Bull. Korean Math. Soc. **52** (2015), No. 1, pp. 287–299
http://dx.doi.org/10.4134/BKMS.2015.52.1.287

SUFFICIENT CONDITION FOR THE EXISTENCE OF THREE DISJOINT THETA GRAPHS

Yunshu Gao and Ding Ma

Abstract. A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. We show that every graph of order $n \geq 12$ and size at least $\lfloor \frac{11n-18}{2} \rfloor$ contains three disjoint theta graphs. As a corollary, every graph of order $n \geq 12$ and size at least $\lfloor \frac{11n-18}{2} \rfloor$ contains three disjoint cycles of even length.

1. Terminology and introduction

In this paper, we only consider finite undirected graphs, without loops or multiple edges. We use [1] for the notation and terminology not defined here. A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. Let n be a positive integer, let K_n denote the complete graph of order n and K_4^- be the graph obtained by removing exactly one edge from K_4. For a graph G, we denote its vertex set, edge set, minimum degree by $V(G)$, $E(G)$ and $\delta(G)$, respectively. The order and size of a graph G, are defined by $|V(G)|$ and $|E(G)|$, respectively. A set of subgraphs is said to be vertex-disjoint or independent, if no two of them have any common vertex in G, and we use disjoint to stand for vertex-disjoint throughout this paper. If u is a vertex of G and H is either a subgraph of G or a subset of $V(G)$, we define $N_H(u)$ to be the set of neighbors of u contained in H, and $d_H(u) = |N_H(u)|$. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. In particular, we often use $[U]$ to stand for $G[U]$. If S is a set of subgraphs of G, we write $G \supseteq S$, it means that S is isomorphic to a subgraph of G, in particular, we use mS to represent a set of m vertex-disjoint copies of S. When $S = \{x_1, x_2, \ldots, x_t\}$, we may also use $[x_1, x_2, \ldots, x_t]$ to denote $[\{x_1, x_2, \ldots, x_t\}]$. Let V_1, V_2 be two disjoint subsets or subgraphs of G, we use $E(V_1, V_2)$ to denote the set of edges in G with one end-vertex in V_1, while the other in V_2, for simplicity, let $E(x, V_2)$ stand for $E(\{x\}, V_2)$, $E(V_1, x)$

Received February 5, 2014; Revised July 10, 2014.
2010 *Mathematics Subject Classification.* 05C35, 05C70.
Key words and phrases. disjoint theta graphs, sufficient condition, minimum degree.
Supported by the National Natural Science Foundation of China (Grant No. 11161035) and Ningxia Ziran (Grant No.NZ1153).

for $E(V_1, \{x\})$, respectively. A path of order n is denoted by P_n. Throughout this paper, we consider that any cycle has a fixed orientation. Let C be a cycle of G. For $x, y \in V(C)$, we denote by $\overrightarrow{C}[x, y]$ the path from x to y on $\overrightarrow{C}$. A vertex u is called a leaf of G if $d_G(u) = 1$.

Corrádi and Hajnal [3] proved the following well-known result on the existence of vertex-disjoint cycles in graphs.

Theorem 1.1 ([3]). *Let k be a positive integer and G be a graph with order $n \geq 3k$. If $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

Later, Wang [10] and independently Enomoto [5] proved a result stronger than Theorem 1.1 as follows.

Theorem 1.2 ([10]). *Let k be a positive integer and G be a graph with order $n \geq 3k$. Suppose for any pair of nonadjacent u and v in G, $d_G(u) + d_G(v) \geq 4k - 1$, then G contains k disjoint cycles.*

Given a cycle C of a graph G, a chord of C is an edge of $G - E(C)$ which joins two vertices of C. A cycle is called a chorded cycle if it has at least one chord. A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. A chorded cycle is a simple example of a theta graph but, in general a theta graph need not be a chorded cycle. It is obvious that K_4^- is the theta graph with minimum order and every theta graph contains a cycle of even length. Pósa [9] proved that any graph with minimum degree at least three contains a chorded cycle. Motivated by these results, Finkel et al. [6] and Chiba et al. [3] obtained the following results analogous to Theorem 1.2, respectively.

Theorem 1.3 ([6]). *If G is a graph of order $n \geq 4k$ and $\delta(G) \geq 3k$, then G contains k disjoint chorded cycles.*

Theorem 1.4 ([3]). *Let r,s be two nonnegative integers and let G be a graph with order $n \geq 3r + 4s$. Suppose for any pair of nonadjacent u and v in G, $d_G(u) + d_G(v) \geq 4r + 6s - 1$, then G contains $r + s$ disjoint cycles such that s of them are chorded cycles.*

Kawarabayashi [8] considered the minimum degree to ensure the existence of disjoint copies of K_4^- in a general graph G, which can be seen as a specified version of disjoint chorded cycles.

Theorem 1.5 ([8]). *Let k be a positive integer and G be a graph with order $n \geq 4k$. If $\delta(G) \geq \lceil \frac{n+k}{2} \rceil$, then G contains k disjoint copies of K_4^-.*

In this paper, we determine the edge number for a graph to contain three disjoint theta graphs. Our research is motivated by the conjecture put forward by Gao and Ji [7].

Conjecture 1.6 ([7]). *Let $k \geq 2$ be an integer. Every graph of order n and size at least $f(n,k)+1$ contains k disjoint theta graphs, when*

$$f(n,k) = \max\left\{\binom{4k-1}{2} + \frac{3}{2}(n-4k+1), \left\lfloor \frac{2(k-1)(2k-1)+(4k-1)(n-2k+1)}{2} \right\rfloor\right\}.$$

If the conjecture is true, then the bound on size is best possible, which can be seen as following examples in [7]: Let G_1 be $K_1 + (K_{4k-2} \cup \frac{n-4k+1}{2} K_2)$. The order of G_1 is n and size $\binom{4k-1}{2} + \frac{3}{2}(n-4k+1)$, but G_1 does not contain k disjoint theta graphs. Also, let n be an integer such that $n-(2k-1)$ is even. Let $l_1 = \frac{n-(2k-1)}{2}$, $F = K_{2k-1}$, $H_1 = l_1 K_2$ and $G_2 = F + H_1$. It is obvious that the graph G_2 has order n, $|E(G_1\ )| = (k-1)(2k-1)+(4k-1)l_1 = (k-1)(2k-1) + \frac{(4k-1)(n-2k+1)}{2} = \left\lfloor \frac{2(k-1)(2k-1)+(4k-1)(n-2k+1)}{2} \right\rfloor$. Gao and Ji [7] verified Conjecture 1.6 for the case $k=2$.

Theorem 1.7 ([7]). *Every graph of order $n \geq 8$ and size at least $f(n)$ contains two disjoint theta graphs, if*

$$f(n) = \begin{cases} 23 & \text{if } n = 8 \\ \lfloor \frac{7n-13}{2} \rfloor & \text{if } n \geq 9. \end{cases}$$

Based on Theorem 1.7, in this paper, we give a sufficient condition for the existence of three disjoint theta graphs.

Theorem 1.8. *Every graph of order $n \geq 12$ and size at least $\lfloor \frac{11n-18}{2} \rfloor$ contains three disjoint theta graphs.*

Note that there is a small gap on the lower bound of size between Theorem 1.8 and Conjecture 1.6 for $k=3$. However, the following corollary follows from Theorem 1.8.

Corollary 1.9. *Every graph of order $n \geq 12$ and size at least $\lfloor \frac{11n-18}{2} \rfloor$ contains three disjoint cycles of even length.*

2. Basic lemma

Lemma 2.1. *Let G be a graph of order 12 and size at least 57. Then G contains three disjoint copies of K_4^-.*

Proof. Suppose that G does not contain three disjoint copies of K_4^-. If $\delta(G) \geq 8$, then by Theorem 1.5, $G \supseteq 3K_4^-$, a contradiction. Hence, we may assume that $\delta(G) \leq 7$. Let $v_0 \in V(G)$ such that $d_G(v_0) = \delta(G)$. Suppose that $d_G(v_0) = 1$, then $56 = |E(G)| < 57$, a contradiction. Thus, $d_G(v_0) \geq 2$ and let $v_1, v_2 \in N_G(v_0)$. Suppose that $d_G(v_0) = 2$, then choose $w \in V(G - \{v_0, v_1, v_2\})$, since $|E(G - \{v_0\})| \geq 55$, it is obvious that $\{v_0, v_1, v_2, w\} \supseteq K_4^-$ and $[V(G) - \{v_0, v_1, v_2, w\}] \supseteq 2K_4^-$, a contradiction. Hence, we may assume that $d_G(v_0) \geq 3$. Furthermore, since $G - \{v_0\}$ can be obtained from K_{11} by removing at most five edges, it follows that $[N_G(v_0)]$ contains a path of order three, denoted by P_3. That is, $P_3 + \{v_0\}$ contains a subgraph $Q \cong K_4^-$. Note that $|E(G - V(Q) - \{v_0\}| \geq 57 - 7 - (10 + 9 + 8) = 23$, by Theorem 1.7, $G - V(Q) - \{v_0\}$ contains two disjoint copies of K_4^-, which disjoints from Q, this implies that $G \supseteq 3K_4^-$, a contradiction. This proves Lemma 2.1. □

3. Proof of Theorem 1.8

If $n = 12$, then Lemma 2.1 gives us the required conclusion. Hence, it is sufficient to prove that every graph of order $n \geq 13$ and size at least $\lfloor \frac{11n-18}{2} \rfloor$ contains three disjoint theta graph. We employ induction on n.

Assume that for all integers k with $12 \leq k < n$, every graph of order k and size at least $\lfloor \frac{11k-18}{2} \rfloor$ contains three disjoint theta graphs. In the following proof, we always let G be any graph of order n and size at least $\lfloor \frac{11n-18}{2} \rfloor$. By way of contradiction, we suppose that

(1) G does not contain three disjoint theta graphs.

Claim 3.1. $6 \leq \delta(G) \leq 8$.

Proof. By Theorem 1.3, we have $\delta(G) \leq 8$. Suppose that $\delta(G) \leq 5$ and let $v_0 \in V(G)$ such that $d_G(v_0) = \delta(G)$. The graph $G - v_0$ is of order $n-1$ and size $\lfloor \frac{11n-18}{2} \rfloor - d_G(v_0) \geq \lfloor \frac{11n-18}{2} \rfloor - 5 \geq \frac{11n-19-10}{2} = \frac{11(n-1)-18}{2} \geq \left\lfloor \frac{11(n-1)-18}{2} \right\rfloor$, by induction hypothesis, $G - v_0$ contains three disjoint theta graphs, and so does G, which contradicts (1). Therefore, $\delta(G) \geq 6$. □

Let v_0 be a vertex in G such that $d_G(v_0) = \delta(G)$. In what following, we always assume that $N_G(v_0) = \{v_1, \ldots, v_l\}$ and $H = [v_1, \ldots, v_l]$, where $l = d_G(v_0)$. By Claim 3.1, $6 \leq l \leq 8$. If $l = 6$, then let $\varepsilon_l = 1$; if $l = 7$, then let $\varepsilon_l = 2$; if $l = 8$, then let $\varepsilon_l = 3$. Note that $l = 5 + \varepsilon_l$.

Claim 3.2. *For each* $1 \leq i \leq l$, $d_H(v_i) \geq l - \varepsilon_l$.

Proof. Suppose that there exists $1 \leq i \leq l$ such that $d_H(v_i) \leq l - \varepsilon_l - 1 = (l-1) - \varepsilon_l$. Without loss of generality, we may assume that $i = l$, and we may also assume that $v_j v_l \notin E(G)$ for each $1 \leq j \leq \varepsilon_l$ (otherwise, we can relabel the index of $V(H)$). Define the edge set $X = \{v_j v_l : 1 \leq j \leq \varepsilon_l\}$ and construct the graph $G' = (G - v_0) + X$, which is a graph with order $n-1$ and $|E(G')| = \lfloor \frac{11n-18}{2} \rfloor - l + \varepsilon_l \geq \frac{11n-19}{2} - l + \varepsilon_l = \frac{11(n-1)-18}{2} \geq \lfloor \frac{11(n-1)-18}{2} \rfloor$, because of $l = 5 + \varepsilon_l$. By induction hypothesis, G' contains three disjoint theta graphs, say T_1, T_2 and T_3, respectively. Clearly, at least two of them, say T_1 and T_2, do not contain vertex v_l, since T_1, T_2 and T_3 are disjoint theta graphs, then $E(T_1) \cap X = \emptyset$, $E(T_2) \cap X = \emptyset$ and by (1), $E(T_3) \cap X \neq \emptyset$.

Suppose that $|E(T_3) \cap X| = 1$, we may assume that $E(T_3) \cap X = \{v_l v_1\}$. Then ${T_3}' = (T_3 - \{v_l v_1\}) + \{v_1 v_0, v_l v_0\}$ is a theta graph in G, T_1, T_2 and ${T_3}'$ are disjoint in G, which contradicts (1). Therefore, it remains the case

$E(T_3) \cap X = \{v_1v_l, v_2v_l\}$ or $E(T_3) \cap X = \{v_1v_l, v_2v_l, v_3v_l\}$, as $\varepsilon_l \le 3$. Let

$$T_3' = \begin{cases} (T_3 - \{v_1v_l, v_2v_l\}) + \{v_0v_1, v_0v_2\}, & \text{if } d_{T_3}(v_l) = 2 \\ (T_3 - \{v_1v_l, v_2v_l\}) + \{v_0v_1, v_0v_2, v_0v_3\}, & \text{if } d_{T_3}(v_l) = 3 \text{ and} \\ & E(T_3) \cap X = \{v_1v_l, v_2v_l\} \\ (T_3 - \{v_1v_l, v_2v_l, v_3v_l\}) + \{v_0v_1, v_0v_2, v_0v_3\}, & \text{otherwise.} \end{cases}$$

It is obvious that T_1, T_2 and $T_3{}'$ are three disjoint theta graphs in G, which contradicts (1). □

By Claim 3.2, Theorem 1.5 and the definition of ε_l, when $7 \le l \le 8$, for each subset S of $V(H)$ with $|S| \ge 7$, we obtain

$$[\{v_0\} \cup S] \supseteq 2K_4^-. \tag{2}$$

In particular, if $l = 6$, then

$$[\{v_0\} \cup V(H)] \cong K_7. \tag{3}$$

We take a vertex $v \in V(G - H - \{v_0\})$ such that $|E(v, V(H))|$ is maximum. When $l = 6$, by (3) and the definition of v, denote $W = V(H) \cup \{v\}$, we claim that

$$[\{v_0\} \cup W] \supseteq 2K_4^-. \tag{4}$$

Proof. By way of contradiction, suppose that $[\{v_0\} \cup W]$ does not contain two disjoint K_4^-. By (3) and the assumption that $[\{v_0\} \cup W] \not\supseteq 2K_4^-$, for each $w \in V(G - \{v_0\} - V(H))$, there is at most one edge between w and $V(H)$. If $n = 13$, then $62 \le |E(G)| \le \frac{7\times 6}{2} + 6 + \frac{6\times 5}{2} = 42$, a contradiction. If $n = 14$, then $68 \le |E(G)| \le \frac{7\times 6}{2} + 7 + \frac{7\times 6}{2} = 49$, a contradiction. If $n = 15$, then $73 \le |E(G)| \le \frac{7\times 6}{2} + 8 + \frac{8\times 7}{2} = 57$, a contradiction. If $n = 16$, then $84 \le |E(G)| \le \frac{7\times 6}{2} + 9 + \frac{9\times 8}{2} = 66$, a contradiction. Therefore, we see that $n \ge 17$. Since

$$\begin{aligned} |E(G - \{v_0\} - V(H))| &\ge |E(G)| - \frac{7\times 6}{2} - (n - 7) \\ &\ge \frac{11n - 19}{2} - n - 14 \\ &\ge \frac{7n - 13}{2}, \end{aligned}$$

by Theorem 1.7, $G - \{v_0\} - V(H)$ contains two disjoint theta graphs, together with (3), G contains three disjoint theta graphs, a contradiction. □

Let

$$G^* = \begin{cases} G - (\{v_0\} \cup V(H)), & \text{if } 7 \le l \le 8 \\ G - (\{v_0, v\} \cup V(H)), & \text{if } l = 6. \end{cases}$$

Let F^* be the set of components of G^*. By (2) and (4), it follows from (1) that every graph in F^* contains no theta graph. In the following proof, let F denote arbitrary component in F^*, then, each block of F is either a K_2 or a cycle.

Claim 3.3. *Let $F \in F^*$ with $|V(F)| \geq 4$. Then each end block of F is isomorphic to K_2.*

Proof. Otherwise, suppose that there exists an end block B of F, such that B is a cycle. Let C denote the set of cut vertices of F. Let u_1 and u_2 be two distinct vertices in $V(B) - C$. Next, we choose two distinct vertices u_3 and u_4 (both are distinct with u_1 and u_2) as follows: If $F = B$, then let $\{u_3, u_4\} \subseteq V(F - \{u_1, u_2\})$; otherwise, F contains another end blocks B' which is different from B, let $u_3 \in V(B')$ such that $u_3 \notin C$ and choose $u_4 \in V(F) \setminus C$ if possible, unless F contains exactly two end blocks B and B', such that B is a triangle and $B' \cong K_2$. For each i with $1 \leq i \leq 3$, since $d_F(u_i) \leq 2$, if $7 \leq l \leq 8$, then $|E(u_i, V(H))| \geq \delta(G) - 2 = l - 2$, if $l = 6$, then $|E(u_i, V(H) \cup \{v\})| \geq l - 2$. This implies that there exists a vertex $v' \in V(H)$ ($v' \in V(H) \cup \{v\}$ if $l = 6$), such that $u_1v', u_2v' \in E(G)$. As B is a cycle, it is easy to see that $[B \cup \{v'\}]$ contains a theta graph. When $F = B$, without loss of generality, we may assume that u_1, u_2, u_3 and u_4 occur along the direction of B.

If $l = 8$, by applying (2) and Theorem 1.5, $[\{v_0\} \cup V(H) - \{v'\}]$ contains two theta graphs, that is, G contains three disjoint theta graphs, which contradicts (1). If $l = 7$, we may assume that $\{v_2, v_3, v_4, v_5, v_6\} \subseteq N_G(u_3)$ and $v' \neq v_4, v_5$ and v_6, then $[\{v_4, v_5, v_6, u_3\}] \supseteq K_4^-$ by Claim 3.2. If $u_3 \notin V(B)$, that is, u_3 belongs to another end block by our choice, notice that $[V(H - \{v_4, v_5, v_6, v'\}) \cup \{v_0\}] \supseteq K_4^-$ and $[B \cup \{v'\}]$ contains a theta graph, we obtain a contradiction to (1). Therefore, we see that $u_3 \in V(B)$ and so $F = B$ by our choice. We may assume that $\{v_1, v_2, v_3\} \subseteq N_G(u_1) \cap N_G(u_2)$ because we don't use the assumption of $\{v_2, v_3, v_4, v_5, v_6\} \subseteq N_G(u_3)$. Suppose for the moment, there exists at most one $v_i \in \{v_1, v_2, v_3\}$, such that $v_iu_3, v_iu_4 \in E(G)$. Then there exist $v_p, v_q \in V(H - \{v_1, v_2, v_3\})$ with $p \neq q$, such that $\{v_p, v_q\} \subseteq N_G(u_3) \cap N_G(u_4)$. However, by Claim 3.2, $[\{v_0\} \cup V(H - \{v_1, v_2, v_p, v_q\})] \supseteq K_4^-$, notice that $[\{v_1, v_2\} \cup V(\overrightarrow{B}[u_1, u_2])]$ and $[\{v_p, v_q\} \cup V(\overrightarrow{B}[u_3, u_4])]$ contain two disjoint theta graphs, this implies that G contains three disjoint theta graphs, a contradiction. Thus, without loss of generality, say $\{v_1, v_2\} \subseteq N_H(u_3) \cap N_H(u_4)$. As $|E(u_3, V(H))| \geq 5$, without loss of generality, we may assume that $v_4u_3, v_5u_3 \in E(G)$. As $d_H(v_1) \geq 5$ by Claim 3.2, we may assume that $v_1v_4 \in E(G)$. This implies that $[v_4, v_1, u_4, u_3] \supseteq K_4^-$, notice that $[\{v_0, v_5, v_6, v_7\}] \supseteq K_4^-$ and $[\{v_2, v_3\} \cup V(\overrightarrow{B}[u_1, u_2])]$ contains a theta graph, then G contains three disjoint theta graphs, a contradiction. Now, it remains the case $l = 6$. As $d_F(u_i) \leq 2$ for $i \in \{1, 2, 3\}$, so $|E(u_i, V(H) \cup \{v\})| \geq l - 2 = 4$. Furthermore, by our choice of u_4, $d_F(u_i) \leq 3$ and $|E(u_4, V(H) \cup \{v\})| \geq 3$.

Suppose for the moment that $u_1v, u_2v, u_3v \in E(G)$, then $[B \cup \{v\}]$ contains a theta graph. If $u_3 \notin V(B)$, by the choice of u_3 and (3), $H + \{v_0, u_3\} \supseteq 2K_4^-$, this implies that G contains three disjoint theta graphs, which contradicts (1). Thus, $u_3 \in V(B)$ and so $F = B$. However, $[\{v\} \cup V(\overrightarrow{B}[u_1, u_3])]$ contains a theta graph and $[\{u_4, v_0\} \cup V(H)] \supseteq 2K_4^-$, a contradiction. Thus, there exists

$i \in \{1,2,3\}$, such that $u_i v \notin E(G)$. By the definition of v, $|E(v, V(H))| \geq 4$. Without loss of generality, we assume that $\{v_1, v_2, v_3, v_4\} \subseteq N_H(v)$.

If $u_1 v, u_2 v \in E(G)$, then $F = B$. Without loss of generality, we may assume that $v_4 u_1 \in E(G)$ and so $[\{v, v_4\} \cup V(\overrightarrow{B}[u_1, u_2])]$ contains a theta graph. Notice that $u_3 v \notin E(G)$ and $u_4 v \notin E(G)$, then $[\{u_3, u_4, v_0\} \cup V(H - \{v_4\})] \supseteq 2K_4^-$, a contradiction. Therefore, we assume that $u_1 v \notin E(G)$ by symmetry.

Suppose that $u_3 \notin V(B)$. Then $|E(u_3, \{v_1, v_2, v_3, v_4\})| \geq 2$; otherwise, $u_3 v \in E(G)$ and we may further assume that $u_3 v_4, u_3 v_5, u_3 v_6 \in E(G)$. Since $|E(u_1, V(H))| \geq 4$ and $|E(u_2, V(H))| \geq 3$, there exists $i \in \{1,2,3,4\}$, such that $u_i \in N_H(u_2) \cap N_H(u_1)$, then $[B \cup \{u_i\}]$ is a theta graph, since $[\{u_1, u_2, u_3, u_4, v\} - \{u_i\}] \supseteq K_4^-$ and $[v_0, v_5, v_6, u_3] \supseteq K_4^-$, which contradicts (1). By symmetry, we assume that $v_1 u_3, v_2 u_3 \in E(G)$. If $v' \neq v_1$ and $v' \neq v_2$, then $[v, u_3, v_1, v_2] \supseteq K_4^-$ and $[\{v_0\} \cup (V(H) - \{v_1, v_2, v'\})] \supseteq K_4^-$, which disjoints from $[B \cup \{v'\}]$, a contradiction. Hence, we may assume that $v' = v_1$. By Claim 3.2 and (1), we may assume that $v_5 u_3, v_6 u_3 \in E(G)$. However, $[v_0, v_5, v_6, u_3] \supseteq K_4^-$ and $[v, v_2, v_3, v_4] \supseteq K_4^-$, which disjoints from $[B \cup \{v_1\}]$, a contradiction. Thus, $u_3 \in V(B)$ and so $F = B$ by our choice.

By symmetry, we may assume that $u_2 v \notin E(G)$ and $u_3 v \notin E(G)$. By pigeonhole principle, there exists $\{v_p, v_q\} \subseteq V(H)$ such that $\{v_p, v_q\} \subseteq N_H(u_1) \cap N_H(u_2)$. If $u_3 v_p \in E(G)$, then $[V(\overrightarrow{B}[u_1, u_3]) \cup \{v_p\}]$ contains a theta graph, notice that $[V(H - \{v_p\}) \cup \{v, v_0, u_4\}] \supseteq 2K_4^-$, G contains three disjoint theta graphs, a contradiction. Thus, $u_3 v_p \notin E(G)$ and by symmetry, $u_3 v_q \notin E(G)$, $u_4 v_p \notin E(G)$ and $u_4 v_q \notin E(G)$. This implies that there exist $v_i, v_j \in V(H) - \{v_p, v_q\}$, such that $\{v_i, v_j\} \subseteq N_H(u_3) \cap N_H(u_4)$. By (1), we see that $|\{p,q\} \cap \{1,2,3,4\}| \leq 1$ and $|\{i,j\} \cap \{1,2,3,4\}| \leq 1$. Therefore, $[\{v, v_0\} \cup V(H - \{v_p, v_q, v_i, v_j\})] \supseteq K_4^-$. Notice that $[\{v_p, v_q\} \cup V(\overrightarrow{B}[u_1, u_2])]$ and $[\{v_i, v_j\} \cup V(\overrightarrow{B}[u_3, u_4])]$ contains two disjoint theta graphs, which contradicts (1). This completes the proof that B is not an end block, and in particular, we see that every end block of F is isomorphic K_2. □

Claim 3.4. *Let $F \in F^*$ with $|V(F)| \geq 4$. Then each block of F is isomorphic to K_2.*

Proof. Since $|V(F)| \geq 4$, F contains at least two end block, say F_1 and F_2. Note $F_i \cong K_2$ for each $1 \leq i \leq 2$. Let $u_1 \in V(F_1)$ such that $d_{F_1}(u_1) = 1$ and let $u_3 \in V(F_2)$ such that $d_{F_2}(u_3) = 1$. Suppose that the conclusion of Claim 3.4 is false, we may assume that B is the nearest block to u_1 in F, such that B is a cycle. By Claim 3.3, B is not an end block of F. We choose two distinct vertices u_2 and u_4 such that both of them are distinct with u_1 and u_3 as follows: Let $u_2 \in V(B)$ and u_2 is not a cut vertex of F, and choose u_4 such that u_4 is not a cut vertex of F, unless F contains exactly three blocks F_1, F_2 and $B \cong K_3$, then choose $u_4 \in V(F_2) - \{u_3\}$. Notice that if there exists v' such that $u_1 v', u_2 v' \in E(G)$, then using these blocks of F from F_1 to B, we see that $[V(F - \{u_3\}) \cup \{u_1\}]$ contains a theta graph. Now, since u_1 and u_3 are in different blocks, with the same role of u_1, u_2, u_3 and u_4, we continue part of the process in the proof of Claim 3.3, we can complete the proof. This proves Claim 3.4. $\square$

Claim 3.5. $|V(F)| \le 3$ *for each* $F \in F^*$.

Proof. Otherwise, suppose that there exists $F \in F^*$ such that $|V(F)| \ge 4$. By Claim 3.4, F must be a tree.

Suppose for the moment that there exists three distinct leaves in $V(F)$, say u_1, u_2 and u_3. Then for each $1 \le i \le 3$, $|E(u_i, V(H))| \ge l-1$ if $7 \le l \le 8$, and $|E(u_i, V(H) \cup \{v\})| \ge l-1$ if $l = 6$. As $|V(F)| \ge 4$, by Claim 3.4, we choose $u_4 \in V(F - \{u_1, u_2, u_3\})$ as follows: if F contains at least four leaves, then let u_4 denote the leave different from u_1, u_2 and u_3; otherwise, let u_4 and u_1 belongs to the same block of F. It is obvious that $|E(u_4, V(H))| \ge l-3$ if $7 \le l \le 8$, and $|E(u_4, V(H) \cup \{v\})| \ge l-3$ if $l = 6$.

Suppose that $l = 8$. Notice that there exist $v', v'' \in V(H)$ with $v' \ne v''$ and $v'v'' \in E(G)$ such that $\{v', v''\} \subseteq N_H(u_1) \cap N_H(u_2)$. It is obvious that $[v', v'', u_1, u_2] \supseteq K_4^-$. by Claim 3.1, $H - \{v', v''\} + \{v_0, u_3\}$ induces a graph with minimum degree at least five, and therefore contains two disjoint copies of K_4^- by Theorem 1.5, a contradiction. Next, suppose that $l = 7$, by pigeonhole principle, we can find two distinct vertices $v_i, v_j \in V(H)$ such that $\{v_i, v_j\} \subseteq N_H(u_3) \cap N_H(u_4)$. Since there is a path P in F which connecting u_3 and u_4, thus, $[V(P) \cup \{v_i, v_j\}]$ contains a theta graph. Notice that there exist $v', v'' \in V(H - \{v_i, v_j\})$ with $v' \ne v''$ and $v'v'' \in E(G)$, such that $\{v', v''\} \subseteq N_H(u_1) \cap N_H(u_2)$. It is obvious that $[v', v'', u_1, u_2] \supseteq K_4^-$. As $[\{v_0\} \cup V(H - \{v', v'', v_i, v_j\})] \supseteq K_4^-$, which contradicts (1). Thus, $l = 6$.

We show $N_H(u_1) \cap N_H(u_4) \ne \emptyset$. Suppose not, without loss of generality, we may assume that $N_G(u_1) \cap (V(H) \cup \{v\}) = \{v, v_1, v_2, v_3, v_4\}$ and $N_G(u_4) \cap (V(H) \cup \{v\}) = \{v, v_5, v_6\}$. If $u_3v \in E(G)$, then $[V(F - \{u_2\}) \cup \{v\}]$ contains a theta graph, as $[V(H) \cup \{v_0, u_2\}] \supseteq 2K_4^-$, which contradicts (1). Hence, $u_3v \notin E(G)$ and $u_2v \notin E(G)$ by symmetry. Furthermore, by the choice of v, we have $|E(v, V(H))| \ge 4$ and so $N_H(v) \cap N_H(u_1) \ne \emptyset$, without loss of generality, say $vv_1 \in E(G)$. Then $[v, v_1, u_1, u_4] \supseteq K_4^-$, since $|N_H(u_2) \cap N_H(u_3)| \ge 3$, it follows that $[V(H - \{v_1\}) \cup \{u_2, u_3, v_0\}] \supseteq 2K_4^-$, which contradicts (1).

Now, by symmetry, say $v_6 \in N_H(u_1) \cap N_H(u_4)$. If $u_2v_6 \in E(G)$, then $[\{v_6\} \cup V(F - \{u_3\})]$ contains a theta graph, as $[V(H - \{v_6\}) \cup \{v, u_3\}] \supseteq 2K_4^-$, which contradicts (1). Thus, $v_6u_2 \notin E(G)$ and $v_6u_3 \notin E(G)$ by symmetry. As $|E(u_2, V(H))| \ge 4$ and $|E(u_3, V(H))| \ge 4$, we may assume that $\{v_1, v_2, v_3, v_4\} \subseteq N_H(u_2)$ and $\{v_1, v_2, v_3\} \subseteq N_H(u_2) \cap N_H(u_3)$. If $v_5u_1 \in E(G)$, then $[\{v_6, v_5, u_1, u_4\}] \supseteq K_4^-$. Notice that $[V(H - \{v_5, v_6\}) \cup \{v_0, v, u_3, u_2\}] \supseteq 2K_4^-$ by the definition of v and (3), which contradicts (1). Thus, $v_5u_1 \notin E(G)$. If $u_1v_4 \in E(G)$, then $u_2v, u_3v \in E(G)$. Otherwise, say $u_2v \notin E(G)$. Then $u_2v_5 \in E(G)$ and $|E(v, V(H))| \ge 5$ by the choice of v. By symmetry, we may assume that $\{v_1, v_2\} \subseteq N_H(v) \cap N_H(u_3)$. Then $[v, v_1, v_2, u_3] \supseteq K_4^-$, $[u_1, u_4, v_4, v_6] \supseteq K_4^-$ and $[u_2, v_3, v_5, v_0] \supseteq K_4^-$, which contradicts (1). Hence, by (1), $vv_i \notin E(G)$ for each $i \in \{1,2,3\}$, that is, $|E(v, V(H))| \leq 3$, which contradicts the choice of v. Therefore, $u_1v_4 \notin E(G)$ and so $\{v_1, v_2, v_3\} \subseteq N_H(u_1)$ and $u_1v \in E(G)$. By (1) and (3), $u_2v, u_3v \in E(G)$ and $|E(v, V(H))| \leq 3$, which contradicts the choice of v. Consequently, F contains exactly two leaves and F must be a path with order at least four.

Let $F = u_1u_2 \cdots u_{p-1}u_p$ and $p \geq 4$. Suppose that $7 \leq l \leq 8$, then continue the process as above, we can find three disjoint theta graphs, a contradiction. Hence, $l = 6$. Then $|E(u_1, V(H) \cup \{v\})| \geq 5$, $|E(u_p, V(H) \cup \{v\})| \geq 5$, $|E(u_2, V(H) \cup \{v\})| \geq 4$ and $|E(u_{p-1}, V(H) \cup \{v\})| \geq 4$.

Suppose $u_1v, u_pv \in E(G)$. Then $u_2v \notin E(G)$ or $u_{p-1}v \notin E(G)$, otherwise, $[v, u_1, u_2, u_{p-1}] \supseteq K_4^-$, as $[V(H) \cup \{v_0, u_p\}] \supseteq 2K_4^-$ by Claim 3.2, which contradicts (1). By symmetry, say $u_2v \notin E(G)$ and so $|E(u_2, V(H))| \geq 4$. Without loss of generality, by pigeonhole principle, we may assume that $v_1 \in N_H(u_2) \cap N_H(u_{p-1})$ and $\{v_1, v_2, v_3, v_4\} \subseteq N_H(u_2)$. Suppose for a moment that $|N_H(u_2) \cap N_H(u_{p-1})| \geq 2$. Without loss of generality, say $v_2u_{p-1} \in E(G)$. Then $[u_2, u_{p-1}, v_1, v_2] \supseteq K_4^-$. We prove that $vv_1 \notin E(G)$ and $vv_2 \notin E(G)$. Otherwise, by symmetry, say $vv_1 \in E(G)$. If $u_1v_1 \in E(G)$, then $[v, v_1, u_1, u_2] \supseteq K_4^-$, since $[\{u_p, v_0\} \cup V(H - \{v_1\})] \supseteq 2K_4^-$, which contradicts (1). Hence, $u_1v_1 \notin E(G)$. Next, we show that $u_1v_2 \notin E(G)$. Suppose that $u_1v_2 \in E(G)$. Then $[V(F - \{u_p\}) \cup \{v_2\}]$ contains a theta graph, as $[\{v, u_p, v_0\} \cup V(H - \{v_2\})] \supseteq 2K_4^-$, a contradiction once again. Until now, we see that $N_H(u_1) = \{v_3, v_4, v_5, v_6\}$. According to this, we have $u_pv_1 \notin E(G)$ and $u_pv_2 \notin E(G)$. This implies that $N_H(u_1) = N_H(u_p)$. If $vv_2 \in E(G)$, then $[v, v_1, v_2, u_{p-1}] \supseteq K_4^-$, notice that $[V(H - \{v_1, v_2\}) \cup \{u_1, u_2, u_p, v_0\}] \supseteq 2K_4^-$, which contradicts (1). Thus, $vv_2 \notin E(G)$ and it follows that there exists $i \in \{3, 4\}$ such that $v_iv \in E(G)$. Without loss of generality, say $i = 3$, then $[v, v_3, u_1, u_2] \supseteq K_4^-$, as $[V(H - \{v_3\}) \cup \{u_{p-1}, u_p, v_0\}] \supseteq 2K_4^-$, which contradicts (1) and completes the proof of $vv_1 \notin E(G)$. Then $\{v_3, v_4, v_5, v_6\} \subseteq N_H(v)$ and so $[V(H - \{v_1, v_2\}) \cup \{v, u_p, u_1, v_0\}] \supseteq 2K_4^-$, a contradiction. This proves that $N_H(u_2) \cap N_H(u_{p-1}) = \{v_1\}$ and so $v_5u_{p-1}, v_6u_{p-1}, u_{p-1}v \in E(G)$. Suppose that $v_1u_1 \in E(G)$, then let $P' = P - \{u_p\}$, then $[V(P') \cup \{v_1\}]$ contains a theta graph, by (3), $[V(H - \{v_1\}) \cup \{v, u_p\}] \supseteq 2K_4^-$, which contradicts (1). Thus, $v_1u_1 \notin E(G)$ and so $|N_{H-v_1}(u_1) \cap N_{H-v_1}(u_p)| \geq 2$. If $vv_1 \in E(G)$, then $[V(P - \{u_1, u_p\}) \cup \{v, v_1\}]$ contains a theta graph, as $[V(H - \{v_1\}) \cup \{u_1, u_p\}] \supseteq 2K_4^-$, which contradicts (1). Thus, $vv_1 \notin E(G)$. As $|E(v, V(H))| \geq 4$, by the symmetry role of v_5 and v_6, we may assume that $vv_5 \in E(G)$, then $[v, v_5, u_{p-1}, u_p] \supseteq K_4^-$, since u_1 and u_2 has at least two common neighbors in $V(H) - \{v_1, v_5, v_6\}$, $[V(H - \{v_5\}) \cup \{u_1, u_2\}] \supseteq 2K_4^-$, which contradicts (1). Consequently, we may assume that $u_1v \notin E(G)$ by symmetry. This gives us $|E(u_1, V(H))| \geq 5$ and so $|E(v, V(H))| \geq 5$ by the maximality of v. Without loss of generality, we may assume that $\{v_1, v_2, v_3, v_4, v_5\} \subseteq N_H(v)$ and $\{v_1, v_2, v_3, v_4\} \subseteq N_H(u_1) \cap N_H(v)$. Because of $|E(u_1, V(H))| \geq 5$, we divide the proof into two cases.

Case 1. $u_1v_5 \in E(G)$.

Without loss of generality, say $v_4u_2, v_5u_2 \in E(G)$, because of $|E(u_2, V(H))| \geq 3$. If $u_{p-1}v_4 \in E(G)$, then $[u_1, u_2, \ldots, u_{p-1}, v_4]$ contains a theta graph, since $[V(H - \{v_4\}) \cup \{v, u_p, v_0\}] \supseteq 2K_4^-$, which contradicts (1) and proves that $u_{p-1}v_4 \notin E(G)$. Similarly, $u_{p-1}v_5 \notin E(G)$. If there exists $v_i \in \{v_1, v_2, v_3\}$, say $i = 1$, such that $v_1u_2 \in E(G)$, then $u_{p-1}v_1 \notin E(G)$, $N_H(u_{p-1}) = \{v_2, v_3, v_6\}$ and $u_{p-1}v \in E(G)$. Suppose that there exist $v_i, v_j \in \{v_1, v_4, v_5\}$ such that $u_pv_i, u_pv_j \in E(G)$, then $[v_i, v_j, u_2, u_p] \supseteq K_4^-$. For simplicity, say $i = 4$ and $j = 5$. Since $[v, v_2, v_6, u_{p-1}] \supseteq K_4^-$ and $[v_0, u_1, v_1, v_3] \supseteq K_4^-$, this contradicts (1) and proves that u_p has at most one neighbor in $\{v_1, v_4, v_5\}$. This implies that $u_pv_6, u_pv \in E(G)$. Hence, $[v, u_{p-1}, u_p, v_6] \supseteq K_4^-$, notice that $[V(H - \{v_6\}) \cup \{v_0, u_1, u_2\}] \supseteq 2K_4^-$, a contradiction. This proves that u_2 has no neighbor in $\{v_1, v_2, v_3\}$ and so $u_2v_6, u_2v \in E(G)$. As $|E(u_{p-1}, V(H))| \geq 3$, we may assume that $v_2u_{p-1}, v_3u_{p-1} \in E(G)$. Since $[v, v_4, u_1, u_2] \supseteq K_4^-$ and $[v_0, v_2, v_3, u_{p-1}] \supseteq K_4^-$, $|E(u_p, \{v_1, v_5, v_6\})| \leq 1$ by (1) and (3). Therefore, $\{v_2, v_3, v_4\} \in N_H(u_p)$ and $u_pv \in E(G)$. However, $[v, v_5, v_6, u_2] \supseteq K_4^-$, $[u_{p-1}, v_2, v_3, u_p] \supseteq K_4^-$ and $[v_0, v_1, v_4, u_1] \supseteq K_4^-$, a contradiction. This proves Case 1.

Case 2. $u_1v_6 \in E(G)$.

Suppose that $u_2v \in E(G)$. Then for each v_i with $1 \leq i \leq 4$, $v_iu_2 \notin E(G)$, otherwise, $[v, v_i, u_2, u_1] \supseteq K_4^-$, it is obvious that $[V(H - \{v_i\}) \cup \{v_0, u_p, u_{p-1}\}] \supseteq 2K_4^-$, which contradicts (1). However, this gives us $|E(u_2, V(H) \cup \{v\})| \leq 3$, a contradiction. Thus, $u_2v \notin E(G)$ and $|E(u_2, V(H))| \geq 4$. By symmetry, we may assume that $u_2v_3, u_2v_4 \in E(G)$. According to (1), $u_{p-1}v_3 \notin E(G)$ and $u_{p-1}v_4 \notin E(G)$. If there exists $v_i \in \{v_1, v_2\}$, say $i = 1$, such that $v_1u_2 \in E(G)$, then $u_{p-1}v_1 \notin E(G)$, $N_H(u_{p-1}) = \{v_2, v_5, v_6\}$ and $u_{p-1}v \in E(G)$. This together with (1) tell us u_p has at most one neighbor in $\{v_1, v_3, v_4\}$ and thus $\{v_2, v_5, v_6\} \subseteq N_H(u_p)$ and $u_pv \in E(G)$. We see that $[v, u_p, u_{p-1}, v_6] \supseteq K_4^-$, $[u_1, u_2, v_3, v_4] \supseteq K_4^-$ and $[v_0, v_1, v_2, v_5] \supseteq K_4^-$, a contradiction. This proves that u_2 has no neighbor in $\{v_1, v_2\}$ and so $u_2v_5, u_2v_6 \in E(G)$. As $|E(u_{p-1}, V(H))| \geq 3$, by the symmetry role of v_1 and v_2, we may assume that $v_1u_{p-1} \in E(G)$. Suppose that $u_{p-1}v_6 \in E(G)$. If $v_6u_p \in E(G)$, then $[u_{p-1}, u_p, v_1, v_6] \supseteq K_4^-$, $[v, v_2, v_3, u_1] \supseteq K_4^-$ and $[v_0, u_2, v_4, v_5] \supseteq K_4^-$, a contradiction. Therefore, $v_6u_p \notin E(G)$ and then there exist $v_i, v_j \in \{v_2, v_3, v_4, v_5\}$, such that $v_iu_p, v_ju_p \in E(G)$. If $2 \in \{i, j\}$, then $[v, v_i, v_j, u_p] \supseteq K_4^-$, $[v_1, u_1, u_{p-1}, v_6] \supseteq K_4^-$ and $[V(H - \{v_1, v_i, v_j, v_6\}) \cup \{v_0, u_2\}] \supseteq K_4^-$, a contradiction. Hence, $2 \notin \{i, j\}$. Then $[u_2, v_i, v_j, u_p] \supseteq K_4^-$, $[v_1, u_1, v_6, u_{p-1}] \supseteq K_4^-$ and $[V(H - \{v_1, v_i, v_j, v_6\}) \cup \{v_0, v\}] \supseteq K_4^-$, a contradiction. This proves that $u_{p-1}v_6 \notin E(G)$ and it follows that $v_2u_{p-1}, v_5u_{p-1} \in E(G)$. By (1), $u_pv_5 \notin E(G)$. Since $[V(F - \{u_1, u_p\}) \cup \{v_5, v_6\}]$ contains a theta graph and u_p has at least two neighbors in $\{v_1, v_2, v_3, v_4\}$, we see that $[V(H - \{v_5, v_6\}) \cup \{v, u_1, u_p, v_0\}] \supseteq 2K_4^-$, a contradiction. This completes the proof of Case 2 and the proof of Claim 3.5. □

Since $n \geq 13$ and $6 \leq |V(H)| \leq 8$, it follows from Claim 3.5 that $|F^*| \geq 2$.

Claim 3.6. *$|V(F)| \leq 2$ for each $F \in F^*$.*

Proof. By way of contradiction. Suppose that there exists $F \in F^*$ such that $|V(F)| \geq 3$. According to Claim 3.5, $|V(F)| = 3$. If F is a triangle, then the proof of Claim 3.3 works, because of $|F^*| \geq 2$. Thus, F is a path of order three and write $F = u_1u_4u_3$. Let $F' \in F^* - F$ and $u_4 \in V(F')$ such that u_2 is an end vertex of F'. It is obvious that $d_{F'}(u_2) = 1$. Suppose that $7 \leq l \leq 8$. It is obvious that there exists $v_i \in V(H)$, such that $u_1v_i, u_4v_i, u_3v_i \in E(G)$, that is, $[v_i, u_1, u_4, u_3] \supseteq K_4^-$, since $[V(H - \{v_i\}) \cup \{v_0, u_2\}] \supseteq 2K_4^-$, a contradiction. Thus, $l = 6$, then continue the same proof in Claim 3.5 (when $|F| \geq 4$ and contains at least three leaves). □

Claim 3.7. *For each graph $F \in \mathcal{F}$ such that $|V(F)| = 2$, there exists $S \subset V(H)$ with $|S| = 2$ and $[V(F) \cup S] \supseteq K_4^-$.*

Proof. Let $F \in \mathcal{F}$ such that $|V(F)| = 2$, label $V(F) = \{u_1, u_2\}$. Since $|E(u_i, V(H))| \geq l - 1$ if $7 \leq l \leq 8$ and $|E(u_i, V(H) \cup \{v\})| \geq l - 1$ for each i with $1 \leq i \leq 2$, it follows from the pigeonhole principle that there exists a subset $S \subset V(H)$ with $|S| = 2$ and $S \subseteq N_H(u_1) \cap N_H(u_2)$. By (3), we know $[V(F) \cup S] \supseteq K_4^-$. □

Claim 3.8. *For any $u \in V(G^*)$, $|E(u, \{v_0\} \cup V(H))| = |E(u, V(H))| \leq l - 1$ if $7 \leq l \leq 8$; $|E(u, V(H) \cup \{v\})| \leq l$ if $l = 6$.*

Proof. Suppose that there exists $u \in V(G^*)$ such that $|E(u, V(H))| \geq l$ if $7 \leq l \leq 8$, and $|E(u, V(H) \cup \{v\})| \geq l + 1$ if $l = 6$. By Claim 3.6, we may assume that F^* contains two components F_1 and F_2 with $|V(F_i)| \leq 2$ for each $1 \leq i \leq 2$, such that $u \in V(F_1)$. Suppose that $|V(F_2)| = 2$ and label $F_2 = u_2u_3$. Note that $|E(u_i, V(H))| \geq l - 1$ for each $i \in \{2, 3\}$. By Claim 3.7, there exist $v_i, v_j \in V(H)$ such that $[u_2, u_3, v_i, v_j] \supseteq K_4^-$. If $7 \leq l \leq 8$, combining with (2) and (3), $[V(H - \{v_i, v_j\}) \cup \{u, v_0\}] \supseteq 2K_4^-$, which contradicts (1). Therefore, $l = 6$. By the choice of v, $|E(v, V(H))| = 6$. Notice that $v_pv_q \in E(G)$, thus, $[v_p, v_q, v, u] \supseteq K_4^-$. Since $F^* \setminus (F_1 \cup F_2) \neq \emptyset$, choose $u_4 \in V(F^* \setminus (F_1 \cup F_2))$. By Claim 3.6, $|E(u_4, V(H))| \geq 4$, choose $\{v_p, v_q\} \subseteq N_H(u_4) \cap N_H(v) - \{v_i, v_j\}$ such that $p \neq q$. Now, $[v_p, v_q, u_4, v_0] \supseteq K_4^-$ and $[V(H - \{v_i, v_j, v_p, v_q\}) \cup \{u, v\}] \supseteq K_4^-$, which contradicts (1). This shows the order of each components of $F^* \setminus F_1$ is one. Now, note that $|F^* \setminus F_1| \geq 3$, we can choose three different vertices u_1, u_2, u_3, such that $|E(u_i, V(H))| \geq 5$ for each $1 \leq i \leq 3$. As above, it is obvious that $[V(H) \cup \{v, u, v_0, u_1, u_2, u_3\}] \supseteq 3K_4^-$, a contradiction. □

Now we are in the position to complete the proof of Theorem 1.8. By Claim 3.6 and Claim 3.8, $|V(F)| = 2$ for all $F \in F^*$, we have

$$\sum_{F \in F^*} |E(F)| = \begin{cases} \frac{n-1-l}{2}, & \text{if } 7 \leq l \leq 8 \\ \frac{n-8}{2}, & \text{if } l = 6. \end{cases}$$

Suppose that $7 \leq l \leq 8$. We may assume that u_1u_2 and u_3u_4 are two component of G^*, since $|E(u_i, V(H))| \geq l-1$, by Claim 3.2, it is obvious that $[V(H) \cup \{v_0, u_1, u_2, u_3, u_4\}] \supseteq 3K_4^-$, a contradiction. Thus, $l = 6$, and according to Claim 3.8, we obtain

$$\begin{aligned}
|E(G)| &= |E([\{v_0, v\} \cup V(H)])| + |E(V(G^*), \{v_0, v\} \cup V(H))| + \sum_{F \in F^*} |E(F)| \\
&\leq 27 + 5|V(G^*)| + \sum_{F \in F^*} |E(F)| \\
&= 27 + 5(n-8) + \frac{n-8}{2} \\
&= \frac{11n-34}{2},
\end{aligned}$$

this is an obvious contradiction and completes the proof of Theorem 1.8.

Acknowledgements. The authors wish to thank the referees for their suggestions for revising the manuscript, especially, one of the referees helps us to construct an extremal graph in Conjecture 1.6.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, the second edition, Springer, 2008.

[2] S. Chiba, S. Fujita, Y. Gao, and G. Li, *On a sharp degree sum condition for disjoint chorded cycles in graphs*, Graphs Combin. **26** (2010), no. 2, 173–186.

[3] K. Corrádi and A. Hajnal, *On the maximal number of independent circuits in a graph*, Acta Math. Acad. Sci. Hungar. **14** (1963), 423–439.

[4] G. A. Dirac, *On the maximal number of independent triangles*, Abh. Math. Sem. Univ. Hamburg **26** (1963), 78–82.

[5] H. Enomoto, *On the existence of disjoint cycles in a graph*, Combinatorica **18** (1998), no. 4, 487–492.

[6] D. Finkel, *On the number Of independent chorded cycle in a graph*, Discrete Math. **308** (2008), no. 22, 5265–5268.

[7] Y. Gao and N. Ji, *The extremal function for two disjoint cycles*, Bull. Malays. Math. Sci. Soc. DOI:10.1007/s40840-014-0102-0, 2014.

[8] K. Kawarabayashi, *K_4^--factor in graphs*, J. Graph Theory **39** (2002), no. 2, 111–128.

[9] L. Pósa, *On the circuits of finite graphs*, Magyar Tud. Akad. Mat. Kut. Int. Közl. **8** (1963), 355–361.

[10] H. Wang, *On the maximum number of independent cycles in a graph*, Discrete Math. **205** (1999), no. 1-3, 183–190.

Yunshu Gao
School of Mathematics and Computer Science
Ningxia University
Yinchuan,750021, P. R. China
E-mail address: gysh2004@gmail.com

Ding Ma
School of Mathematics and Computer Science
Ningxia University
Yinchuan,750021, P. R. China
E-mail address: mading0202@sina.com