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ON SPACELIKE ROTATIONAL SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract. In this paper, we study a class of spacelike rotational surfaces in the Minkowski 4-space $\mathbb{E}_1^4$ with meridian curves lying in 2-dimensional spacelike planes and having pointwise 1-type Gauss map. We obtain all such surfaces with pointwise 1-type Gauss map of the first kind. Then we prove that the spacelike rotational surface with flat normal bundle and pointwise 1-type Gauss map of the second kind is an open part of a spacelike 2-plane in $\mathbb{E}_1^4$.

1. Introduction

The notion of finite type submanifolds of Euclidean spaces was introduced by B.-Y. Chen in late 1970's [2]. Since then many works have been done to characterize or classify submanifolds of Euclidean space or pseudo-Euclidean space in terms of finite type. Also, B.-Y. Chen and P. Piccinni extended the notion of finite type to differentiable maps, in particular, to Gauss map of submanifolds in [4]. A smooth map ϕ on a submanifold M of a Euclidean space or a pseudo-Euclidean space is said to be of *finite type* if ϕ can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of M, that is, $\phi = \phi_0 + \sum_{i=1}^{k} \phi_i$, where ϕ_0 is a constant map, $\phi_1, \ldots, \phi_k$ non-constant maps such that $\Delta\phi_i = \lambda_i\phi_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$.

If a submanifold M of a Euclidean space or a pseudo-Euclidean space has 1-type Gauss map ν, then ν satisfies $\Delta\nu = \lambda(\nu + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. In [4], B.-Y. Chen and P. Piccinni studied compact submanifolds of Euclidean spaces with finite type Gauss map. However, the Laplacian of the Gauss map of several surfaces and hypersurfaces such as helicoids of the 1st, 2nd, and 3rd kind, conjugate Enneper's surface of the second kind and B-scrolls in a 3-dimensional Minkowski space $\mathbb{E}_1^3$, generalized catenoids, spherical n-cones, hyperbolical n-cones and Enneper's hypersurfaces in $\mathbb{E}_1^{n+1}$ take the

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form

$$\Delta\nu = f(\nu + C) \tag{1}$$

for some smooth function f on M and some constant vector C ([11, 18]). A submanifold of a pseudo-Euclidean space is said to have *pointwise 1-type Gauss map* if its Gauss map satisfies (1) for some smooth function f on M and some constant vector C. In particular, if C is zero, it is said to be of *the first kind*. Otherwise, it is said to be of *the second kind* (cf. [1, 3, 5, 6, 7, 10, 12, 14, 17, 19]).

The complete classification of ruled surfaces in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the first kind was obtained in [18]. Recently, ruled surfaces in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind were studied in [8, 13]. Also, a complete classification of rational surfaces of revolution in $\mathbb{E}_1^3$ satisfying (1) was given in [17], and it was proved that a right circular cone and a hyperbolic cone in $\mathbb{E}_1^3$ are the only rational surfaces of revolution in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind. The rotational hypersurfaces in Lorentz-Minkowski space with pointwise 1-type Gauss map was studied in [11]. Moreover, in [20] a complete classification of cylindrical and non-cylindrical surfaces in $\mathbb{E}_1^m$ with pointwise 1-type Gauss map of the first kind was obtained.

Recently, the author and Turgay have studied some characterization and classifications on spacelike surfaces in the Minkowski space $\mathbb{E}_1^4$ with pointwise 1-type Gauss map [15, 16].

In this work, we study a class of spacelike rotational surfaces in the Minkowski 4-space $\mathbb{E}_1^4$ defined by (11) with meridian curves lying in 2-dimensional spacelike planes and having pointwise 1-type Gauss map. We obtain all such surfaces with pointwise 1-type Gauss map of the first kind. We conclude that there exists no non-planar maximal spacelike rotational surface in $\mathbb{E}_1^4$ with pointwise 1-type Gauss map of the first kind. We also prove that the spacelike rotational surface with flat normal bundle and pointwise 1-type Gauss map of the second kind is an open part of a spacelike 2-plane in $\mathbb{E}_1^4$.

2. Preliminaries

Let $\mathbb{E}_1^m$ denote m-dimensional Minkowski space with the canonical metric tensor given by

$$g = dx_1^2 + dx_2^2 + \cdots + dx_{m-1}^2 - dx_m^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}_1^m$.

A vector $\zeta \in \mathbb{E}_1^m$ is called spacelike (resp., time-like or light-like) if $\langle \zeta, \zeta \rangle > 0$ or $\zeta = 0$ (resp., $\langle \zeta, \zeta \rangle < 0$ or $\langle \zeta, \zeta \rangle = 0$ with $\zeta \neq 0$). A submanifold M of $\mathbb{E}_1^m$ is said to be spacelike if every non-zero tangent vector on M is spacelike.

Let M be an oriented n-dimensional submanifold in an $(n+2)$-dimensional Minkowski space $\mathbb{E}_1^{n+2}$. We choose an oriented local orthonormal frame $\{e_1, \ldots, e_{n+2}\}$ on M with $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$ such that $e_1, \ldots, e_n$ are tangent to M

and e_{n+1}, e_{n+2} are normal to M. We use the following convention on the range of indices: $1 \le i, j, k, \ldots \le n$, $n+1 \le r, s, t, \ldots \le n+2$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}_1^{n+2}$ and ∇ the induced connection on M. Denote by $\{\omega^1, \ldots, \omega^{n+2}\}$ the dual frame and by $\{\omega_{AB}\}$, $A, B = 1, \ldots, n+2$, the connection forms associated to $\{e_1, \ldots, e_{n+2}\}$. Then we have

$$\begin{aligned}
\widetilde{\nabla}_{e_k} e_i &= \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{n+2} \varepsilon_r h_{ik}^r e_r, \\
\widetilde{\nabla}_{e_k} e_s &= -A_r(e_k) + \sum_{r=n+1}^{n+2} \varepsilon_r \omega_{sr}(e_k) e_r, \\
D_{e_k} e_s &= \sum_{r=n+1}^{n+2} \varepsilon_r \omega_{sr}(e_k) e_r,
\end{aligned}$$

where D is the normal connection, h_{ij}^r the coefficients of the second fundamental form h, and A_r the Weingarten map in the direction e_r.

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined, respectively, by

$$H = \frac{1}{n} \sum_{r,i} \varepsilon_i \varepsilon_r h_{ii}^r e_r \tag{2}$$

and

$$\|h\|^2 = \sum_{r,i,j} \varepsilon_i \varepsilon_j \varepsilon_r h_{ij}^r h_{ji}^r. \tag{3}$$

A submanifold M is said to have parallel mean curvature vector H if the mean curvature vector satisfies $DH = 0$ identically.

The gradient of a smooth function f on M is defined by $\nabla f = \sum_{i=1}^{n} \varepsilon_i e_i(f) e_i$, and the Laplace operator acting on M is $\Delta = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} e_i - e_i e_i)$.

The Codazzi equation of M in E_1^{n+2} is given by

$$\begin{aligned}
&h_{ij,k}^r = h_{jk,i}^r, \\
&h_{jk,i}^r = e_i(h_{jk}^r) + \sum_{s=n+1}^{n+2} \varepsilon_s h_{jk}^s \omega_{sr}(e_i) - \sum_{\ell=1}^{n} \varepsilon_\ell \left(\omega_{j\ell}(e_i) h_{\ell k}^r + \omega_{k\ell}(e_i) h_{\ell j}^r \right).
\end{aligned} \tag{4}$$

Also, from the Ricci equation of M in E_1^{n+2}, we have

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_s}](e_j), e_k \rangle = \sum_{i=1}^{n} \varepsilon_i \left(h_{ik}^r h_{ij}^s - h_{ij}^r h_{ik}^s \right), \tag{5}$$

where R^D is the normal curvature tensor.

A spacelike submanifold M in $\mathbb{E}_1^m$ is said to have flat normal bundle if its normal curvature tensor R^D vanishes identically.

Let $G(m-n, m)$ be the Grassmannian manifold consisting of all oriented $(m-n)$-planes through the origin of an m-dimensional pseudo-Euclidean space

$\mathbb{E}_t^m$ with index t and $\bigwedge^{m-n}\mathbb{E}_t^m$ the vector space obtained by the exterior product of $m-n$ vectors in $\mathbb{E}_t^m$. Let $f_{i_1}\wedge\cdots\wedge f_{i_{m-n}}$ and $g_{i_1}\wedge\cdots\wedge g_{i_{m-n}}$ be two vectors in $\bigwedge^{m-n}\mathbb{E}_t^m$, where $\{f_1,f_2,\ldots,f_m\}$ and $\{g_1,g_2,\ldots,g_m\}$ are two orthonormal bases of $\mathbb{E}_t^m$. Define an indefinite inner product $\langle,\rangle$ on $\bigwedge^{m-n}\mathbb{E}_t^m$ by

$$\langle f_{i_1}\wedge\cdots\wedge f_{i_{m-n}}, g_{i_1}\wedge\cdots\wedge g_{i_{m-n}}\rangle = \det(\langle f_{i_\ell}, g_{j_k}\rangle). \tag{6}$$

Therefore, for some positive integer s, we may identify $\bigwedge^{m-n}\mathbb{E}_t^m$ with some pseudo-Euclidean space $\mathbb{E}_s^N$, where $N=\binom{m}{m-n}$. Let $e_1,\ldots,e_n,e_{n+1},\ldots,e_m$ be an oriented local orthonormal frame on an n-dimensional pseudo-Riemannian submanifold M in $\mathbb{E}_t^m$ with $\varepsilon_B=\langle e_B,e_B\rangle=\pm1$ such that $e_1,\ldots,e_n$ are tangent to M and $e_{n+1},\ldots,e_m$ are normal to M. The map $\nu: M\to G(m-n,m)\subset\mathbb{E}_s^N$ from an oriented pseudo-Riemannian submanifold M into $G(m-n,m)$ defined by

$$\nu(p)=(e_{n+1}\wedge e_{n+2}\wedge\cdots\wedge e_m)(p) \tag{7}$$

is called the *Gauss map* of M that is a smooth map which assigns to a point p in M the oriented $(m-n)$-plane through the origin of $\mathbb{E}_t^m$ and parallel to the normal space of M at p [19].

We put $\varepsilon=\langle\nu,\nu\rangle=\varepsilon_{n+1}\varepsilon_{n+2}\cdots\varepsilon_m=\pm1$ and

$$\widetilde{M}_s^{N-1}(\varepsilon)=\begin{cases}\mathbb{S}_s^{N-1}(1) & \text{in } \mathbb{E}_s^N \text{ if } \varepsilon=1,\\ \mathbb{H}_{s-1}^{N-1}(-1) & \text{in } \mathbb{E}_s^N \text{ if } \varepsilon=-1.\end{cases}$$

Then the Gauss image $\nu(M)$ can be viewed as $\nu(M)\subset\widetilde{M}_s^{N-1}(\varepsilon)$.

2.1. Rotational surfaces in E_1^4

In [21], Moore introduced general rotational surfaces in the Euclidean space E^4. A rotational surface in E^4 is a surface left invariant by a rotation in E^4 which is defined as a linear transformation of positive determinant preserving distance and leaving one point fixed. Let $\beta(s)=(x(s),y(s),z(s),w(s))$ be a regular smooth curve on an open interval I in $\mathbb{R}$, and let a and b be some real numbers. Then, a general rotational surface M in E^4 with the meridian curve β and the rates of rotation a and b is given by

$$\begin{aligned}X(s,t)=\Big(&x(s)\cos at-y(s)\sin at,\ x(s)\sin at+y(s)\cos at,\\ &z(s)\cos bt-w(s)\sin bt,\ z(s)\sin bt+w(s)\cos bt\Big).\end{aligned} \tag{8}$$

If a or b is zero, then a surface M defined by (8) is called a simple rotational surface as the rotation subgroup which produces M to be a simple rotation [9].

Using the idea of Moore we consider a class of rotational surfaces in E_1^4 which is invariant under the following subgroup of linear isometries of the Minkowski

space E_1^4,

$$G_{(a,b)} = \Big\{ B_t(a,b) = \begin{pmatrix} \cos at & \sin at & 0 & 0 \\ -\sin at & \cos at & 0 & 0 \\ 0 & 0 & \cosh bt & \sinh bt \\ 0 & 0 & \sinh bt & \cosh bt \end{pmatrix} : t \in \mathbb{R} \Big\}. \tag{9}$$

Let $\beta(s) = (x(s), y(s), z(s), w(s)),\ s \in I$ be a spacelike or timelike curve in E_1^4. Then we consider a Moore type rotational surface M with the meridian curve β given by

$$\begin{aligned} X(s,t) &= \beta(s) B_t(a,b) \\ &= \Big(x(s)\cos at - y(s)\sin at,\ x(s)\sin at + y(s)\cos at, \\ &\quad z(s)\cosh bt + w(s)\sinh bt,\ z(s)\sinh bt + w(s)\cosh bt \Big) \end{aligned} \tag{10}$$

which is invariant under the given above subgroup, where a and b are constants. It is also called a *double rotational surface*.

In this work, we study double spacelike rotational surfaces defined by (10) in E_1^4 whose meridians lie in spacelike 2-planes. By choosing $\beta(s) = (x(s), 0, z(s), 0)$ in the x_1x_3-plane, we have from (10) a rotational surface E_1^4 given by

$$F(s,t) = (x(s)\cos at, x(s)\sin at, z(s)\cosh bt, z(s)\sinh bt), \tag{11}$$

where $s \in I \subset \mathbb{R}$, $t \in (0, 2\pi)$. This surface is spacelike if $a^2x^2(s) - b^2z^2(s) > 0$ on I.

Suppose that s is the arc length parameter of β. Then, $x'^2(s) + z'^2(s) = 1$, and the curvature function κ of β is given by $\kappa(s) = x'(s)z''(s) - x''(s)z'(s)$, $s \in I$.

Let M be a rotational surface E_1^4 defined by (11). We consider the following orthonormal moving frame field $\{e_1, e_2, e_3, e_4\}$ on M such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M:

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{q}\frac{\partial}{\partial t}, \tag{12}$$

$$e_3 = (-z'\cos at, -z'\sin at, x'\cosh bt, x'\sinh bt), \tag{13}$$

$$e_4 = \frac{1}{q}(-bz\sin at, bz\cos at, ax\sinh bt, ax\cosh bt), \tag{14}$$

where $\varepsilon = \operatorname{sgn}(a^2x^2(s) - b^2z^2(s)) = \pm 1$ and $q = \sqrt{\varepsilon(a^2x^2(s) - b^2z^2(s))} \neq 0$. Then $\varepsilon_1 = 1,\ \varepsilon_2 = \varepsilon,\ \varepsilon_3 = 1$ and $\varepsilon_4 = -\varepsilon$.

By a direct computation we have the components of the second fundamental form and the connection forms as follows

$$h_{11}^3 = \kappa, \qquad h_{22}^3 = \frac{a^2xz' + b^2zx'}{a^2x^2 - b^2z^2}, \qquad h_{12}^3 = 0, \tag{15}$$

$$h_{12}^4 = \frac{ab(zx' - xz')}{a^2x^2 - b^2z^2}, \qquad h_{11}^4 = 0, \qquad h_{22}^4 = 0, \tag{16}$$

$$\omega_{12}(e_1)=0, \qquad \omega_{12}(e_2)=\frac{a^2xx'-b^2zz'}{a^2x^2-b^2z^2}, \tag{17}$$

$$\omega_{34}(e_1)=0, \qquad \omega_{34}(e_2)=-\frac{ab(xx'+zz')}{a^2x^2-b^2z^2}. \tag{18}$$

Thus, the shape operators of M are of the form

$$A_3=\begin{pmatrix} h^3_{11} & 0 \\ 0 & h^3_{22}\end{pmatrix} \text{ and } A_4=\begin{pmatrix} 0 & h^4_{12} \\ h^4_{12} & 0\end{pmatrix} \tag{19}$$

from which we obtain the mean curvature vector and the normal curvature of M as

$$H=\frac{1}{2}(h^3_{11}+\varepsilon h^3_{22})e_3, \tag{20}$$

$$R^D(e_1,e_2;e_3,e_4)=h^4_{12}(\varepsilon h^3_{22}-h^3_{11}). \tag{21}$$

On the other hand, from the Codazzi equation (4) we have

$$e_1(h^3_{22})=\omega_{12}(e_2)\left(h^3_{11}-\varepsilon h^3_{22}\right)+\varepsilon h^4_{12}\omega_{34}(e_2), \tag{22}$$

$$e_1(h^4_{12})=-2\varepsilon\omega_{12}(e_2)h^4_{12}+h^3_{11}\omega_{34}(e_2). \tag{23}$$

3. Double spacelike rotational surfaces with pointwise 1-type Gauss map of the first kind

In this section, we obtain spacelike rotational surfaces defined by (11) with pointwise 1-type Gauss map of the first kind.

The Laplacian of the Gauss map ν for an n-dimensional submanifold M in a pseudo-Euclidean space $\mathbb{E}^{n+2}_t$ was given:

Lemma 3.1 ([16]). *Let M be an n-dimensional submanifold of a pseudo-Euclidean space E^{n+2}_t. Then, the Laplacian of the Gauss map $\nu=e_{n+1}\wedge e_{n+2}$ is given by*

$$\begin{aligned}\Delta\nu=&||h||^2\nu+2\sum_{j<k}\varepsilon_j\varepsilon_kR^D(e_j,e_k;e_{n+1},e_{n+2})e_j\wedge e_k\\&+\nabla(\mathrm{tr}A_{n+1})\wedge e_{n+2}+e_{n+1}\wedge\nabla(\mathrm{tr}A_{n+2})\\&+n\sum_{j=1}^{n}\varepsilon_j\omega_{(n+1)(n+2)}(e_j)H\wedge e_j,\end{aligned} \tag{24}$$

where $||h||^2$ is the squared length of the second fundamental form, R^D the normal curvature tensor, and $\nabla(\mathrm{tr}A_r)$ the gradient of $\mathrm{tr}A_r$.

In [16], the following results were given for the characterization of spacelike surfaces in E^4_1 with pointwise 1-type Gauss map of the first kind.

Theorem 3.2 ([16]). *Let M be an oriented maximal surface in the Minkowski space $\mathbb{E}^4_1$. Then M has pointwise 1-type Gauss map ν of the first kind if and only if M has flat normal bundle. Hence the Gauss map ν satisfies* (1) *for $f=\|h\|^2$ and $C=0$.*

Theorem 3.3 ([16]). *Let M be an oriented non-maximal spacelike surface in $\mathbb{E}_1^4$. Then M has pointwise 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector.*

We will classify spacelike rotational surfaces in E_1^4 defined by (11) with pointwise 1-type Gauss map of the first kind by using the above theorems. From now on we take $\varepsilon = 1$, that is, $a^2x^2(s) - b^2z^2(s) > 0$.

Theorem 3.4. *Let M be a spacelike rotational surface in E_1^4 defined by* (11). *Then, M is maximal, and its normal bundle is flat if and only if M is an open part of a spacelike plane in $\mathbb{E}_1^4$.*

Proof. Let M be a spacelike rotational surface given by (11). Then, we have an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M in E^4 given by (12)-(14), and the shape operators A_3 and A_4 are given by (19). If M is maximal, and its normal bundle is flat, then (20) and (21) imply, respectively,

$$\kappa + h_{22}^3 = 0, \tag{25}$$

$$h_{12}^4(h_{22}^3 - \kappa) = 0 \tag{26}$$

as $h_{11}^3 = \kappa$, where κ is the curvature of the meridian curve of M. By using these equations we get $h_{12}^4\kappa = 0$. Let $\mathbf{O} = \{p \in M \,|\, h_{12}^4 \neq 0\}$. Suppose that $\mathbf{O} \neq \emptyset$. Then, from $h_{12}^4\kappa = 0$ we have $\kappa(s) = x'(s)z''(s) - x''(s)z'(s) = 0$ which implies that

$$z(s) = c_0x(s) + c_1, \tag{27}$$

where c_0 and c_1 are constants. That is, the meridian curve of M is a line.

Now, from (25) we also have $h_{22}^3 = 0$. By using the second equation in (15) and (27) we obtain that

$$h_{22}^3 = \frac{x'[(a^2 + b^2)c_0x + b^2c_1]}{a^2x^2 - b^2z^2} = 0$$

which gives $c_0 = c_1 = 0$ as $x' \neq 0$. If $x' = 0$, then x and z would be constants, hence the surface M would be degenerate. Therefore, $z = 0$ which implies that M is an open part of the spacelike x_1x_2-plane, that is, M and hence $\mathbf{O}$ are totally geodesic. This is a contradiction, and thus $h_{12}^4 = 0$.

So, from the first equation in (16) we have $xz' - x'z = 0$, i.e., $z = c_0x$, where c_0 is a constant. Hence, β is an open part of a line passing through the origin. Since the curvature κ is zero we have the above case. By a similar argument it is seen that M is an open part of the spacelike x_1x_2-plane.

The converse of the proof of the theorem is trivial. □

By Theorem 3.2 and Theorem 3.4 we state:

Theorem 3.5. *There exists no non-planar maximal spacelike rotational surface in E_1^4 defined by* (11) *with pointwise* 1*-type Gauss map of the first kind.*

Now we investigate non-maximal spacelike rotational surfaces in E_1^4 with parallel mean curvature vector to obtain surfaces in E_1^4 with pointwise 1-type Gauss map of the first kind.

Theorem 3.6. *A non-maximal spacelike rotational surface M in E_1^4 defined by* (11) *has parallel mean curvature vector if and only if it is an open part of the spacelike surface defined by*

$$F(s,t) = (r_0\cos(\frac{s}{r_0})\cos at,\ r_0\cos(\frac{s}{r_0})\sin at,\ r_0\sin(\frac{s}{r_0})\cosh bt, \\ r_0\sin(\frac{s}{r_0})\sinh bt) \tag{28}$$

which is maximal in the de Sitter space $S_1^3(r_0) \subset E_1^4$, where $\tan^2(s/r_0) < (a/b)^2$.

Proof. Let M be a non-maximal spacelike rotational surface in E_1^4 defined by (11). Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal moving frame on M in E_1^4 given by (12)-(14). From (19) we have $H = \frac{1}{2}(h_{11}^3 + h_{22}^3)e_3$. Suppose that the mean curvature vector H is parallel. Then, $\nabla_{e_i}^{\perp} H = 0$ for $i = 1, 2$, and by considering (18) we obtain that

$$\nabla_{e_2}^{\perp} H = -\frac{ab(h_{11}^3 + h_{22}^3)(xx' + zz')}{2(a^2x^2 - b^2z^2)}e_4 = 0.$$

Since M is non-maximal, this equation yields $xx' + zz' = 0$, i.e., $x^2 + z^2 = r_0^2$, where r_0 is a positive real number. Hence, the meridian curve β is an open part of a circle which is parametrized by

$$x(s) = r_0\cos\frac{s}{r_0}, \quad z(s) = r_0\sin\frac{s}{r_0}.$$

The surface is spacelike if $\tan^2(s/r_0) < (a/b)^2$. Therefore, M is an open part of the spacelike surface given by (28).

The converse of the proof follows from a direct calculation. □

By Theorem 3.3 and Theorem 3.6 we have:

Corollary 3.7. *A non-maximal spacelike rotational surface M in E_1^4 defined by* (11) *has pointwise 1-type Gauss map of the first kind if and only if it is an open part of the surface given by* (28)*.*

By combining the results obtained in this section we state a classification theorem:

Theorem 3.8. *Let M be a spacelike rotational surface in E_1^4 defined by* (11)*. Then M has pointwise 1-type Gauss map of the first kind if and only if M is an open part of a spacelike plane or the surface given by* (28)*. Moreover, the Gauss map $\nu = e_3 \wedge e_4$ of the rotational surface* (28) *satisfies* (1) *for $C = 0$ and the function*

$$f = \|h\|^2 = \frac{2}{r_0^2}\Big(1 + \frac{a^2b^2}{(a^2\cos^2(\frac{s}{r_0}) - b^2\sin^2(\frac{s}{r_0}))^2}\Big),$$

where $\tan^2(s/r_0) < (a/b)^2$.

Note that there is no non-planar spacelike rotational surface in E_1^4 defined by (11) with global 1-type Gauss map of the first kind.

4. Double spacelike rotational surfaces with pointwise 1-type Gauss map of the second kind

In this section, we study spacelike rotational surfaces in the Minkowski space $\mathbb{E}_1^4$ with pointwise 1-type Gauss map of the second kind.

Let M be a spacelike surface in $\mathbb{E}_1^4$. We choose a local orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ on M such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M. Let C be a vector field in $\Lambda^2\mathbb{E}_1^4 \equiv \mathbb{E}_3^6$. Since the set $\{e_A \wedge e_B \,|\, 1 \leq A < B \leq 4\}$ is an orthonormal basis for $\mathbb{E}_3^6$, C can be expressed as

$$C = \sum_{1\leq A<B\leq 4} \varepsilon_A \varepsilon_B C_{AB}\, e_A \wedge e_B, \tag{29}$$

where $C_{AB} = \langle C, e_A \wedge e_B\rangle$. As $e_1,\ e_2$ are spacelike, we have $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_4 = -\varepsilon_3$.

For the constancy of C, the following lemma was given in [15]:

Lemma 4.1. *A vector C in $\Lambda^2\mathbb{E}_1^4 \equiv \mathbb{E}_3^6$ written by* (29) *is constant if and only if the following equations are satisfied for $i = 1, 2$*

$$e_i\,(C_{12}) = \varepsilon_3 h^3_{i2} C_{13} - \varepsilon_3 h^4_{i2} C_{14} - \varepsilon_3 h^3_{i1} C_{23} + \varepsilon_3 h^4_{i1} C_{24}, \tag{30}$$

$$e_i\,(C_{13}) = -\,h^3_{i2} C_{12} - \varepsilon_3 \omega_{34}(e_i) C_{14} + \omega_{12}(e_i) C_{23} + \varepsilon_3 h^4_{i1} C_{34}, \tag{31}$$

$$e_i\,(C_{14}) = -\,h^4_{i2} C_{12} - \varepsilon_3 \omega_{34}(e_i) C_{13} + \omega_{12}(e_i) C_{24} + \varepsilon_3 h^3_{i1} C_{34}, \tag{32}$$

$$e_i\,(C_{23}) = h^3_{i1} C_{12} - \omega_{12}(e_i) C_{13} - \varepsilon_3 \omega_{34}(e_i) C_{24} + \varepsilon_3 h^4_{i2} C_{34}, \tag{33}$$

$$e_i\,(C_{24}) = h^4_{i1} C_{12} - \omega_{12}(e_i) C_{14} - \varepsilon_3 \omega_{34}(e_i) C_{23} + \varepsilon_3 h^3_{i2} C_{34}, \tag{34}$$

$$e_i\,(C_{34}) = h^4_{i1} C_{13} - h^3_{i1} C_{14} + h^4_{i2} C_{23} - h^3_{i2} C_{24}. \tag{35}$$

Theorem 4.2. *A spacelike rotational surface M in E_1^4 defined by* (11) *with flat normal bundle has pointwise 1-type Gauss map of the second kind if and only if M is an open part of a spacelike plane in E_1^4.*

Proof. Let M be a spacelike rotational surface in E_1^4 defined by (11). Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal moving frame on M in E_1^4 given by (12)-(14). Then the shape operators A_3 and A_4 are given by (19). Since M has flat normal bundle we have $R^D = h^4_{12}(h^3_{22} - h^3_{11}) = 0$ which implies that $h^4_{12} = 0$ or $h^3_{22} = h^3_{11}$.

Let $\mathbf{O} = \{p \in M \,|\, h^4_{12} \neq 0\}$. Suppose that $\mathbf{O} \neq \emptyset$. Then, $h^3_{22} = h^3_{11}$ on $\mathbf{O}$. Considering (24) and using the Codazzi equation (22) we obtain that

$$\Delta\nu = ||h||^2 \nu + 2h^4_{12}\omega_{34}(e_2) e_1 \wedge e_4 - 2h^3_{11}\omega_{34}(e_2) e_2 \wedge e_3. \tag{36}$$

We assume that the Gauss map on $\mathbf{O}$ is of pointwise 1-type of the second kind. According to the assumption, (1) is satisfied for some function $f \neq 0$ and non-zero constant vector $C \in \mathbb{E}_3^6$. From (1), (29) and (36) we have

$$f(1 - C_{34}) = \|h\|^2, \tag{37}$$

$$fC_{14} = -2h_{12}^4\omega_{34}(e_2), \tag{38}$$

$$fC_{23} = -2h_{11}^3\omega_{34}(e_2), \tag{39}$$

$$C_{12} = C_{13} = C_{24} = 0. \tag{40}$$

From (38) and (39) it is seen that $C_{14} \neq 0$ and $C_{23} \neq 0$. Now (38) and (39) imply that

$$h_{11}^3 C_{14} - h_{12}^4 C_{23} = 0. \tag{41}$$

Since C is a nonzero constant vector, its components satisfy (30)-(35) for $i = 1, 2$. From (30) for $i = 1$, we also obtain that

$$h_{12}^4 C_{14} + h_{11}^3 C_{23} = 0. \tag{42}$$

So, (41) and (42) give that $h_{12}^4 = h_{11}^3 = 0$ which is a contradiction. Therefore, $h_{12}^4 = 0$.

Now, from the first equation in (16) we get $z = cx$. Then, for $x > 0$ and $|c| < |a/b|$ M is a spacelike regular cone in E_1^4. For $c = 0$, M is a part of the spacelike x_1x_2-plane. We suppose that $c \neq 0$. If we parametrize the line $z = cx$ with respect to arc length parameter s, we then have $x = \frac{1}{\sqrt{1+c^2}}s + x_0$, $z = \frac{c}{\sqrt{1+c^2}}s + cx_0,\ s > -x_0\sqrt{1+c^2},\ x_0 \in \mathbb{R}$. Thus, from (15)-(18) we obtain that

$$\begin{aligned}
h_{11}^3 = 0, \qquad & h_{22}^3 = \frac{c(a^2+b^2)}{\sqrt{1+c^2}(a^2-b^2c^2)x}, \\
h_{12}^3 = 0, \qquad & h_{ij}^4 = 0,\ i,j = 1,2, \\
\omega_{12}(e_1) = 0, \quad & \omega_{12}(e_2) = \frac{1}{x\sqrt{1+c^2}}, \\
\omega_{34}(e_1) = 0, \quad & \omega_{34}(e_2) = -\frac{ab\sqrt{1+c^2}}{(a^2-b^2c^2)x}.
\end{aligned} \tag{43}$$

Hence the Laplacian of the Gauss map $\nu = e_3 \wedge e_4$ from (24) is given by

$$\Delta\nu = ||h||^2\nu + e_1(h_{22}^3)e_1 \wedge e_4 - h_{22}^3\omega_{34}(e_2)e_2 \wedge e_3. \tag{44}$$

We assume the cone has pointwise 1-type Gauss map of the second kind. Therefore, from (1), (29), (44) and the Codazzi equation (22) we have

$$f(1 - C_{34}) = \|h\|^2 = (h_{22}^3)^2, \tag{45}$$

$$fC_{14} = -e_1(h_{22}^3) = h_{22}^3\omega_{12}(e_2), \tag{46}$$

$$fC_{23} = -h_{22}^3\omega_{34}(e_2), \tag{47}$$

$$C_{12} = C_{13} = C_{24} = 0. \tag{48}$$

It follows form (43), (46), and (47) that $C_{14} \neq 0$ and $C_{23} \neq 0$. Now, from (46) and (47) we have

$$\omega_{34}(e_2)C_{14} + \omega_{12}(e_2)C_{23} = 0. \tag{49}$$

On the other hand, by considering (48), equation (31) for $i = 2$ implies

$$\omega_{34}(e_2)C_{14} - \omega_{12}(e_2)C_{23} = 0. \tag{50}$$

Thus, considering (43) the solution of equations (49) and (50) gives $C_{14} = C_{23} = 0$ which is a contradiction. That is, $c = 0$, and thus $z = 0$. Therefore M is an open part of a spacelike x_1x_2-plane. □

Let $z = cx,\ x > 0$ and $|c| < |a/b|$. Then, the rotational surface M in E_1^4 defined by

$$F(x,t) = (x\cos at, x\sin at, cx\cosh bt, cx\sinh bt) \tag{51}$$

is a spacelike regular cone in E_1^4 with vertex at the origin.

Following the proof of Theorem 4.2 we conclude:

Corollary 4.3. *The spacelike rotational cone M in E_1^4 defined by* (51) *has no pointwise 1-type Gauss map of the second kind.*

Corollary 4.4. *There exists no a non-planar spacelike rotational surface M in E_1^4 defined by* (11) *with flat normal bundle and pointwise 1-type Gauss map of the second kind.*

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