the hypothesis, we show that $S(TM)$ is a Riemannian vector bundle. By Remark 1, M is locally a product manifold $L \times M^*$, where L is a null curve tangent to $Rad(TM)$ and M^* is a leaf of $S(TM)$. Applying ∇_X to $B(Y,\xi) = 0$ and using (2.6), (2.9) and (2.10), we have

$$g(A^*_\xi X, A^*_\xi Y) = \lambda g(A^*_\xi X, Y). \tag{3.1}$$

By (2.12), ξ is an eigenvector field of A^*_ξ corresponding to the eigenvalue 0. As A^*_ξ is $S(TM)$-valued real self-adjoint operator, A^*_ξ have m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of A^*_ξ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$ and $A^*_\xi E_i = \lambda_i E_i$ for each i. Put $X = Y = E_i$ in (3.1), each λ_i is a solution of the equation

$$x^2 - \lambda\, x = 0. \tag{3.2}$$

(3.2) has at most two distinct solutions 0 and λ. Assume that there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = 0$ and $\lambda_{p+1} = \cdots = \lambda_m = \lambda$, by renumbering if necessary.

Case 1. $p = 0$ or $p = m$: As $S(TM)$ is totally geodesic, we have $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$, where $M^* = M_o$ and $M_\lambda = \{x\}$. Thus this theorem is true.

Case 2. $0 < p < m$: Consider the distributions D_o, D_λ, D_o^s and D_λ^s on M;

$$D_o = \{X \in \Gamma(TM) \mid A_\xi^* X = 0 \text{ and } PX \neq 0\}, \qquad D_o^s = PD_o,$$
$$D_\lambda = \{U \in \Gamma(TM) \mid A_\xi^* U = \lambda PU \text{ and } PU \neq 0\}, \qquad D_\lambda^s = PD_\lambda.$$

Clearly we show that $D_o \cap D_\lambda = \{0\}$ and $D_o^s \cap D_\lambda^s = \{0\}$ as $\lambda \neq 0$.

For any $X \in \Gamma(D_o)$ and $U \in \Gamma(D_\lambda)$, we get $A_\xi^* PX = A_\xi^* X = 0$ and $A_\xi^* PU = A_\xi^* U = \lambda PU$. This imply $PX \in \Gamma(D_o^s)$ and $PU \in \Gamma(D_\lambda^s)$. Thus P maps $\Gamma(D_o)$ onto $\Gamma(D_o^s)$ and $\Gamma(D_\lambda)$ onto $\Gamma(D_\lambda^s)$. Since PX and PU are eigenvector fields of the real self-adjoint operator A_ξ^* corresponding to the different eigenvalues 0 and λ respectively, we have $g(PX,\, PU) = 0$. From the facts $g(X,U) = g(PX, PU) = 0$ and $B(X,U) = g(A_\xi^* X, U) + \lambda g(X,U) = \lambda g(X,U) = 0$, we show that $D_o \perp_g D_\lambda$ and $D_o \perp_B D_\lambda$ respectively.

Since $\{E_i\}_{1 \leq i \leq p}$ and $\{E_a\}_{p+1 \leq a \leq m}$ are vector fields of D_o^s and D_λ^s respectively and D_o^s and D_λ^s are mutually orthogonal vector subbundle of $S(TM)$, D_o^s and D_λ^s are non-degenerate distributions of rank p and rank $(m-p)$ respectively. Thus $S(TM) = D_o^s \oplus_{orth} D_\lambda^s$.

From (3.1), we show that $A_\xi^*(A_\xi^* - \lambda P) = (A_\xi^* - \lambda P)A_\xi^* = 0$. Let $Y \in Im\, A_\xi^*$, then there exists $X \in \Gamma(TM)$ such that $Y = A_\xi^* X$. Then we have $(A_\xi^* - \lambda P)Y = 0$ and $Y \in \Gamma(D_\lambda)$. Thus $Im\, A_\xi^* \subset \Gamma(D_\lambda)$. Since the morphism A_ξ^* maps $\Gamma(TM)$ onto $\Gamma(S(TM))$, we have $Im\, A_\xi^* \subset \Gamma(D_\lambda^s)$. By duality, we also have $Im(A_\xi^* - \lambda P) \subset \Gamma(D_o^s)$.

For any $X, Y \in \Gamma(D_o)$ and $U, V \in \Gamma(D_\lambda)$, applying ∇_X to $B(U,V) = 2\lambda g(U,V)$ and ∇_U to $B(X,Y) = \lambda g(X,Y)$ and then, using (2.7), (2.10) and the facts $\nabla B = 0$ and $D_o \perp_g D_\lambda$; $D_o \perp_B D_\lambda$, we have $(X\lambda)g(U,V) = 0$ and $(U\lambda)g(X,Y) = 0$, i.e., $X\lambda = 0$ and $U\lambda = 0$. This imply $Z\lambda = 0$ for all $Z \in \Gamma(D_o \oplus_{orth} D_\lambda)$. Thus λ is a constant on $S(TM)$.

For any $X, Y, Z \in \Gamma(D_o^s)$, applying ∇_Z to $B(X,Y) = \lambda g(X,Y)$ and using (2.7), (2.10) and the facts $\nabla B = 0$ and λ is a constant on $S(TM)$, we have $(\nabla_Z g)(X,Y) = 0$, i.e.,

$$\pi(X)g(Y,Z) + \pi(Y)g(X,Z) = 0. \tag{3.3}$$

Using this and the fact D_o^s is non-degenerate, we have

$$\pi(X)Y = -\pi(Y)X. \tag{3.4}$$

Taking the skew-symmetric part of (3.3) for X and Z, we get $\pi(X)g(Y,Z) = \pi(Z)g(X,Y)$, from which we have

$$\pi(X)Y = \pi(Y)X. \tag{3.5}$$

From (3.4) and (3.6), we obtain $\pi(X) = 0$ for all $X \in \Gamma(D_o^s)$. By duality, we have $\pi(U) = 0$ for all $U \in \Gamma(D_\lambda^s)$. Thus $\pi = 0$ on $S(TM)$ and $\nabla_X g = 0$ for all $X \in \Gamma(S(TM))$.

For any $X,\, Y \in \Gamma(D_o^s)$ and $U,\, V \in \Gamma(D_\lambda^s)$, applying ∇_X to $B(Y,U) = 0$ and ∇_V to $B(Y,U) = 0$ and then, using (2.7), (2.10) and the facts $\nabla B = 0$ and $\nabla_X g = 0$ for all $X \in \Gamma(S(TM))$, we have

$$g(A_\xi^* \nabla_X Y,\, U) = 0, \quad g((A_\xi^* - \lambda P)\nabla_V U,\, Y) = 0.$$

Since D_λ^s is non-degenerate and $Im\, A_\xi^* \subset \Gamma(D_\lambda^s)$, we have $A_\xi^* \nabla_X Y = 0$. Thus $\nabla_X Y \in \Gamma(D_o)$. By duality, we have $\nabla_V U \in \Gamma(D_\lambda)$. As $S(TM)$ is totally geodesic in M, this results imply that $\nabla_X Y \in \Gamma(D_o^s)$ for all $X,\, Y \in \Gamma(D_o^s)$ and $\nabla_V U \in \Gamma(D_\lambda^s)$ for all $U,\, V \in \Gamma(D_\lambda^s)$. Thus D_o^s and D_λ^s are integrable and auto-parallel distributions.

Since the leaf M^* of $S(TM)$ is a Riemannian manifold and $S(TM) = D_o^s \oplus_{orth} D_\lambda^s$, where D_o^s and D_λ^s are auto-parallel distributions with respect to the induced connection ∇ on $S(TM)$, by the decomposition theorem of de Rham [3], we have $M^* = M_o \times M_\lambda$, where M_o and M_λ are leaves of D_o^s and D_λ^s respectively. Thus we have Theorem 1.1.

Concluding remark. Let M be a half lightlike submanifold [5] of a Lorentz manifold $\bar{M}$ with a semi-symmetric non-metric connection subject to the conditions; (1) the screen distribution $S(TM)$ is totally geodesic in M and (2) the second fundamental form B of M is parallel. Then, by a procedure same as for Theorem 1.1 from the structure equations

$$\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X Y + B(X,Y)N + D(X,Y)L,\\
\bar{\nabla}_X N &= -A_N X + \tau(X)N + \rho(X)L,\\
\bar{\nabla}_X L &= -A_L X + \phi(X)N,\\
\nabla_X PY &= \nabla_X^* PY + C(X,PY)\xi,\\
\nabla_X \xi &= -A_\xi^* X - \sigma(X)\xi,\\
(\nabla_X g)(Y,Z) &= B(X,Y)\eta(Z) + B(X,Z)\eta(Y)\\
&\quad - \pi(Y)g(X,Z) - \pi(Z)g(X,Y),\\
T(X,Y) &= \pi(Y)X - \pi(X)Y,\\
B(X,\xi) &= 0, \quad D(X,\xi) = -\epsilon\, \phi(X),\\
g(A_\xi^* X, Y) &= B(X,Y) - \lambda g(X,Y), \quad \bar{g}(A_\xi^* X, N) = 0,
\end{aligned}$$

the following result will be established:

Theorem 3.1. *Let M be a half lightlike submanifold of a Lorentz manifold $\bar{M}$ admitting a semi-symmetric non-metric connection. If the screen distribution $S(TM)$ is totally geodesic in M and the lightlike second fundamental form B of M is parallel, then M is locally a product manifold $L \times M_o \times M_\lambda$, where L is a null curve tangent to the radical distribution $Rad(TM)$, and M_o and M_λ are leaves of some integrable distributions of M.*

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