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FINITE SETS WITH FAKE OBSERVABLE CARDINALITY

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Abstract. Let X be a compact metric space and let $|A|$ denote the cardinality of a set A. We prove that if $f\colon X \to X$ is a homeomorphism and $|X| = \infty$, then for all $\delta > 0$ there is $A \subset X$ such that $|A| = 4$ and for all $k \in \mathbb{Z}$ there are $x, y \in f^k(A)$, $x \neq y$, such that $\operatorname{dist}(x, y) < \delta$. An observer that can only distinguish two points if their distance is grater than δ, for sure will say that A has at most 3 points even knowing every iterate of A and that f is a homeomorphism. We show that for hyper-expansive homeomorphisms the same δ-observer will not fail about the cardinality of A if we start with $|A| = 3$ instead of 4. Generalizations of this problem are considered via what we call (m, n)-expansiveness.

Introduction

Since 1950, when Utz [16] initiated the study of expansive homeomorphism, several variations of the definition appeared in the literature. Let us recall that a homeomorphism $f\colon X \to X$ of a compact metric space (X, dist) is *expansive* if there is an *expansive constant* $\delta > 0$ such that if $x \neq y$, then $\operatorname{dist}(f^k(x), f^k(y)) > \delta$ for some $k \in \mathbb{Z}$. Some variations of this definition are weaker, as for example *continuum-wise expansiveness* [6] and N-*expansiveness* [9] (see also [3,8,13]). A branch of research in topological dynamics investigates the possibility of extending known results for expansive homeomorphisms to these versions. See for example [2,5,10,12,14].

Other related definitions are stronger than expansiveness as for example *positive expansiveness* [15] and *hyper-expansiveness* [1]. Both definitions are so strong that their examples are almost trivial. It is known [15] that if a compact metric space admits a positive expansive homeomorphism, then the space has only a finite number of points. Recall that $f\colon X \to X$ is *positive expansive* if there is $\delta > 0$ such that if $x \neq y$, then $\operatorname{dist}(f^k(x), f^k(y)) > \delta$ for some $k \geq 0$. Therefore, we have that if the compact metric space X is not a finite set, then for every homeomorphism $f\colon X \to X$ and for all $\delta > 0$ there are $x \neq y$ such that $\operatorname{dist}(f^k(x), f^k(y)) < \delta$ for all $k \geq 0$. This is a very general result about the dynamics of homeomorphisms of compact metric spaces.

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Another example of this phenomenon is given in [1], where it is proved that no uncountable compact metric space admits a hyper-expansive homeomorphism (see Definition 3). Therefore, if X is an uncountable compact metric space, as for example a compact manifold, then for every homeomorphism $f\colon X \to X$ and for all $\delta > 0$ there are two compact subsets $A, B \subset X$, $A \neq B$, such that $\operatorname{dist}_H(f^k(A), f^k(B)) < \delta$ for all $k \in \mathbb{Z}$. The distance dist_H is called *Hausdorff metric* and its definition is recalled in equation (3) below.

According to Lewowicz [7] we can explain the meaning of expansiveness as follows. Let us say that a δ-*observer* is someone that cannot distinguish two points if their distance is smaller than δ. If $\operatorname{dist}(x, y) < \delta$ a δ-observer will not be able to say that the set $A = \{x, y\}$ has two points. But if the homeomorphism is expansive, with expansive constant greater than δ, and if the δ-observer knows all of the iterates $f^k(A)$ with $k \in \mathbb{Z}$, then he will find that A contains two different points, because if $\operatorname{dist}(f^k(x), f^k(y)) > \delta$, then he will see two points in $f^k(A)$. Let us be more precise.

Definition 1. For $\delta \geq 0$, a set $A \subset X$ is δ-*separated* if for all $x \neq y$, $x, y \in A$, it holds that $\operatorname{dist}(x, y) > \delta$. The δ-*cardinality* of a set A is

$$|A|_\delta = \sup\{|B| : B \subset A \text{ and } B \text{ is } \delta\text{-separated}\},$$

where $|B|$ denotes the cardinality of the set B.

Notice that the δ-cardinality is always finite because X is compact. The δ-cardinality of a set represents the maximum number of different points that a δ-observer can identify in the set.

In this paper we introduce a series of definitions, some weaker and other stronger than expansiveness, extending the notion of N-expansiveness of [9]. Let us recall that given $N \geq 1$, a homeomorphism is N-*expansive* if there is $\delta > 0$ such that if $\operatorname{diam}(f^k(A)) < \delta$ for all $k \in \mathbb{Z}$, then $|A| \leq N$. In terms of our δ-observer we can say that f is N-expansive if there is $\delta > 0$ such that if $|A| = N + 1$, a δ-observer will be able to say that A has at least two points given that he knows all of the iterates $f^k(A)$ for $k \in \mathbb{Z}$, i.e., $|f^k(A)|_\delta > 1$ for some $k \in \mathbb{Z}$. Let us introduce our main definition.

Definition 2. Given integer numbers $m > n \geq 1$ we say that $f\colon X \to X$ is (m, n)-*expansive* if there is $\delta > 0$ such that if $|A| = m$, then there is $k \in \mathbb{Z}$ such that $|f^k(A)|_\delta > n$.

The first problem under study is the classification of these definitions. We prove that (m, n)-expansiveness implies N-expansiveness if $m \leq (N + 1)n$. In particular, if $m \leq 2n$, then (m, n)-expansiveness implies expansiveness. These results are stated in Corollary 1.7. It is known that even on surfaces, N-expansiveness does not imply expansiveness for $N \geq 2$, see [2]. Here we show that (m, n)-expansiveness does not imply expansiveness if $n \geq 2$. For example, Anosov diffeomorphisms are known to be expansive and a consequence of Theorem 5.1 is that Anosov diffeomorphisms are not (m, n)-expansive for all $n \geq 2$.

It is a fundamental problem in dynamical systems to determine which spaces admit expansive homeomorphisms (or Anosov diffeomorphisms). In this paper we prove that no Peano continuum admits a (m,n)-expansive homeomorphism if $2m \geq 3n$, see Theorem 3.2. We also show that if X admits a $(n+1,n)$-expansive homeomorphism with $n \geq 3$, then X is a finite set. Examples of $(3,2)$-expansive homeomorphisms are given on countable spaces (hyper-expansive homeomorphisms), see Theorem 4.1.

The article is organized as follows. In Section 1 we prove basic properties of (m,n)-expansive homeomorphisms. In Section 2 we prove the first statement of the abstract, i.e., no infinite compact metric space admits a $(4,3)$-expansive homeomorphism. In Section 3 we show that no Peano continuum admits a (m,n)-expansive homeomorphism if $2m \geq 3n$. In Section 4 we show that hyper-expansive homeomorphisms are $(3,2)$-expansive. Such homeomorphisms are defined on compact metric spaces with a countable number of points. In Section 5 we prove that a homeomorphism with the shadowing property and with two points x, y satisfying

$$0 = \liminf_{k\to\infty} \operatorname{dist}(f^k(x), f^k(y)) < \limsup_{k\to\infty} \operatorname{dist}(f^k(x), f^k(y))$$

cannot be $(m,2)$-expansive if $m > 2$.

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1. Separating finite sets

Let (X, dist) be a compact metric space and consider a homeomorphism $f\colon X \to X$. Let us recall that for integer numbers $m > n \geq 1$ a homeomorphism f is (m,n)-*expansive* if there is $\delta > 0$ such that if $|A| = m$, then there is $k \in \mathbb{Z}$ such that $|f^k(A)|_\delta > n$. In this case we say that δ is a (m,n)-*expansive constant.* The idea of (m,n)-expansiveness is that our δ-observer will find more than n points in every set of m points if he knows all of its iterates.

Remark 1.1. From the definitions it follows that a homeomorphisms is $(N+1,1)$-expansive if and only if it is N-expansive in the sense of [9]. In particular, $(2,1)$-expansiveness is equivalent with expansiveness.

Remark 1.2. Notice that if X is a finite set, then every homeomorphism of X is (m,n)-expansive.

Proposition 1.3. *If $n' \leq n$ and $m-n \leq m'-n'$, then (m,n)-expansive implies (m',n')-expansive with the same expansive constant.*

Proof. The case $|X| < \infty$ is trivial, so, let us assume that $|X| = \infty$. Consider $\delta > 0$ as a (m,n)-expansive constant. Given a set A with $|A| = m'$ we will show that there is $k \in \mathbb{Z}$ such that $|f^k(A)|_\delta > n'$, i.e., the same expansive constant works. We divide the proof in two cases.

First assume that $m' \geq m$. Let $B \subset A$ with $|B| = m$. Since f is (m,n)-expansive, there is $k \in \mathbb{Z}$ such that $|f^k(B)|_\delta > n$. Therefore $|f^k(A)|_\delta > n \geq n'$, proving that f is (m', n')-expansive.

Now suppose that $m' < m$. Given that $|A| = m'$ and $|X| = \infty$ there is $C \subset X$ such that $A \cap C = \emptyset$ and $|A \cup C| = m$. By (m,n)-expansiveness, there is $k \in \mathbb{Z}$ such that $|f^k(A \cup C)|_\delta > n$. Then, there is a δ-separated set $D \subset f^k(A \cup C)$ with $|D| > n$. Notice that

$$|f^k(A) \cap D| = |D \setminus f^k(C)| \geq |D| - |f^k(C)| > n - (m - m')$$

and since $n - (m - m') \geq n'$ by hypothesis, we have that $f^k(A) \cap D$ is a δ-separated subset of $f^k(A)$ with more than n' points, that is $|f^k(A)|_\delta > n'$. This proves the $(m'n')$-expansiveness of f in this case too. □

As a consequence of Proposition 1.3 we have that

(1) (m,n)-expansive implies $(m+1, n)$-expansive and
(2) (m,n)-expansive implies $(m-1, n-1)$-expansive.

In Table 1 below we can easily see all these implications. The following proposition allows us to draw more arrows in this table, for example: $(4,2) \Rightarrow (2,1)$.

Table 1. Basic hierarchy of (m,n)-expansiveness. Each pair (m,n) in the table stands for "(m,n)-expansive". In the first position, (2,1), we have expansiveness. The first line, of the form $(N+1, 1)$, we have N-expansive homeomorphisms.

$$\begin{array}{ccccccc}
(2,1) & \Rightarrow & (3,1) & \Rightarrow & (4,1) & \Rightarrow & \dots \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
(3,2) & \Rightarrow & (4,2) & \Rightarrow & (5,2) & \Rightarrow & \dots \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
(4,3) & \Rightarrow & (5,3) & \Rightarrow & (6,3) & \Rightarrow & \dots \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
\dots & & \dots & & \dots & &
\end{array}$$

Proposition 1.4. *If $a, n \geq 2$ and $f\colon X \to X$ is an (an, n)-expansive homeomorphism, then f is $(a, 1)$-expansive.*

In order to prove it, let us introduce two previous results.

Lemma 1.5. *If $A, B \subset X$ are finite sets and $\delta > 0$ satisfies $|A| = |A|_\delta$ and $|B|_\delta = 1$, then for all $\varepsilon > 0$ it holds that*

$$|A \cup B|_{\delta+\varepsilon} \leq |A|_\varepsilon + |B|_\delta - |A \cap B|.$$

Proof. If $A \cap B = \emptyset$, then the proof is easy because

$$|A \cup B|_{\delta+\varepsilon} \leq |A|_{\delta+\varepsilon} + |B|_{\delta+\varepsilon} \leq |A|_\varepsilon + |B|_\delta.$$

Assume now that $A \cap B \neq \emptyset$. Since $|A| = |A|_\delta$ we have that A is δ-separated. Therefore $|A \cap B| = 1$ because $|B|_\delta = 1$. Assume that $A \cap B = \{y\}$. Let us prove that $|A \cup B|_{\delta+\varepsilon} \leq |A|_\varepsilon$ and notice that it is sufficient to conclude the proof of the lemma.

Let $C \subset A \cup B$ be a $(\delta + \varepsilon)$-separated set such that $|C| = |A \cup B|_{\delta+\varepsilon}$. If $C \subset A$, then

$$|A \cup B|_{\delta+\varepsilon} = |A|_{\delta+\varepsilon} \leq |A|_\varepsilon.$$

Therefore, let us assume that there is $x \in C \setminus A$. Define the set

$$D = (C \cup \{y\}) \setminus \{x\}.$$

Notice that $|C| = |D|$ and $D \subset A$.

We will show that D is ε-separated. Take $p, q \in D$ and arguing by contradiction assume that $p \neq q$ and $\operatorname{dist}(p, q) \leq \varepsilon$. If $p, q \in C$ there is nothing to prove because C is $(\delta + \varepsilon)$-separated. Assume now that $p = y$. We have that $\operatorname{dist}(x, p) \leq \delta$ because $x, p \in B$ and $|B|_\delta = 1$. Thus

$$\operatorname{dist}(x, q) \leq \operatorname{dist}(x, p) + \operatorname{dist}(p, q) \leq \varepsilon + \delta.$$

But this is a contradiction because $x, q \in C$ and C is $(\varepsilon + \delta)$-separated. □

Lemma 1.6. *If f is $(m + l, n + 1)$-expansive, then f is (m, n)-expansive or $(l, 1)$-expansive.*

Proof. Let us argue by contradiction and take an $(m + l, n + 1)$-expansive constant $\alpha > 0$. Since f is not (m, n)-expansive for $\varepsilon \in (0, \alpha)$ there is a set $A \subset X$ such that $|A| = m$ and $|f^k(A)|_\varepsilon \leq n$ for all $k \in \mathbb{Z}$. Take $\delta > 0$ such that $|A| = |A|_\delta$ and $\delta + \varepsilon < \alpha$.

Since f is not $(l, 1)$-expansive there is B such that $|B| = l$ and $|f^k(B)|_\delta = 1$ for all $k \in \mathbb{Z}$. By Lemma 1.5 we have that

$$|f^k(A \cup B)|_{\delta+\varepsilon} \leq |f^k(A)|_\varepsilon + |f^k(B)|_\delta - |A \cap B| \leq n + 1 - |A \cap B|$$

for all $k \in \mathbb{Z}$. Also, we know that $|A \cup B| = m + l - |A \cap B|$. If we denote $r = |A \cap B|$, then f is not $(m + l - r, n + 1 - r)$-expansive. And by Proposition 1.3 we conclude that f is not $(m+l, n+1)$-expansive. This contradiction proves the lemma. □

Proof of Proposition 1.4. Assume by contradiction that f is not $(a, 1)$-expansive. Since f is (an, n)-expansive, by Lemma 1.6 we have that f has to be $(a(n - 1), n - 1)$-expansive. Arguing inductively we can prove that f is $(a(n - j), n - j)$-expansive for $j = 1, 2, \ldots, n - 1$. In particular, f is $(a, 1)$-expansive, which is a contradiction that proves the proposition. □

Corollary 1.7. *If $m \leq an$ and f is (m, n)-expansive, then f is $(a, 1)$-expansive (i.e., $(a - 1)$-expansive in the sense of* [9]*). In particular, if $m \leq 2n$ and f is (m, n)-expansive, then f is expansive.*

Proof. By Proposition 1.3 we have that f is (an, n)-expansive. Therefore, by Proposition 1.4 we have that f is $(a, 1)$-expansive. □

2. Separating 4 points

In this section we prove that $(n+1,n)$-expansiveness with $n \geq 3$ implies that X is finite.

Theorem 2.1. *If X is a compact metric space admitting a $(4,3)$-expansive homeomorphism, then X is a finite set.*

Proof. By contradiction assume that f is a $(4,3)$-expansive homeomorphism of X with $|X| = \infty$ and take an expansive constant $\delta > 0$. We know that f cannot be positive expansive (see [4,7] for a proof). Therefore there are x_1, x_2 such that $x_1 \neq x_2$ and

$$\operatorname{dist}(f^k(x_1), f^k(x_2)) < \delta \tag{1}$$

for all $k \geq 0$. Analogously, f^{-1} is not positive expansive, and we can take y_1, y_2 such that $y_1 \neq y_2$ and

$$\operatorname{dist}(f^k(y_1), f^k(y_2)) < \delta \tag{2}$$

for all $k \leq 0$. Consider the set $A = \{x_1, x_2, y_1, y_2\}$. We have that $2 \leq |A| \leq 4$ (we do not know if the 4 points are different). By inequalities (1) and (2) we have that $|f^k(A)|_\delta < |A|$ for all $k \in \mathbb{Z}$. If $n = |A|$, then we have that f is not $(n, n-1)$-expansive. In any case, $n = 2, 3$ or 4, by Proposition 1.3 (see Table 1) we conclude that f is not $(4,3)$-expansive. This contradiction finishes the proof. □

Remark 2.2. If X is a compact metric space admitting a $(n+1,n)$-expansive homeomorphism with $n \geq 3$, then X is a finite set. It follows by Theorem 2.1 and Proposition 1.3.

Corollary 2.3. *If $f\colon X \to X$ is a homeomorphism of a compact metric space and $|X| = \infty$, then for all $\delta > 0$ and $m \geq 4$ there is $A \subset X$ with $|A| = m$ such that $|f^k(A)|_\delta < |A|$ for all $k \in \mathbb{Z}$.*

Proof. It is just a restatement of Remark 2.2. □

3. On Peano continua

In this section we study (m,n)-expansiveness on Peano continua. Let us start recalling that a *continuum* is a compact connected metric space and a *Peano continuum* is a locally connected continuum. A singleton space ($|X| = 1$) is a *trivial* Peano continuum. For $x \in X$ and $\delta > 0$ define the *stable* and *unstable* set of x as

$$W^s_\delta(x) = \{y \in X : \operatorname{dist}(f^k(x), f^k(y)) \leq \delta \,\forall\, k \geq 0\},$$
$$W^u_\delta(x) = \{y \in X : \operatorname{dist}(f^k(x), f^k(y)) \leq \delta \,\forall\, k \leq 0\}.$$

Remark 3.1. Notice that (m,n)-expansiveness implies continuum-wise expansiveness for all $m > n \geq 1$. Recall that f is *continuum-wise expansive* if there is $\delta > 0$ such that if $\operatorname{diam}(f^k(A)) < \delta$ for all $k \in \mathbb{Z}$ and some continuum $A \subset X$, then $|A| = 1$.

Theorem 3.2. *If X is a non-trivial Peano continuum, then no homeomorphism of X is (m, n)-expansive if $2m \geq 3n$.*

Proof. Let δ be a positive real number and assume that f is (m, n)-expansive. As we remarked above, f is a continuum-wise expansive homeomorphism. It is known (see [5, 6]) that for such homeomorphisms on a Peano continuum, every point has non-trivial stable and unstable sets. Take n different points $x_1, \dots, x_n \in X$ and let $\delta' \in (0, \delta)$ be such that $\text{dist}(x_i, x_j) > 2\delta'$ if $i \neq j$. For each $i = 1, \dots, n$, we can take $y_i \in W^s_{\delta'}(x_i)$ and $z_i \in W^u_{\delta'}(x_i)$ with $x_i \neq y_i$ and $x_i \neq z_i$. Consider the set $A = \{x_1, y_1, z_1, \dots, x_n, y_n, z_n\}$. Since $\text{dist}(x_i, x_j) > 2\delta'$ if $i \neq j$, and $y_i, z_i \in B_{\delta'}(x_i)$ we have that $|A| = 3n$. If A_i denotes the set $\{x_i, y_i, z_i\}$ we have that $|f^k(A_i)|_{\delta'} \leq 2$ for all $k \in \mathbb{Z}$. This is because if $k \geq 0$, then $\text{dist}(f^k(x_i), f^k(y_i)) \leq \delta'$ and if $k \leq 0$, then $\text{dist}(f^k(x_i), f^k(z_i)) \leq \delta'$. Therefore $|f^k(A)|_{\delta'} \leq 2n$, and then $|f^k(A)|_\delta \leq 2n$. Since $\delta > 0$ and $n \geq 1$ are arbitrary, we have that f is not $(3n, 2n)$ expansive for all $n \geq 1$. Finally, by Proposition 1.3, we have that f is not (m, n)-expansive if $2m \geq 3n$. □

Corollary 3.3. *If $f \colon X \to X$ is a homeomorphism and X is a non-trivial Peano continuum, then for all $\delta > 0$ there is $A \subset X$ such that $|A| = 3$ and $|f^k(A)|_\delta \leq 2$ for all $k \in \mathbb{Z}$.*

Proof. By Theorem 3.2 we know that f is not $(3, 2)$-expansive. Therefore, the proof follows by definition. □

4. Hyper-expansive homeomorphisms

Denote by $\mathcal{K}(X)$ the set of compact subsets of X. This space is usually called as the *hyper-space* of X. We recommend the reader to see [11] for more on the subject of hyper-spaces and the proofs of the results that we will cite below. In the set $\mathcal{K}(X)$ we consider the Hausdorff distance dist_H making $(\mathcal{K}(X), \text{dist}_H)$ a compact metric space. Recall that

$$\text{dist}_H(A, B) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon(B) \text{ and } B \subset B_\varepsilon(A)\}, \tag{3}$$

where $B_\varepsilon(C) = \cup_{x \in C} B_\varepsilon(x)$ and $B_\varepsilon(x)$ is the usual ball of radius ε centered at x. As usual, we let f to act on $\mathcal{K}(X)$ as $f(A) = \{f(a) : a \in A\}$.

Definition 3. We say that f is *hyper-expansive* if $f \colon \mathcal{K}(X) \to \mathcal{K}(X)$ is expansive, i.e., there is $\delta > 0$ such that given two compact sets $A, B \subset X$, $A \neq B$, there is $k \in \mathbb{Z}$ such that $\text{dist}_H(f^k(A), f^k(B)) > \delta$ where dist_H is the Hausdorff distance.

In [1] it is shown that $f \colon X \to X$ is hyper-expansive if and only if f has a finite number of orbits (i.e., there is a finite set $A \subset X$ such that $X = \cup_{k \in \mathbb{Z}} f^k(A)$) and the non-wandering set is a finite union of periodic points which are attractors or repellers. Recall that a point x is in the *non-wandering set* if for every neighborhood U of x there is $k > 0$ such that $f^k(U) \cap U \neq \emptyset$. A point x is *periodic* if for some $k \geq 0$ it holds that $f^k(x) = x$. The orbit

$\gamma = \{x, f(x), \dots, f^{k-1}(x)\}$ is a *periodic orbit* if x is a periodic point. A periodic orbit γ is an *attractor* (*repeller*) if there is a compact neighborhood U of γ such that $f^k(U) \to \gamma$ in the Hausdorff distance as $k \to \infty$ (resp. $k \to -\infty$).

Theorem 4.1. *If $f\colon X \to X$ is a hyper-expansive homeomorphism and $|X| = \infty$, then f is (m,n)-expansive for some $m > n \geq 1$ if and only if $m \leq 3$.*

Proof. Let us start with the direct part of the theorem. Let P_a be the set of periodic attractors, P_r the set of periodic repellers and take $x_1, \dots, x_j$ one point in each wandering orbit. (Recall that, as we said above, hyper-expansiveness implies that f has just a finite number of orbits.) Define $Q = \{x_1, \dots, x_j\}$. Take $\delta > 0$ such that

(1) if $p, q \in P_a \cup P_r$ and $p \neq q$, then $\operatorname{dist}(p,q) > \delta$,
(2) if $x_i \in Q$, then $B_\delta(x_i) = \{x_i\}$ (recall that wandering points are isolated by [1]),
(3) if $p \in P_a$, $x_i \in Q$ and $k \leq 0$, then $\operatorname{dist}(p, f^k(x_i)) > \delta$,
(4) if $q \in P_r$, $x_i \in Q$ and $k \geq 0$, then $\operatorname{dist}(p, f^k(x_i)) > \delta$ and
(5) if $x, y \in Q$ and $k > 0 > l$, then $\operatorname{dist}(f^k(x), f^l(y)) > \delta$.

Let us prove that such δ is a $(3,2)$-expansive constant. Take $a, b, c \in X$ with $|\{a,b,c\}| = 3$. The proof is divided by cases:

- If $a, b, c \in P = P_a \cup P_r$, then item 1 above concludes the proof.
- If $a, b \in P$ and $c \notin P$, then there is $k \in \mathbb{Z}$ such that $f^k(c) \in Q$. In this case items 1 and 2 conclude the proof.
- Assume now that $a \in P$ and $b, c \notin P$. Without loss of generality let us suppose that a is a repeller. Let $k_b, k_c \in \mathbb{Z}$ be such that $f^{k_b}(b), f^{k_c}(c) \in Q$. Define $k = \min\{k_b, k_c\}$. In this way: $\operatorname{dist}(f^k(a), f^k(b)), \operatorname{dist}(f^k(a), f^k(c)) \geq \delta$ by item 4 and $\operatorname{dist}(f^k(b), f^k(c)) \geq \delta$ by item 2.
- If $a, b, c \notin P$, then consider $k_a, k_b, k_c \in \mathbb{Z}$ such that $f^{k_a}(a), f^{k_b}(b), f^{k_c}(c) \in Q$. Assume, without loss, that $k_a \leq k_b \leq k_c$. Take $k = k_b$. In this way, items 2 and 5 finishes the direct part of the proof.

To prove the converse, we will show that f is not $(m,3)$-expansive for all $m > 3$. Take $\delta > 0$. Notice that since $X = \infty$ there is at least one wandering point x. Without loss of generality assume that $\lim_{k\to\infty} f^k(x) = p_a$ an attractor fixed point and $\lim_{k\to-\infty} f^k(x) = p_r$ a repeller fixed point. Take $k_1, k_2 \in \mathbb{Z}$ such that $\operatorname{dist}(f^k(x), p_r) < \delta$ for all $k \leq k_1$ and $\operatorname{dist}(f^k(x), p_a) < \delta$ for all $k \geq k_2$. Let $l = k_2 - k_1$ and define $x_1 = f^{-k_1}(x)$, and $x_{i+1} = f^l(x_i)$ for all $i \geq 1$. Consider the set $A = \{x_1, \dots, x_m\}$. By construction we have that $|A| = m$ and $|f^k(A)|_\delta \leq 3$ for all $k \in \mathbb{Z}$. Thus, proving that f is not $(m,3)$-expansive if $m > 3$. □

Remark 4.2. In light of the previous proof one may wonder if a *smart* δ-observer will not be able to say that A has more than 3 points. We mean, we are assuming that a δ-observer will say that A has n' points with

$$n' = \max_{k\in\mathbb{Z}} |f^k(A)|_\delta.$$

According to the dynamic of the set A in the previous proof, we guess that with more reasoning a smarter δ-observer will find that A has more than 3 points.

Theorem 4.1 gives us examples of $(3,2)$-expansive homeomorphisms on infinite countable compact metric spaces. A natural question is: does $(3,2)$-expansiveness implies hyper-expansiveness? I do not know the answer, but let us remark some facts that may be of interest. If f is $(3,2)$-expansive, then:

- For all $x \in X$ either the stable or the unstable set must be trivial. It follows by the arguments of the proof of Theorem 3.2.
- If x, y are doubly asymptotic, i.e., $\operatorname{dist}(f^k(x), f^k(y)) \to 0$ as $k \to \pm\infty$, then they are isolated points of the space. Suppose that x were an accumulation point. Given $\delta > 0$ take k_0 such that if $|k| > k_0$, then $\operatorname{dist}(f^k(x), f^k(y)) < \delta$. Take a point z close to x such that $\operatorname{dist}(f^k(x), f^k(z)) < \delta$ if $|k| \leq k_0$ (we are just using the continuity of f). Then x, y, z contradicts $(3,2)$-expansiveness.

Proposition 4.3. *There are* $(4,2)$*-expansive homeomorphisms that are not* $(3,2)$*-expansive.*

Proof. Let us prove it giving an example. Consider a countable compact metric space X and a homeomorphism $f\colon X \to X$ with the following properties:

(1) f has 5 orbits,
(2) $a, b, c \in X$ are fixed points of f,
(3) there is $x \in X$ such that $\lim_{k\to-\infty} f^k(x) = a$ and $\lim_{k\to+\infty} f^k(x) = b$,
(4) there is $y \in X$ such that $\lim_{k\to-\infty} f^k(y) = b$ and $\lim_{k\to+\infty} f^k(y) = c$.

In order to see that f is not $(3,2)$-expansive consider $\varepsilon > 0$. Take $k_0 \in \mathbb{Z}$ such that for all $k \geq k_0$ it holds that $\operatorname{dist}(f^k(x), b) < \varepsilon$ and $\operatorname{dist}(f^{-k}(y), b) < \varepsilon$. Define $u = f^{k_0}(x)$ and $v = f^{-k_0}(y)$. In this way $\|\{f^k(u), b, f^k(v)\}\|_\varepsilon \leq 2$ for all $k \in \mathbb{Z}$. This proves that f is not $(3,2)$-expansive.

Let us now indicate how to prove that f is $(4,2)$-expansive. Consider $\varepsilon > 0$ such that if $i \geq 0$ and $j \in \mathbb{Z}$, then $\operatorname{dist}(f^{-i}(x), f^j(y)) > \varepsilon$ and $\operatorname{dist}(f^j(x), f^i(y)) > \varepsilon$. Now, a similar argument as in the proof of Theorem 4.1, shows that f is $(4,2)$-expansive. □

5. With the shadowing property

In this section we prove that an important class of homeomorphisms are not (m,n)-expansive for all $m > n \geq 2$. In order to state this result let us recall that a δ-*pseudo orbit* is a sequence $\{x_k\}_{k\in\mathbb{Z}}$ such that $\operatorname{dist}(f(x_k), x_{k+1}) \leq \delta$ for all $k \in \mathbb{Z}$. We say that a homeomorphism has the *shadowing property* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\{x_k\}_{k\in\mathbb{Z}}$ is a δ-pseudo orbit, then there is x such that $\operatorname{dist}(f^k(x), x_k) < \varepsilon$ for all $k \in \mathbb{Z}$. In this case we say that x ε-*shadows* the δ-pseudo orbit.

Theorem 5.1. *Let $f\colon X \to X$ be a homeomorphism of a compact metric space X. If f has the shadowing property and there are $x, y \in X$ such that*

$$0 = \liminf_{k\to\infty} \operatorname{dist}(f^k(x), f^k(y)) < \limsup_{k\to\infty} \operatorname{dist}(f^k(x), f^k(y)),$$

then f is not (m,n)-expansive if $m > n \geq 2$.

Proof. By Proposition 1.3 it is enough to prove that f cannot be $(m,2)$-expansive if $m > 2$. Consider $\varepsilon > 0$. We will define a set A with $|A| = \infty$ such that for all $k \in \mathbb{Z}$, $f^k(A) \subset B_\varepsilon(f^k(x)) \cup B_\varepsilon(f^k(y))$, proving that f is not $(m,2)$-expansive for all $m > 2$.

Consider two increasing sequences $a_l, b_l \in \mathbb{Z}$, $\rho \in (0,\varepsilon)$ and $\delta > 0$ such that

$$\begin{aligned}
&a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \cdots,\\
&\operatorname{dist}(f^{a_l}(x), f^{a_l}(y)) < \delta,\\
&\operatorname{dist}(f^{b_l}(x), f^{b_l}(y)) > \rho
\end{aligned}$$

for all $l \geq 1$ and assume that every δ-pseudo orbit can be $(\rho/2)$-shadowed. For each $l \geq 1$ define the δ-pseudo orbit z^l_k as

$$z^l_k = \begin{cases} f^k(x) & \text{if } k < a_l,\\ f^k(y) & \text{if } k \geq a_l. \end{cases}$$

Then, for every $l \geq 1$ there is a point w^l whose orbit $(\rho/2)$-shadows the δ-pseudo orbit $\{z^l_k\}_{k\in\mathbb{Z}}$. Let us now prove that if $1 \leq l < s$, then $w^l \neq w^s$. We have that $a_l < b_l < a_s$. Therefore $z^l_{b_l} = f^{b_l}(y)$ and $z^s_{b_l} = f^{b_l}(x)$. Since w^l and w^s $(\rho/2)$-shadows the pseudo orbits z^l and z^s, respectively, we have that

$$\operatorname{dist}(f^{b_l}(w^l), f^{b_l}(y)), \operatorname{dist}(f^{b_l}(w^s), f^{b_l}(x)) < \rho/2.$$

We conclude that $w^l \neq w^s$ because $\operatorname{dist}(f^{b_l}(x), f^{b_l}(y)) > \rho$. Therefore, if we define $A = \{w^l : l \geq 1\}$ we have that $|A| = \infty$. Finally, since $\rho < \varepsilon$, we have that $f^k(A) \subset B_\varepsilon(f^k(x)) \cup B_\varepsilon(f^k(y))$ for all $k \in \mathbb{Z}$. Therefore, $|f^k(A)|_\varepsilon \leq 2$ for all $k \in \mathbb{Z}$. □

Remark 5.2. Examples where Theorem 5.1 can be applied are Anosov diffeomorphisms and symbolic shift maps. Also, if $f\colon X \to X$ is a homeomorphism with an invariant set $K \subset X$ such that $f\colon K \to K$ is conjugated to a symbolic shift map, then Theorem 5.1 holds because the (m,n)-expansiveness of f in X implies the (m,n)-expansiveness of f restricted to any compact invariant set $K \subset X$ as can be easily checked.

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