East Asian Math. J.
Vol. 31 (2015), No. 1, pp. 001–018
http://dx.doi.org/10.7858/eamj.2015.001

CHARACTERIZATIONS OF HEMIRINGS BY $(\in, \in \vee q)$-FUZZY IDEALS

Muhammad Shabir, Yasir Nawaz, and Tahir Mahmood

Abstract. In this paper we characterize different classes of hemirings by the properties of their $(\in, \in \vee q)$-fuzzy ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals and $(\in, \in \vee q)$-fuzzy bi-ideals.

1. Introduction

There are many generalizations of associative rings. Some of them, in particular, nearrings and semirings are very useful for solving problems in different areas of applied mathematics and information sciences. Semirings which are common generalizations of associative rings and distributive lattices are introduced by H. S. Vandiver in 1934 [27]. Semirings are very useful for studying optimization theory, graph theory, theory of discrete event dynamical systems, matrices, determinants, generalized fuzzy computation, automata theory, formal language theory, coding theory, analysis of computer programs, and so on (see [12, 13, 14, 22]). Hemirings, which are semirings with commutative addition and zero element appears in a natural manner in some applications to the theory of automata and formal languages (see [13, 14]).

Ideals play important role in the study of hemirings and are useful for many purposes. But they don't coincide with ring ideals. Thus many results of ring theory have no analogues in semirings using only ideals. In order to overcome this deficiency, Henriksen [15] defined a class of ideals in semirings, called k-ideals. These ideals have the property that if the semiring R is a ring then a subset of R is a k-ideal if and only if it is a ring ideal. A more restricted class of ideals in hemiring is defined by Iizuka [16]. La Torre [19] thoroughly studied h-ideals and k-ideals and established some analogues ring results for semirings.

The concept of fuzzy subset introduced by Zadeh [29] is a useful tool to describe situation in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. Many researchers used this concept to generalize some notions of algebra.

Received August 13, 2011; Revised September 5, 2014; Accepted September 26, 2014.
2010 *Mathematics Subject Classification.* 16Y60, 08A72, 03G25, 03E72.
Key words and phrases. $(\in, \in \vee q)$-fuzzy ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals, $(\in, \in \vee q)$-fuzzy bi-ideals.

(pISSN 1226-6973, eISSN 2287-2833)

In [2] J. Ahsan initiated the study of fuzzy semirings (see also [1]). Fuzzy k-ideals in semirings are studied in [11] and fuzzy h-ideals are studied in [10, 17, 20, 25, 26, 28, 30].

The concept of "belongingness" and "quasicoincidence" of a fuzzy point with a fuzzy set are given in [23, 24]. Many authors used these concepts to generalize some concepts of algebra, for example [3, 4, 5, 6, 7, 8, 18]. In [9, 21] (α, β)-fuzzy ideals of hemirings are defined. In this paper we characterize different classes of hemirings by the properties of their $(\in, \in \vee q)$-fuzzy h-ideals, $(\in, \in \vee q)$-fuzzy h-quasi ideals, and $(\in, \in \vee q)$-fuzzy h-bi-ideals.

2. Preliminaries

A semiring is a non-empty set R together with two associative binary operations addition " $+$ " and multiplication " $\cdot$ " such that " $\cdot$ " is distributive over " $+$ " in R. An element $0 \in R$ is called a zero or additive identity of the semiring $(R, +, \cdot)$ if $0x = x0 = 0$ and $0 + x = x + 0 = x$ for all $x \in R$. An additively commutative semiring with zero is called a hemiring. An element 1 of a hemiring R is called the identity or unity of R if $1.x = x.1 = x$ for all $x \in R$. A hemiring with commutative multiplication is called a commutative hemiring. A non-empty subset A of a hemiring R is called a sub hemiring of R if it contains zero and is closed with respect to the addition and multiplication of R. A non-empty subset I of a hemiring R is called a left (right) ideal of R if I is closed under addition and $RI \subseteq I$ ($IR \subseteq I$). A non-empty subset I of a hemiring R is called an ideal of R if it is both a left ideal and a right ideal of R. A subset Q of a hemiring R is called a quasi-ideal of R if Q is closed under addition and $RQ \cap QR \subseteq Q$. A sub hemiring B of a hemiring R is called a bi-ideal of R if $BRB \subseteq B$. Every one sided ideal of a hemiring R is a quasi-ideal and every quasi-ideal is a bi-ideal but the converse is not true.

A left (right) ideal I of a hemiring R is called a left (right) h-ideal if for all $x, z \in R$ and for any $a, b \in I$ from $x + a + z = b + z$ it follows $x \in I$. A bi-ideal B of a hemiring R is called an h-bi-ideal of R if for all $x, z \in R$ and $a, b \in B$ from $x + a + z = b + z$ it follows $x \in B$ (cf. [28]).

The h-closure $\overline{A}$ of a non-empty subset A of a hemiring R is defined as

$$\overline{A} = \{x \in R \mid x + a + z = b + z \text{ for some } a, b \in A, z \in R\}.$$

A quasi-ideal Q of a hemiring R is called an h-quasi-ideal of R if $\overline{RQ} \cap \overline{QR} \subseteq Q$ and $x + a + z = b + z$ implies $x \in Q$, for all $x, z \in R$ and $a, b \in Q$ (cf. [28]). Every left (right) h-ideal of a hemiring R is an h-quasi-ideal of R and every h-quasi-ideal is an h-bi-ideal of R. However, the converse is not true in general.

For $A \subseteq X$, characteristic function $C_A : X \to \{0, 1\}$ defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

A fuzzy subset f of a universe X is a function from X into the unit closed interval $[0,1]$, that is $f : X \Rightarrow [0,1]$. A fuzzy subset f of a universe X of the form

$$f(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set f of a set X, Pu and Liu [24] gave meaning to the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set f written $x_t \in f$ (resp. $x_t q f$) if $f(x) \geq t$ (resp. $f(x) + t > 1$), and in this case, $x_t \in \vee q f$ (resp. $x_t \in \wedge q f$) means that $x_t \in f$ or $x_t q f$ (resp. $x_t \in f$ and $x_t q f$). To say that $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold. For any two fuzzy subsets f and g of X, $f \leq g$ means that, for all $x \in X$, $f(x) \leq g(x)$. The symbols $f \wedge g$, and $f \vee g$ will mean the following fuzzy subsets of X

$$(f \wedge g)(x) = f(x) \wedge g(x)$$
$$(f \vee g)(x) = f(x) \vee g(x)$$

for all $x \in X$. More generally, if $\{f_i : i \in \Lambda\}$ is a family of fuzzy subsets of X, then $\wedge_{i \in \Lambda} f_i$ and $\vee_{i \in \Lambda} f_i$ are defined by

$$(\wedge_{i \in \Lambda} f_i)(x) = \wedge_{i \in \Lambda}(f_i(x))$$
$$(\vee_{i \in \Lambda} f_i)(x) = \vee_{i \in \Lambda}(f_i(x))$$

and are called the intersection and the union of the family $\{f_i : i \in \Lambda\}$ of fuzzy subsets of X, respectively.

2.1. Definition [28]

Let f and g be two fuzzy subsets of a hemiring R. The h-intrinsic product of f and g is defined by

$$(f \odot_h g)(x) = \begin{cases} \displaystyle\bigvee_{\substack{x + \Sigma_{i=1}^m a_i b_i + z \\ = \Sigma_{j=1}^n a_j' b_j' + z}} \left\{ \left(\bigwedge_{i=1}^m f(a_i) \right) \wedge \left(\bigwedge_{i=1}^m g(b_i) \right) \wedge \left(\bigwedge_{j=1}^n f(a_j') \right) \wedge \left(\bigwedge_{j=1}^n g(b_j') \right) \right\} \\ 0 \qquad \text{if } x \text{ cannot be expressed as } x + \Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n a_j' b_j' + z \end{cases}$$

2.2. Proposition [28]

Let R be a hemiring and f, g, h, k be any fuzzy subsets of R. If $f \leq g$ and $h \leq k$, then $f \odot_h h \leq g \odot_h k$.

2.3. Lemma [28]

Let R be a hemiring and $A, B \subseteq R$. Then we have

(1) $A \subseteq B$ if and only if $C_A \leq C_B$.

(2) $C_A \wedge C_B = C_{A \cap B}$.

(3) $C_A \odot_h C_B = C_{\overline{AB}}$.

2.4. Definition

A fuzzy subset f of a hemiring R is called a fuzzy h-subhemiring of R if for all $x, y, z, a, b \in R$, we have
(i) $f(x+y) \geq \min\{f(x), f(y)\}$,
(ii) $f(xy) \geq \min\{f(x), f(y)\}$,
(iii) $x + a + z = b + z \Rightarrow f(x) \geq \min\{f(a), f(b)\}$.

2.5. Definition [17]

A fuzzy subset f of a hemiring R is called a fuzzy left (right) h-ideal of R if for all $x, y, z, a, b \in R$, we have
(i) $f(x+y) \geq \min\{f(x), f(y)\}$,
(ii) $f(xy) \geq f(y) \quad (f(xy) \geq f(x))$,
(iii) $x + a + z = b + z \Rightarrow f(x) \geq \min\{f(a), f(b)\}$.
A fuzzy subset f of R is called a fuzzy h-ideal of R if it is both a fuzzy left and a fuzzy right h-ideal of R.

2.6. Definition [28]

A fuzzy subset f of a hemiring R is called a fuzzy h-bi-ideal of R if for all $x, y, z, a, b \in R$, we have
(i) $f(x+y) \geq \min\{f(x), f(y)\}$,
(ii) $f(xy) \geq \min\{f(x), f(y)\}$,
(iii) $f(xyz) \geq \min\{f(x), f(z)\}$,
(iv) $x + a + z = b + z \Rightarrow f(x) \geq \min\{f(a), f(b)\}$.

2.7. Definition [28]

A fuzzy subset f of a hemiring R is called a fuzzy h-quasi-ideal of R if for all $x, y, z, a, b \in R$, we have
(i) $f(x+y) \geq \min\{f(x), f(y)\}$,
(ii) $(f \odot_h \mathcal{R}) \wedge (\mathcal{R} \odot_h f) \leq f$
(iii) $x + a + z = b + z \Rightarrow f(x) \geq \min\{f(a), f(b)\}$.
where $\mathcal{R}$ is the fuzzy subset of R mapping every element of R on 1.
Note that if f is a fuzzy left h-ideal (right h-ideal, h-bi-ideal, h-quasi-ideal) , then $f(0) \geq f(x)$ for all $x \in R$.

2.8. Definition [28]

A hemiring R is said to be h-hemiregular if for each $x \in R$, there exist $a, b, z \in R$ such that $x + xax + z = xbx + z$.

2.9. Definition [28]

A hemiring R is said to be h-intra-hemiregular if for each $x \in R$, there exist $a_i, a_i', b_j, b_j', z \in R$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=1}^{n} b_j x^2 b_j' + z$.

2.10. Lemma [28]

If I and L are respectively, right and left h-ideals of a hemiring R, then $\overline{IL} \subseteq I \cap L$.

2.11. Lemma [28]

A hemiring R is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $\overline{IL} = I \cap L$.

2.12. Theorem [28]

A hemiring R is h-hemiregular if and only if for any fuzzy right h-ideal μ and any fuzzy left h-ideal ν of R we have $\mu \odot_h \nu = \mu \wedge \nu$.

2.13. Lemma [28]

A hemiring R is h-intra-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $I \cap L \subseteq \overline{LI}$.

2.14. Theorem [28]

A hemiring R is h-intra-hemiregular if and only if for any fuzzy right h-ideal μ and any fuzzy left h-ideal ν of R we have $\mu \wedge \nu \subseteq \nu \odot_h \mu$.

2.15. Lemma [28]

The following conditions are equivalent for a hemiring R.
(i) R is both h-hemiregular and h-intra-hemiregular.
(ii) $B = \overline{B^2}$ for every h-bi-ideal B of R.
(iii) $Q = \overline{Q^2}$ for every h-quasi-ideal Q of R.

2.16. Lemma [28]

Let R be a hemiring. Then the following conditions are equivalent.
(1) R is h-hemiregular.
(2) $B = \overline{BRB}$ for every h-bi-ideal B of R.
(3) $Q = \overline{QRQ}$ for every h-quasi-ideal Q of R.

2.17. Theorem [28]

The following conditions are equivalent for a hemiring R.
(i) R is both h-hemiregular and h-intra-hemiregular.
(ii) $f = f \odot_h f$ for every fuzzy h-bi-ideal f of R.
(iii) $f = f \odot_h f$ for every fuzzy h-quasi-ideal f of R.

3. $(\in, \in \vee q)$-Fuzzy ideals

3.1. Definition

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy subhemiring of R if it satisfies

$(1')$ $x_t, y_r \in f \Rightarrow (x+y)_{\min\{t,r\}} \in \vee q f$

$(2')$ $x_t, y_r \in f \Rightarrow (xy)_{\min\{t,r\}} \in \vee q f$

for all $x, y \in R$ and $t, r \in (0, 1]$.

3.2. Definition

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy right (left) h-ideal of R if it satisfies $(1')$ and

$(3')$ $x_t \in f$ and $y \in R \Rightarrow (xy)_t \in \vee q f$ ($(4')$ $x_t \in f$ and $y \in R \Rightarrow (yx)_t \in \vee q f$)

$(5')$ $x + a + y = b + y$ and $a_t, b_r \in f \Rightarrow (x)_{\min\{t,r\}} \in \vee q f$

for all $x, y \in R$ and $t, r \in (0, 1]$.

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy h-ideal of R if it is both $(\in, \in \vee q)$-fuzzy right and $(\in, \in \vee q)$-fuzzy left h-ideal of R.

3.3. Definition

An $(\in, \in \vee q)$-fuzzy subhemiring of R is called $(\in, \in \vee q)$-fuzzy h-subhemiring of R if it satisfies $(5')$.

3.4. Definition [21]

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy h-bi-ideal of R if it satisfies $(1')$, $(2')$, $(5')$ and

$(6')$ $x_t, z_r \in f$ implies $(xyz)_{\min\{t,r\}} \in \vee q f$

for all $x, y, z \in R$ and $t, r \in (0, 1]$.

3.5. Definition [21]

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy h-quasi-ideal of R, if it satisfies $(1'), (5')$ and

$(7')$ $x_t \in f \odot \mathcal{R}, x_r \in \mathcal{R} \odot f \Rightarrow (x)_{\min\{t,r\}} \in \vee q f$

for all $x \in R$ and $t, r \in (0, 1]$.

Where $\mathcal{R}$ is the fuzzy subset of R mapping every element of R on 1.

It is shown in [9, 21] that for any fuzzy subset f of R and for all $x, y \in R$, $(1')$ is equivalent to $(1'')$, $(2')$ is equivalent to $(2'')$, $(3')$ is equivalent to $(3'')$ and $(4')$ is equivalent to $(4'')$ where:

$(1'')$ $f(x+y) \geq \min\{f(x), f(y), 0.5\}$

$(2'')$ $f(xy) \geq \min\{f(x), f(y), 0.5\}$

$(3'')$ $f(xy) \geq \min\{f(x), 0.5\}$

$(4'')$ $f(yx) \geq \min\{f(x), 0.5\}$.

It is also shown in [9, 21] that for any fuzzy subset f of R and for all $a, b, x, y \in R$ such that $x + a + y = b + y$, then $(5')$ is equivalent to $(5'')$ where:

$(5'')$ $f(x) \geq \min\{f(a), f(b), 0.5\}$.
Also $(6')$ is equivalent to $(6'')$ and $(7')$ is equivalent to $(7'')$where:
$(6'')$ $f(xyz) \geq \min\{f(x), f(z), 0.5\}$
$(7'')$ $f(x) \geq \min\{(f \odot \mathcal{R})(x), (\mathcal{R} \odot f))(x), 0.5\}$
for all $x, y, z \in R$.
From the above discussion we deduce;

3.6. Definition

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy h-subhemiring of R if it satisfies $(1''), (2'')$ and $(5'')$.

3.7. Definition

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy right (left) h-ideal of R if it satisfies $(1''), (3'')((4''))$ and $(5'')$.

3.8. Definition [21]

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy h-bi-ideal of R if it satisfies $(1''), (2''), (5'')$ and $(6'')$.

3.9. Definition [21]

A fuzzy subset f of a hemiring R is called an $(\in, \in \vee q)$-fuzzy h-quasi-ideal of R, if it satisfies $(1''), (5'')$ and $(7'')$.

3.10. Theorem

Let f be an $(\in, \in \vee q)$-fuzzy h-bi-ideal of a hemiring R, then $f \wedge 0.5$ is an $(\in, \in \vee q)$-fuzzy h-bi-ideal of R where $(f \wedge 0.5)(x) = f(x) \wedge 0.5$ for all $x \in R$.

Proof. Suppose f is an $(\in, \in \vee q)$-fuzzy h-bi-ideal of a hemiring R and let $a, b, x, y \in R$. Then
$(f \wedge 0.5)(x + y) = f(x + y) \wedge 0.5 \geq (\min\{f(x), f(y), 0.5\}) \wedge 0.5$
$= \min\{f(x) \wedge 0.5, f(y) \wedge 0.5, 0.5\} = \min\{(f \wedge 0.5)(x), (f \wedge 0.5)(y), 0.5\}$
Similarly we can show that
$(f \wedge 0.5)(xy) \geq \min\{(f \wedge 0.5)(x), (f \wedge 0.5)(y), 0.5\}$ and
$(f \wedge 0.5)(xyz) \geq \min\{(f \wedge 0.5)(x), (f \wedge 0.5)(z), 0.5\}$.
Now let $x + a + y = b + y$, then
$(f \wedge 0.5)(x) = f(x) \wedge 0.5 \geq (\min\{f(a), f(b), 0.5\}) \wedge 0.5$
$= \min\{f(a) \wedge 0.5, f(b) \wedge 0.5, 0.5\} = \min\{(f \wedge 0.5)(a), (f \wedge 0.5)(b), 0.5\}$.
This shows that $(f \wedge 0.5)$ is an $(\in, \in \vee q)$-fuzzy h-bi-ideal of R. ∎

Similarly we can show that:

3.11. Theorem

If f is an $(\in, \in \vee q)$-fuzzy left (right) h-ideal of a hemiring R, then $(f \wedge 0.5)$ is an $(\in, \in \vee q)$-fuzzy left (right) h-ideal of R.

3.12. Definition [21]

Let f, g be fuzzy subsets of a hemiring R. Then the fuzzy subsets $f \wedge_{0.5} g$ and $f \odot_{0.5} g$ of R are defined as following:
$f \wedge_{0.5} g(x) = \min\{f(x), g(x), 0.5\}$
$(f \odot_{0.5} g)(x) = (f \odot g)(x) \wedge 0.5$ for all $x \in R$.
Now we define addition of two fuzzy subsets of a hemiring.

3.13. Definition

Let f, g be fuzzy subsets of a hemiring R. The fuzzy subset $f + g$ of R is defined by
$$(f+g)(x) = \bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f(a_1) \wedge f(a_2) \wedge g(b_1) \wedge g(b_2)\}$$
for all $a_1, a_2, b_1, b_2, x, z \in R$.
We also define $f +_{0.5} g = (f + g) \wedge 0.5$.

3.14. Lemma

Let A, B be subsets of R, then
$$C_A +_{0.5} C_B = C_{\overline{A+B}} \wedge 0.5.$$

Proof. Let A, B be subsets of a hemiring R and $x \in R$. If $x \in \overline{A+B}$ then there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $x + (a_1 + b_1) + z = (a_2 + b_2) + z$ for some $z \in R$. Thus
$$(C_A +_{0.5} C_B)(x)$$
$$= \left(\bigvee_{x+(a_1'+b_1')+z=(a_2'+b_2')+z} \{C_A(a_1') \wedge C_A(a_2') \wedge C_B(b_1') \wedge C_B(b_2')\} \right) \wedge 0.5$$
$$= 1 \wedge 0.5 = C_{\overline{A+B}}(x) \wedge 0.5.$$
If $x \notin \overline{A+B}$ then there do not exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $x + (a_1 + b_1) + z = (a_2 + b_2) + z$ for some $z \in R$. Thus
$(C_A +_{0.5} C_B)(x) = 0 \wedge 0.5 = C_{\overline{A+B}}(x) \wedge 0.5$.
Hence $C_A +_{0.5} C_B = C_{\overline{A+B}} \wedge 0.5$. ∎

3.15. Lemma

A fuzzy subset f of a hemiring R satisfies conditions $(1'')$ and $(5'')$ if and only if it satisfies condition
(8) $f +_{0.5} f \leq f \wedge 0.5$.

Proof. Suppose f satisfies conditions $(1'')$ and $(5'')$. Let $x \in R$, then
$$(f +_{0.5} f)(x) = \left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2)\} \right) \wedge 0.5$$

$$= \left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \left\{ \begin{array}{c} (f(a_1) \wedge f(b_1) \wedge 0.5) \wedge \\ (f(a_2) \wedge f(b_2) \wedge 0.5) \end{array} \right\} \right) \wedge 0.5$$

$$\leq \left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{(f(a_1+b_1)) \wedge (f(a_2+b_2))\} \right) \wedge 0.5$$

by condition $(1'')$

$\leq f(x) \wedge 0.5$ by condition $(5'')$.

Thus $f +_{0.5} f \leq f \wedge 0.5$.

Conversely, assume that $f +_{0.5} f \leq f \wedge 0.5$. Then for each $x \in R$ we have

$f(0) \geq f(0) \wedge 0.5 \geq (f +_{0.5} f)(0)$

$$= \left(\bigvee_{0+(a_1+b_1)+z=(a_2+b_2)+z} \{f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2)\} \right) \wedge 0.5$$

$\geq f(x) \wedge 0.5$ because $0 + (x + x) + 0 = (x + x) + 0$.

Thus $f(0) \geq f(x) \wedge 0.5$.

Let $x, y \in R$, then

$f(x + y) \geq f(x + y) \wedge 0.5 \geq (f +_{0.5} f)(x + y)$

$$= \left(\bigvee_{(x+y)+(a_1+b_1)+z=(a_2+b_2)+z} \{f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2)\} \right) \wedge 0.5$$

$\geq \{f(0) \wedge f(x) \wedge f(0) \wedge f(y)\} \wedge 0.5$

because $(x + y) + (0 + 0) + 0 = (x + y) + 0$.

$= \{f(x) \wedge f(y) \wedge 0.5\}$ because $f(0) \geq f(x) \wedge 0.5$.

Thus f satisfies condition $(1'')$.

Let $a, b, x, z \in R$ such that $x + a + z = b + z$. Then

$f(x) \geq f(x) \wedge 0.5 \geq (f +_{0.5} f)(x)$

$$= \left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2)\} \right) \wedge 0.5$$

$\geq \{f(a) \wedge f(0) \wedge f(b) \wedge 0.5\}$ because $x + (a + 0) + z = (b + 0) + z$

$= \{f(a) \wedge f(b) \wedge 0.5\}$ because $f(0) \geq f(x) \wedge 0.5$.

This shows that f satisfies condition $(5'')$. ∎

3.16. Theorem

A fuzzy subset f of a hemiring R is an $(\in, \in \vee q)$-fuzzy left (resp. right) h-ideal of R if and only if f satisfies conditions

(8) $f +_{0.5} f \leq f \wedge 0.5$

(9) $\mathcal{R} \odot_{0.5} f \leq f \wedge 0.5$ (Resp. $f \odot_{0.5} \mathcal{R} \leq f \wedge 0.5$).

Proof. Suppose f is an $(\in, \in \vee q)$-fuzzy left h-ideal of a hemiring R, then by Lemma 3.15, f satisfies condition (8). Now we show that f satisfies condition (9). Let $x \in R$. If $(\mathcal{R} \odot_{0.5} f)(x) = 0$, then $(\mathcal{R} \odot_{0.5} f)(x) \leq f(x) \wedge 0.5$.

Otherwise, there exist elements $a_i, b_i, c_j, d_j, z \in R$ such that $x+\sum_{i=1}^{m} a_ib_i+z = \sum_{j=1}^{n} c_jd_j + z$. Then we have

$$
\begin{aligned}
&(\mathcal{R} \odot_{0.5} f)(x) \\
&= \bigvee_{x+\Sigma_{i=1}^{m} a_ib_i+z=\Sigma_{j=1}^{n} a_j' b_j' + z} \left\{ \begin{array}{c} \left(\bigwedge_{i=1}^{m} \mathcal{R}(a_i)\right) \wedge \left(\bigwedge_{i=1}^{m} f(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} \mathcal{R}\left(a_j'\right)\right) \wedge \left(\bigwedge_{j=1}^{n} f\left(b_j'\right)\right) \end{array} \right\} \wedge 0.5 \\
&= \left(\bigvee_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a_j' b_j' + z} \left(\left(\bigwedge_{i=1}^{m} f(b_i)\right) \wedge \left(\bigwedge_{j=1}^{n} f\left(b_j'\right)\right)\right)\right) \wedge 0.5 \\
&= \left(\bigvee_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a_j' b_j' + z} \left(\begin{array}{c} \left(\bigwedge_{i=1}^{m} f(b_i) \wedge 0.5\right) \wedge \\ \left(\bigwedge_{j=1}^{n} f\left(b_j'\right) \wedge 0.5\right) \end{array} \right)\right) \wedge 0.5 \\
&\leq \left(\bigvee_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a_j' b_j' + z} \left(\left(\bigwedge_{i=1}^{m} f(a_ib_i)\right) \wedge \left(\bigwedge_{j=1}^{n} f\left(a_j'b_j'\right)\right)\right)\right) \wedge 0.5 \\
&= \left(\bigvee_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a_j' b_j' + z} \left(\begin{array}{c} \left(\bigwedge_{i=1}^{m} f(a_ib_i) \wedge 0.5\right) \wedge \\ \left(\bigwedge_{j=1}^{n} f\left(a_j'b_j'\right) \wedge 0.5\right) \end{array} \right)\right) \wedge 0.5 \\
&\leq \left(\bigvee_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a_j' b_j' + z} \left(\min\{f\left(\textstyle\sum_{i=1}^{m} a_ib_i\right), f\left(\textstyle\sum_{j=1}^{n} a_j'b_j'\right)\}\right)\right) \wedge 0.5 \\
&\leq f(x) \wedge 0.5.
\end{aligned}
$$

This implies that $\mathcal{R} \odot_{0.5} f \leq f \wedge 0.5$.

Conversely, assume that f satisfies conditions (8) and (9). Then by Lemma 3.15, f satisfies conditions $(1'')$ and $(5'')$. We show that f satisfies condition $(4'')$.

Let $x, y \in R$. Then we have

$$\begin{aligned}
f(xy) \wedge 0.5 &\geq (\mathcal{R} \odot_{0.5} f)(xy) \\
&= \left(\bigvee_{xy+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ \begin{array}{c} \left(\bigwedge_{i=1}^{m} \mathcal{R}(a_i) \right) \wedge \left(\bigwedge_{i=1}^{m} f(b_i) \right) \wedge \\ \left(\bigwedge_{j=1}^{n} \mathcal{R}(c_j) \right) \wedge \left(\bigwedge_{j=1}^{n} f(d_j) \right) \end{array} \right\} \right) \wedge 0.5 \\
&= \left(\bigvee_{xy+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ \left(\bigwedge_{i=1}^{m} f(b_i) \right) \wedge \left(\bigwedge_{j=1}^{n} f(d_j) \right) \right\} \right) \wedge 0.5 \\
&\geq f(y) \wedge 0.5 \qquad \text{because } xy + 0y + z = xy + z.
\end{aligned}$$

This shows that f satisfies condition $(4'')$. So f is an $(\in, \in \vee q)$-fuzzy left h-ideal of R. Similarly we can prove the case of an $(\in, \in \vee q)$-fuzzy right h-ideal of R. ∎

3.17. Theorem

A fuzzy subset f of a hemiring R is an $(\in, \in \vee q)$-fuzzy h-quasi-ideal of R if and only if f satisfies conditions (7) and (8).

Proof. Proof is straight forward because conditions $(1'')$ and $(5'')$ are equivalent to condition (8). ∎

3.18. Theorem

Every $(\in, \in \vee q)$-fuzzy left h-ideal *of a hemiring R is an* $(\in, \in \vee q)$-fuzzy h-quasi-ideal *of R.*

Proof. Proof is straight forward because condition (9) $\Rightarrow$ $(7'')$. ∎

3.19. Lemma [21]

Every $(\in, \in \vee q)$-fuzzy h-quasi-ideal *of R is an* $(\in, \in \vee q)$-fuzzy h-bi-ideal *of R.*

3.20. Lemma [21]

If f and g are $(\in, \in \vee q)$-fuzzy right and left h-ideal of R respectively, then $f \odot_{0.5} g \leq f \wedge_{0.5} g$.

3.21. Theorem [21]

For a hemiring R the following conditions are equivalent.

(1) *R is h-hemiregular.*

(2) $(f \wedge_{0.5} g) = (f \odot_{0.5} g)$ *for every* $(\in, \in \vee q)$-fuzzy right h-ideal f *and every* $(\in, \in \vee q)$-fuzzy left h-ideal g of R.

3.22. Theorem [21]

For a hemiring R, the following conditions are equivalent.
(1) R is h-hemiregular.
(2) $f \wedge 0.5 \leq (f \odot_{0.5} \mathcal{R} \odot_{0.5} f)$ *for every* $(\in, \in \vee q)$-fuzzy h-bi-ideal f *of* R.
(3) $f \wedge 0.5 \leq (f \odot_{0.5} \mathcal{R} \odot_{0.5} f)$ *for every* $(\in, \in \vee q)$-fuzzy h-quasi-ideal f *of* R.

3.23. Theorem [21]

For a hemiring R, the following conditions are equivalent.
(1) R is h-hemiregular.
(2) $(f \wedge_{0.5} g) \leq (f \odot_{0.5} g \odot_{0.5} f)$ *for every* $(\in, \in \vee q)$-fuzzy h-bi-ideal f *and every* $(\in, \in \vee q)$-fuzzy h-ideal g *of* R.
(3) $(f \wedge_{0.5} g) \leq (f \odot_{0.5} g \odot_{0.5} f)$ *for every* $(\in, \in \vee q)$-fuzzy h-quasi-ideal f *and every* $(\in, \in \vee q)$-fuzzy h-ideal g *of* R.

3.24. Theorem

For a hemiring R, the following conditions are equivalent.
(1) R is h-hemiregular.
(2) $(f \wedge_{0.5} g) \leq (f \odot_{0.5} g)$ *for every* $(\in, \in \vee q)$-fuzzy h-bi-ideal f *and every* $(\in, \in \vee q)$-fuzzy left h-ideal g *of* R.
(3) $(f \wedge_{0.5} g) \leq (f \odot_{0.5} g)$ *for every* $(\in, \in \vee q)$-fuzzy h-quasi-ideal f *and every* $(\in, \in \vee q)$-fuzzy left h-ideal g *of* R.
(4) $(f \wedge_{0.5} g) \leq (f \odot_{0.5} g)$ *for every* $(\in, \in \vee q)$-fuzzy *right* h-ideal f *and every* $(\in, \in \vee q)$-fuzzy h-bi-ideal g *of* R.
(5) $(f \wedge_{0.5} g) \leq (f \odot_{0.5} g)$ *for every* $(\in, \in \vee q)$-fuzzy *right* h-ideal f *and every* $(\in, \in \vee q)$-fuzzy h-quasi-ideal g *of* R.
(6) $(f \wedge_{0.5} g \wedge_{0.5} h) \leq (f \odot_{0.5} g \odot_{0.5} h)$ *for every* $(\in, \in \vee q)$-fuzzy *right* h-ideal f,*every* $(\in, \in \vee q)$-fuzzy h-bi-ideal g *and every* $(\in, \in \vee q)$-fuzzy *left* h-ideal h *of* R.
(7) $(f \wedge_{0.5} g \wedge_{0.5} h) \leq (f \odot_{0.5} g \odot_{0.5} h)$ *for every* $(\in, \in \vee q)$-fuzzy *right* h-ideal f,*every* $(\in, \in \vee q)$-fuzzy h-quasi-ideal g *and every* $(\in, \in \vee q)$-fuzzy *left* h-ideal h *of* R.

Proof. (1) $\Rightarrow$ (2) Let f be any $(\in, \in \vee q)$-fuzzy h-bi-ideal and g any $(\in, \in \vee q)$-fuzzy left h-ideal of R. Since R is h-hemiregular, so for any $a \in R$ there exist

$x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Thus we have

$$\begin{aligned}
&(f \odot_{0.5} g)(a) \\
= &\bigvee_{a+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z} \left(\begin{array}{c} \left(\bigwedge_{i=1}^{m} f(a_i)\right) \wedge \left(\bigwedge_{i=1}^{m} g(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} f\left(a'_j\right)\right) \wedge \left(\bigwedge_{j=1}^{n} g\left(b'_j\right)\right) \end{array} \right) \wedge 0.5 \\
\geq &\{f(a) \wedge g(x_1a) \wedge g(x_2a)\} \wedge 0.5 \text{ because } a + ax_1a + z = ax_2a + z \\
\geq &\{f(a) \wedge g(a) \wedge 0.5\} \wedge 0.5 \\
= &\{f(a) \wedge g(a) \wedge 0.5\} \\
= &(f \wedge_{0.5} g)(a).
\end{aligned}$$

So $(f \odot_{0.5} g) \geq (f \wedge_{0.5} g)$.

$(2) \Rightarrow (3)$ This is obvious because every $(\in, \in \vee q)$-fuzzy h-quasi-ideal is an $(\in, \in \vee q)$-fuzzy h-bi-ideal.

$(3) \Rightarrow (1)$ Let f be an $(\in, \in \vee q)$-fuzzy right h-ideal and g be an $(\in, \in \vee q)$-fuzzy left h-ideal of R. Since every $(\in, \in \vee q)$-fuzzy right h-ideal is an $(\in, \in \vee q)$-fuzzy h-quasi-ideal, so by (3) we have $(f \odot_{0.5} g) \geq (f \wedge_{0.5} g)$. But by Lemma 3.20, $(f \odot_{0.5} g) \leq (f \wedge_{0.5} g)$. Hence $(f \odot_{0.5} g) = (f \wedge_{0.5} g)$ for every $(\in, \in \vee q)$-fuzzy right h-ideal f of R, and every $(\in, \in \vee q)$-fuzzy left h-ideal g of R. Thus by Theorem 3.21, R is h-hemiregular.

Similarly we can show that $(1) \Leftrightarrow (4) \Leftrightarrow (5)$.

$(1) \Rightarrow (6)$ Let f be an $(\in, \in \vee q)$-fuzzy *right* h-ideal, g be an $(\in, \in \vee q)$-fuzzy h-bi-ideal *and* h be an $(\in, \in \vee q)$-fuzzy *left* h-ideal *of* R. Since R is h-hemiregular, so for any $a \in R$ there exist $x_1, x_2, z \in R$ such that $a + ax_1a + z = ax_2a + z$. Thus we have

$$\begin{aligned}
&(f \odot_{0.5} g \odot_{0.5} h)(a) \\
= &\bigvee_{a+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z} \left(\begin{array}{c} \left(\bigwedge_{i=1}^{m} (f \odot_{0.5} g)(a_i)\right) \wedge \left(\bigwedge_{i=1}^{m} h(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} (f \odot_{0.5} g)\left(a'_j\right)\right) \wedge \left(\bigwedge_{j=1}^{n} h\left(b'_j\right)\right) \end{array} \right) \wedge 0.5 \\
\geq &\{(f \odot_{0.5} g)(a) \wedge h(x_1a) \wedge h(x_2a) \wedge 0.5\}
\end{aligned}$$

$$\geq \{ \bigvee_{a+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}a_j'b_j'+z} \left(\begin{array}{c} \left(\bigwedge_{i=1}^{m} f\left(a_i\right)\right)\wedge \\ \left(\bigwedge_{i=1}^{m} g\left(b_i\right)\right)\wedge \\ \left(\bigwedge_{j=1}^{n} f\left(a_j^{'}\right)\right)\wedge \\ \left(\bigwedge_{j=1}^{n} g\left(b_j^{'}\right)\right) \end{array} \right) \wedge 0.5 \wedge h\left(x_1a\right) \wedge h\left(x_2a\right)$$

$\geq \{f(ax_1) \wedge f(ax_2) \wedge g(a) \wedge h\left(x_1a\right) \wedge h\left(x_2a\right) \wedge 0.5\}$
$\geq \{f(a) \wedge g(a) \wedge h\left(a\right) \wedge 0.5\} = \left(f \wedge_{0.5} g \wedge_{0.5} h\right)(a)$.
Thus $(f \wedge_{0.5} g \wedge_{0.5} h) \leq (f \odot_{0.5} g \odot_{0.5} h)$.
(6) $\Rightarrow$ (7) Obvious.
(7) $\Rightarrow$ (1) Let f be an $(\in, \in \vee q)$-fuzzy *right* h-ideal, *and* h be an $(\in, \in \vee q)$-fuzzy *left* h-ideal *of* R. Then
$(f \wedge_{0.5} h) = (f \wedge_{0.5} \mathcal{R} \wedge_{0.5} h) \leq (f \odot_{0.5} \mathcal{R} \odot_{0.5} h) \leq (f \odot_{0.5} h)$.
But $(f \odot_{0.5} h) \leq (f \wedge_{0.5} h)$. Hence $(f \odot_{0.5} h) = (f \wedge_{0.5} h)$ for every $(\in, \in \vee q)$-fuzzy *right* h-ideal f and for every $(\in, \in \vee q)$-fuzzy *left* h-ideal h of R. Thus by Theorem 3.21, R is h-hemiregular. ∎

3.25. Lemma [21]

A hemiring R is h-intra-hemiregular if and only if $f \wedge_{0.5} g \leq f \odot_{0.5} g$ for every $(\in, \in \vee q)$-fuzzy left h-ideal f and for every $(\in, \in \vee q)$-fuzzy right h-ideal g of R.

3.26. Theorem

The following conditions are equivalent for a hemiring $R:$
(1) R is both h-hemiregular and h-intra-hemiregular.
(2) $f \wedge 0.5 = f \odot_{0.5} f$ for every $(\in, \in \vee q)$-fuzzy h-bi-ideal f of R.
(3) $f \wedge 0.5 = f \odot_{0.5} f$ for every $(\in, \in \vee q)$-fuzzy h-quasi-ideal f of R.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\in, \in \vee q)$-fuzzy h-bi-ideal of R and $x \in R$. Since R is both h-hemiregular and h-intra-hemiregular, there exist elements $a_1, a_2, p_i, p_i', q_j, q_j', z \in R$ such that
$x+\sum_{j=1}^{n}(xa_2q_jx)(xq_j'a_1x)+\sum_{j=1}^{n}(xa_1q_jx)(xq_j'a_2x)+\sum_{i=1}^{m}(xa_1p_ix)(xp_i'a_1x)$
$+\sum_{i=1}^{m}(xa_2p_ix)(xp_i'a_2x)+z = \sum_{i=1}^{m}(xa_2p_ix)(xp_i'a_1x)+\sum_{i=1}^{m}(xa_1p_ix)(xp_i'a_2x)$
$+\sum_{j=1}^{n}(xa_1q_jx)(xq_j'a_1x) + \sum_{j=1}^{n}(xa_2q_jx)(xq_j'a_2x) + z$
(As given in Lemma 5.6 [28])
$(f \odot_{0.5} f)(x)$

$$= \bigvee_{x+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}a'_jb'_j+z} \left(\begin{array}{c} \left(\bigwedge_{i=1}^{m} f(a_i)\right) \wedge \left(\bigwedge_{i=1}^{m} f(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} f\left(a'_j\right)\right) \wedge \left(\bigwedge_{j=1}^{n} f\left(b'_j\right)\right) \end{array} \right) \wedge 0.5$$

$$\geq \left(\begin{array}{c} \left(\bigwedge_{j=1}^{n} f(xa_2q_jx)\right) \wedge \left(\bigwedge_{j=1}^{n} f\left(xq'_ja_1x\right)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} f(xa_1q_jx)\right) \wedge \left(\bigwedge_{j=1}^{n} f\left(xq'_ja_2x\right)\right) \end{array} \right) \wedge$$

$$\left(\begin{array}{c} \left(\bigwedge_{i=1}^{m} f(xa_1p_ix)\right) \wedge \left(\bigwedge_{i=1}^{m} f(xp'_ia_1x)\right) \wedge \\ \left(\bigwedge_{i=1}^{m} f(xa_2p_ix)\right) \wedge \left(\bigwedge_{i=1}^{m} f(xp'_ia_2x)\right) \end{array} \right) \wedge 0.5$$

$\geq \min\{f(x), 0.5\} \wedge 0.5 = f(x) \wedge 0.5.$

This implies that $f \odot_{0.5} f \geq f \wedge 0.5.$

On the other hand if $x + \Sigma_{i=1}^{m} a_ib_i + z = \Sigma_{j=1}^{n} a'_jb'_j + z$, we have

$f(x) \wedge 0.5 \geq \min\{f(\Sigma_{i=1}^{m} a_ib_i), f(\Sigma_{j=1}^{n} a'_jb'_j), 0.5\} \wedge 0.5$ by $(5'')$

$$\geq \min\{\left(\bigwedge_{i=1}^{m} f(a_ib_i)\right), \left(\bigwedge_{i=1}^{m} f(a'_jb'_j)\right), 0.5\}$$

$$\geq \min\{\left(\bigwedge_{i=1}^{m} f(a_i) \wedge f(b_i)\right), \left(\bigwedge_{i=1}^{m} f(a'_j) \wedge f(b'_j)\right), 0.5\}$$

because f is an $(\in,\in\vee q)$-fuzzy h-bi-ideal of R.

Thus

$(f \odot_{0.5} f)(x)$

$$= \bigvee_{x+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}a'_jb'_j+z} \left(\begin{array}{c} \left(\bigwedge_{i=1}^{m} f(a_i)\right) \wedge \left(\bigwedge_{i=1}^{m} f(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} f\left(a'_j\right)\right) \wedge \left(\bigwedge_{j=1}^{n} f\left(b'_j\right)\right) \end{array} \right) \wedge 0.5$$

$\leq f(x) \wedge 0.5.$

Consequently $f \wedge 0.5 = f \odot_{0.5} f$.

(2) $\Rightarrow$ (3) This is straightforward because every $(\in,\in\vee q)$-fuzzy h-quasi-ideal of R is an $(\in,\in\vee q)$-fuzzy h-bi-ideal of R.

(3) $\Rightarrow$ (1) Let Q be an h-quasi-ideal of R. Then C_Q is an $(\in,\in\vee q)$-fuzzy h-quasi-ideal of R. Thus by hypothesis

$C_Q \wedge 0.5 = C_Q \odot_{0.5} C_Q = C_Q \odot C_Q \wedge 0.5 = C_{\overline{Q^2}} \wedge 0.5.$

Then it follows $Q = \overline{Q^2}$. Hence by Lemma 2.15, R is both h-hemiregular and h-intra-hemiregular. ∎

3.27. Theorem

The following conditions are equivalent for a hemiring R:

(1) R is both h-hemiregular and h-intra-hemiregular.

(2) $f \wedge_{0.5} g \leq f \odot_{0.5} g$ for all $(\in, \in \vee q)$-fuzzy h-bi-ideals f and g of R.

(3) $f \wedge_{0.5} g \leq f \odot_{0.5} g$ for every $(\in, \in \vee q)$-fuzzy h-bi-ideal f and every $(\in, \in \vee q)$-fuzzy h-quasi-ideals f of R.

(4) $f \wedge_{0.5} g \leq f \odot_{0.5} g$ for every $(\in, \in \vee q)$-fuzzy h-quasi-ideal f and every $(\in, \in \vee q)$-fuzzy h-bi-ideals f of R.

(5) $f \wedge_{0.5} g \leq f \odot_{0.5} g$ for all $(\in, \in \vee q)$-fuzzy h-quasi-ideals f and g of R.

Proof. (1) $\Rightarrow$ (2) Let f and g be $(\in, \in \vee q)$-fuzzy h-bi-ideals of R and $x \in R$. Since R is both h-hemiregular and h-intra-hemiregular, there exist elements $a_1, a_2, p_i, p_i', q_j, q_j', z \in R$ such that

$$x+\sum_{j=1}^{n}(xa_2q_jx)(xq_j'a_1x)+\sum_{j=1}^{n}(xa_1q_jx)(xq_j'a_2x)+\sum_{i=1}^{m}(xa_1p_ix)(xp_i'a_1x) + \sum_{i=1}^{m}(xa_2p_ix)(xp_i'a_2x)+z = \sum_{i=1}^{m}(xa_2p_ix)(xp_i'a_1x)+\sum_{i=1}^{m}(xa_1p_ix)(xp_i'a_2x) + \sum_{j=1}^{n}(xa_1q_jx)(xq_j'a_1x) + \sum_{j=1}^{n}(xa_2q_jx)(xq_j'a_2x) + z$$

(As given in Lemma 5.6 [28])

$(f \odot_{0.5} g)(x)$

$$= \bigvee_{x+\Sigma_{i=1}^{m} a_ib_i+z=\Sigma_{j=1}^{n} a_j'b_j'+z} \begin{pmatrix} \left(\bigwedge_{i=1}^{m} f(a_i)\right) \wedge \left(\bigwedge_{i=1}^{m} g(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} f\left(a_j^{'}\right)\right) \wedge \left(\bigwedge_{j=1}^{n} g\left(b_j^{'}\right)\right) \end{pmatrix} \wedge 0.5$$

$$\geq \begin{pmatrix} \left(\bigwedge_{j=1}^{n} f(xa_2q_jx)\right) \wedge \left(\bigwedge_{j=1}^{n} g(xq_j'a_1x)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} f(xa_1q_jx)\right) \wedge \left(\bigwedge_{j=1}^{n} g(xq_j'a_2x)\right) \end{pmatrix} \wedge$$

$$\begin{pmatrix} \left(\bigwedge_{i=1}^{m} f(xa_1p_ix)\right) \wedge \left(\bigwedge_{i=1}^{m} g(xp_i'a_1x)\right) \wedge \\ \left(\bigwedge_{i=1}^{m} f(xa_2p_ix)\right) \wedge \left(\bigwedge_{i=1}^{m} g(xp_i'a_2x)\right) \end{pmatrix} \wedge 0.5$$

$\geq \min\{f(x), g(x), 0.5\} \wedge 0.5 = f(x) \wedge_{0.5} g(x)$.

This implies that $f \odot_{0.5} g \geq f \wedge_{0.5} g$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (1) Let f be an $(\in, \in \vee q)$-fuzzy left h-ideals of R and g be an $(\in, \in \vee q)$-fuzzy right h-ideal of R. Then f and g are $(\in, \in \vee q)$-fuzzy h-bi-ideals of R. So by hypothesis $f \wedge_{0.5} g \leq f \odot_{0.5} g$ but $f \wedge_{0.5} g \geq f \odot_{0.5} g$ by Lemma 3.20. Thus $f \wedge_{0.5} g = f \odot_{0.5} g$. Hence by Theorem 3.21, R is h-hemiregular. On the other hand by hypothesis we also have $f \wedge_{0.5} g \leq g \odot_{0.5} f$. By Lemma 3.25, R is h-intra-hemiregular. ∎

References

[1] J. Ahsan, Semirings characterized by their fuzzy ideals, J. Fuzzy Math. 6 (1998), 181-192.

[2] J. Ahsan, K. Saifullah, M. F. Khan, Fuzzy Semirings, Fuzzy Sets Syst. 60 (1993), 309-320.

[3] S. K. Bhakat, $(\in \vee q)$-level subset, Fuzzy Sets and Systems, 103 (1999), 529-533.

[4] S. K. Bhakat, $(\in, \in \vee q)$-fuzzy normal, quasinormal and maximal subgroups, Fuzzy Sets and Systems, 112 (2000), 299-312.

[5] S. K. Bhakat, P. Das, On the definition of a fuzzy subgroup, Fuzzy Sets and Systems, 51 (1992), 235-241.

[6] S. K. Bhakat, P. Das, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy Sets and Systems 80 (1996), 359-368.

[7] S. K. Bhakat, P. Das, Fuzzy subrings and ideals redefined, Fuzzy Sets and Systems, 81 (1996), 383-393.

[8] B. Davvaz, $(\in, \in \vee q)$-fuzzy subnearrings and ideals, Soft Comput, 10 (2006), 206-211.

[9] W. A. Dudek, M. Shabir, M. Irfan Ali, (α, β)-Fuzzy ideals of Hemirings, Comput. Math. Appl., 58 (2009), 310-321.

[10] W. A. Dudek, M. Shabir, R. Anjum, Characterizations of hemirings by their h-ideals, Comput. Math. Appl., 59 (2010), 3167-3179.

[11] S. Ghosh, Fuzzy k-ideals of semirings, Fuzzy Sets Syst., 95 (1998), 103-108.

[12] K. Glazek, A guide to litrature on semirings and their applications in mathematics and information sciences: with complete bibliography, Kluwer Acad. Publ. Nederland, 2002.

[13] J. S. Golan, Semirings and their applications, Kluwer Acad. Publ. 1999.

[14] U. Hebisch, H. J. Weinert, Semirings: Algebraic Theory and Applications in the Computer Science, World Scientific, 1998.

[15] M. Henriksen, Ideals in semirings with commutative addition, Amer. Math. Soc. Notices, 6 (1958), 321.

[16] K. Iizuka, On Jacobson radical of a semiring, Tohoku Math. J. 11 (1959), 409-421.

[17] Y. B. Jun, M. A. Özürk, S. Z. Song, On fuzzy h-ideals in hemirings, Inform. Sci. 162 (2004), 211-226.

[18] Y. B. Jun, S. Z. Song, Generalized fuzzy interior ideals in semigroups, Inform. Sci. 176 (2006), 3079-3093.

[19] D. R. La Torre, On h-ideals and k-ideals in hemirings, Publ. Math. Debrecen 12 (1965), 219-226.

[20] X. Ma, J. Zhan, On fuzzy h-ideals of hemirings, J. Syst. Sci. Complexity 20 (2007), 470-478.

[21] X. Ma, J. Zhan, Generalized fuzzy h-bi-ideals and h-quasi-ideals of hemirings, Inform. Sci., 179 (2009), 1249-1268.

[22] J. N. Mordeson, D. S. Malik, Fuzzy Automata and Languages, Theory and Applications, Computational Mathematics Series, Chapman and Hall/CRC, Boca Raton 2002.

[23] V. Murali, Fuzzy points of equivalent fuzzy subsets, Inform. Sci. 158 (2004), 277-288.

[24] P. M Pu, Y. M. Liu, Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980), 571-599.

[25] M. Shabir, T. Mahmood, Characterizations of Hemirings by Interval Valued Fuzzy Ideals, Quasigroups and Related Systems, 19 (2011), 317-329.

[26] M. Shabir, T. Mahmood, Hemirings characterized by the properties of their fuzzy ideals with thresholds, Quasigroups and Related Systems 18 (2010), 195 - 212.

[27] H. S. Vandiver, Note on a simple type of algebra in which cancellation law of addition does not hold, Bull. Amer. Math. Soc., 40 (1934), 914-920.

[28] Y. Q. Yin, H. Li, The charatecrizations of h-hemiregular hemirings and h-intra-hemiregular hemirings, Inform. Sci. 178 (2008), 3451-3464.

[29] L.A. Zadeh, Fuzzy Sets, Information and Control, 8 (1965), 338-353.
[30] J. Zhan, W. A. Dudek, Fuzzy h-ideals of hemirings, Inform. Sci. 177 (2007), 876-886.

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.
E-mail address: mshabirbhatti@yahoo.co.uk.

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.
E-mail address: yasir_maths@yahoo.com

Department of Mathematics, International Islamic Univesity, Islamabad, Pakistan.
E-mail address: tahirbakhat@yahoo.com