Bull. Korean Math. Soc. **52** (2015), No. 1, pp. 275–286
http://dx.doi.org/10.4134/BKMS.2015.52.1.275

AREA OF TRIANGLES ASSOCIATED WITH A CURVE II

Dong-Soo Kim, Wonyong Kim, Young Ho Kim, and Dae Heui Park

Abstract. It is well known that the area U of the triangle formed by three tangents to a parabola X is half of the area T of the triangle formed by joining their points of contact. In this article, we consider whether this property and similar ones characterizes parabolas. As a result, we present three conditions which are necessary and sufficient for a strictly convex curve in the plane to be an open part of a parabola.

1. Introduction

Suppose that X is a regular curve in the plane $\mathbb{R}^2$ with nonvanishing curvature, and $P = A$, A_i, $i = 1, 2$, are three distinct neighboring points on the curve X. Let us denote by ℓ, ℓ_1, ℓ_2 the tangent lines passing through the points A, A_1, A_2 and by B, B_1, B_2 the intersection points $\ell_1 \cap \ell_2$, $\ell \cap \ell_1$, $\ell \cap \ell_2$, respectively. If X is an open part of a parabola, it is well known that the area $U = |\triangle BB_1B_2|$ of the triangle formed by three tangents to the parabola X is half of the area $T = |\triangle AA_1A_2|$ of the triangle formed by joining their points of contact ([1]).

A regular plane curve $X : I \to \mathbb{R}^2$ defined on an open interval is called *convex* if, for all $t \in I$, the trace $X(I)$ lies entirely on one side of the closed half-plane determined by the tangent line at $X(t)$ ([2]).

From now on, we will say that a simple convex curve X in the plane $\mathbb{R}^2$ is *strictly convex* if the curve is smooth (that is, of class $C^{(3)}$) and is of positive curvature κ with respect to the unit normal N pointing to the convex side. Hence, in this case we have $\kappa(s) = \langle X''(s), N(X(s))\rangle > 0$, where $X(s)$ is an arclength parametrization of X.

For a smooth function $f : I \to \mathbb{R}$ defined on an open interval, we will also say that f is *strictly convex* if the graph of f has positive curvature κ with respect

Received January 23, 2014; Revised February 20, 2014.
2010 *Mathematics Subject Classification.* 53A04.
Key words and phrases. triangle, area, parabola, strictly convex curve, plane curvature.
The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0022926). The third author supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2042298).

to the upward unit normal N. This condition is equivalent to the positivity of $f''(x)$ on I.

Suppose that X is a strictly convex curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. For a fixed point $P = A \in X$, and for a sufficiently small $h > 0$, we consider the line m passing through $P + hN(P)$ which is parallel to the tangent ℓ of X at P and the points A_1 and A_2, where the line m intersects the curve X.

We denote by ℓ_1, ℓ_2 the tangent lines of X at the points A_1, A_2 and by B, B_1, B_2 the intersection points $\ell_1 \cap \ell_2$, $\ell \cap \ell_1$, $\ell \cap \ell_2$, respectively. We let $L_P(h)$ and $\ell_P(h)$ denote the length $|A_1A_2|$ and $|B_1B_2|$ of the corresponding segments, respectively.

Now, we consider $T_P(h)$, $U_P(h)$, $V_P(h)$ and $W_P(h)$ defined by the area $|\triangle AA_1A_2|$, $|\triangle BB_1B_2|$, $|\triangle BA_1A_2|$ of corresponding triangles and the area $|\square A_1A_2B_2B_1|$ of trapezoid $\square A_1A_2B_2B_1$, respectively. Then, obviously we have

$$T_P(h) = \frac{h}{2} L_P(h)$$

and

$$V_P(h) = W_P(h) + U_P(h).$$

Let us denote by $S_P(h)$ the area of the region bounded by the curve X and chord A_1A_2. Then, we have ([7])

$$S'_P(h) = L_P(h).$$

It is well known that parabolas satisfy the following properties ([1, 12]).

Lemma 1.1. *Suppose that X is an open part of a parabola. For arbitrary point $P \in X$ and sufficiently small $h > 0$, it satisfies*

$$S_P(h) = \frac{4}{3} T_P(h), \tag{1.1}$$

$$S_P(h) = \frac{2}{3} V_P(h), \tag{1.2}$$

$$S_P(h) = \frac{8}{9} W_P(h), \tag{1.3}$$

and

$$U_P(h) = \frac{1}{2} T_P(h). \tag{1.4}$$

Actually, Archimedes showed that parabolas satisfy (1.1) ([12]). Recently, in [7] the first and third authors of the present paper proved that (1.1) is a characteristic property of parabolas and established some characterizations of parabolas, which are the converses of well-known properties of parabolas originally due to Archimedes ([12]). For the higher dimensional analogues of some results in [7], see [5] and [6].

In this article, we study whether the remaining properties in Lemma 1.1 characterize parabolas.

First of all, in Section 3 we prove the following:

Theorem 1.2. *Let X denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then the following are equivalent.*

1) *For all $P \in X$ and sufficiently small $h > 0$, $S_P(h) = \lambda(P)V_P(h)$, where $\lambda(P)$ is a function of P.*
2) *For all $P \in X$ and sufficiently small $h > 0$, $S_P(h) = \frac{2}{3}V_P(h)$.*
3) *X is an open part of a parabola.*

Next, in Section 4 we study plane curves satisfying (1.4) in Lemma 1.1.

In [10], Krawczyk showed that for a strictly convex $C^{(4)}$ curve X in the plane $\mathbb{R}^2$, the following holds:

$$\lim_{A_1,A_2\to A} \frac{T}{U} = 2. \tag{1.5}$$

His application of (1.5) states that if a strictly convex $C^{(4)}$ curve X in the plane $\mathbb{R}^2$ satisfies for some function $\lambda(P)$

$$U = \lambda(P)T, \tag{1.6}$$

then $\lambda(P) = \frac{1}{2}$ and X is an open part of the graph of a quadratic polynomial.

Extending the results in [10], in [9] the first author and K.-C. Shim showed that if a strictly convex $C^{(3)}$ curve X in the plane $\mathbb{R}^2$ satisfies (1.6) for some function $\lambda(P)$, then $\lambda(P) = \frac{1}{2}$ and X is an open part of a parabola. For a further study, they also posed a question (Question 3 in [9]) whether the property (1.4) is a characteristic property of parabolas.

As a result, in Section 4 we give an affirmative answer to Question 3 in [9] as follows:

Theorem 1.3. *Let X denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then the following are equivalent.*

1) *For all $P \in X$ and sufficiently small $h > 0$, $U_P(h) = \lambda(P)T_P(h)$, where $\lambda(P)$ is a function of P.*
2) *For all $P \in X$ and sufficiently small $h > 0$, $U_P(h) = \frac{1}{2}T_P(h)$.*
3) *X is an open part of a parabola.*

Finally, in Section 5 we prove the following:

Theorem 1.4. *Let X denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then the following are equivalent.*

1) *For all $P \in X$ and sufficiently small $h > 0$, $S_P(h) = \lambda(P)W_P(h)$, where $\lambda(P)$ is a function of P.*
2) *For all $P \in X$ and sufficiently small $h > 0$, $S_P(h) = \frac{8}{9}W_P(h)$.*
3) *X is an open part of a parabola.*

For some characterizations of parabolas or conic sections by properties of tangent lines, see [3] and [8]. In [4], using curvature function κ and support function h of a plane curve, the first and third authors of the present paper gave a characterization of ellipses and hyperbolas centered at the origin.

In [11], B. Richmond and T. Richmond established a dozen necessary and sufficient conditions for the graph of a function to be a parabola by using elementary techniques.

Throughout this article, all curves are of class $C^{(3)}$ and connected, unless otherwise mentioned.

2. Preliminaries

In order to prove theorems in Section 1, we need the following lemma.

Lemma 2.1. *Suppose that X is a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. Then we have*

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}L_P(h)=\frac{2\sqrt{2}}{\sqrt{\kappa(P)}},\tag{2.1}$$

$$\lim_{h\to 0}\frac{1}{h\sqrt{h}}S_P(h)=\frac{4\sqrt{2}}{3\sqrt{\kappa(P)}},\tag{2.2}$$

$$\lim_{h\to 0}\frac{T_P(h)}{h\sqrt{h}}=\frac{\sqrt{2}}{\sqrt{\kappa(P)}}\tag{2.3}$$

and

$$\lim_{h\to 0}\frac{U_P(h)}{h\sqrt{h}}=\frac{\sqrt{2}}{2\sqrt{\kappa(P)}},\tag{2.4}$$

where $\kappa(P)$ is the curvature of X at P with respect to the unit normal N.

Proof. It follows from [7] that (2.1) and (2.2) hold. For a proof of (2.3) and (2.4), see [9]. □

First, we obtain:

Lemma 2.2. *Suppose that X is a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. Then we have*

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}\ell_P(h)=\frac{\sqrt{2}}{\sqrt{\kappa(P)}},\tag{2.5}$$

$$\lim_{h\to 0}\frac{1}{h\sqrt{h}}V_P(h)=\frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\tag{2.6}$$

and

$$\lim_{h\to 0}\frac{1}{h\sqrt{h}}W_P(h)=\frac{3\sqrt{2}}{2\sqrt{\kappa(P)}}. \tag{2.7}$$

Proof. We fix an arbitrary point P on X. Then, we may take a coordinate system (x,y) of $\mathbb{R}^2$: P is taken to be the origin $(0,0)$ and x-axis is the tangent line ℓ of X at P. Furthermore, we may regard X to be locally the graph of a non-negative strictly convex function $f:\mathbb{R}\to\mathbb{R}$ with $f(0)=f'(0)=0$. Then N is the upward unit normal.

Since the curve X is of class $C^{(3)}$, the Taylor's formula of $f(x)$ is given by

$$f(x)=ax^2+f_3(x), \tag{2.8}$$

where $2a=f''(0)$ and $f_3(x)$ is an $O(|x|^3)$ function. From $\kappa(P)=f''(0)>0$, we see that a is positive.

For a sufficiently small $h>0$, we denote by $A_1(s,f(s))$ and $A_2(t,f(t))$ the points, where the line $m: y=h$ meets the curve X with $s<0<t$. Then $f(s)=f(t)=h$ and we get $B_1(s-h/f'(s),0)$, $B_2(t-h/f'(t),0)$ and $B(x_0,y_0)$ with

$$x_0=\frac{tf'(t)-sf'(s)}{f'(t)-f'(s)} \tag{2.9}$$

and

$$y_0=\frac{(t-s)f'(t)f'(s)+h(f'(t)-f'(s))}{f'(t)-f'(s)}. \tag{2.10}$$

Noting that $L_P(h)=t-s$ and

$$\ell_P(h)=t-s+\frac{f'(t)-f'(s)}{f'(s)f'(t)}h, \tag{2.11}$$

one obtains

$$\begin{aligned} 2V_P(h)&=\{t-s\}(h-y_0)\\ &=\frac{-f'(s)f'(t)}{f'(t)-f'(s)}L_P(h)^2 \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} 2W_P(h)&=\{2(t-s)+\frac{f'(t)-f'(s)}{f'(s)f'(t)}h\}h\\ &=2hL_P(h)+\frac{f'(t)-f'(s)}{f'(s)f'(t)}h^2. \end{aligned} \tag{2.13}$$

If we let

$$\alpha_P(h)=\frac{f'(t)-f'(s)}{-f'(s)f'(t)}\sqrt{h}, \tag{2.14}$$

then from (2.11)-(2.13) we have

$$\begin{aligned} \ell_P(h) &= L_P(h) - \alpha_P(h)\sqrt{h}, \\ 2V_P(h) &= \frac{\sqrt{h}}{\alpha_P(h)} L_P(h)^2 \end{aligned} \tag{2.15}$$

and

$$2W_P(h) = 2hL_P(h) - \alpha_P(h)h\sqrt{h}. \tag{2.16}$$

On the other hand, from Lemma 5 in [9] we get

$$\lim_{h\to 0} \alpha_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}. \tag{2.17}$$

Hence, together with Lemma 2.1, (2.15) and (2.16), this completes the proof. □

Next, we get the following lemma which plays a crucial role in the proofs of Theorems 1.3 and 1.4.

Lemma 2.3. *Suppose that X is a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. Then we have*

$$\ell_P(h) = L_P(h) - h\frac{d}{dh}L_P(h). \tag{2.18}$$

Proof. As in the proof of Lemma 2.2, for an arbitrary point P on X we take a coordinate system (x, y) of $\mathbb{R}^2$ so that (2.8) holds. If we put $f(t) = h$ for sufficiently small $t > 0$, then the line $m : y = h$ meets the curve X at the points $A_1(s(t), h)$ and $A_2(t, h)$ with $s = s(t) < 0 < t$. Hence we have

$$f(s(t)) = f(t) \tag{2.19}$$

and

$$B_1 = (s(t) - \frac{h}{f'(s(t))}, 0), \quad B_2 = (t - \frac{h}{f'(t)}, 0). \tag{2.20}$$

Thus, we get

$$L_P(h) = t - s(t) \tag{2.21}$$

and

$$\ell_P(h) = t - s(t) - \{\frac{1}{f'(t)} - \frac{1}{f'(s(t))}\}h. \tag{2.22}$$

Noting $h = f(t)$, one obtains from (2.21) that

$$\frac{d}{dh}L_P(h) = \frac{1 - s'(t)}{f'(t)}. \tag{2.23}$$

Therefore, it follows from (2.19) that

$$\frac{d}{dh}L_P(h) = \frac{1}{f'(t)} - \frac{1}{f'(s(t))}. \tag{2.24}$$

Together with (2.21) and (2.22), this completes the proof. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

It is trivial to show that any open part of parabolas satisfy 1) and 2) in Theorem 1.2.

Conversely, suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies $S_P(h) = \lambda(P)V_P(h)$ for all $P \in X$ and sufficiently small $h > 0$. Then, it follows from Lemmas 2.1 and 2.2 that $\lambda(P) = \frac{2}{3}$.

We fix an arbitrary point A_1 on X. Then, we may take a coordinate system (x, y) of $\mathbb{R}^2$ so that A_1 is the origin $(0, 0)$ and x-axis is the tangent line of X at A_1. Furthermore, we may regard X to be locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$. Then N is the upward unit normal.

Since the curve X is of class $C^{(3)}$, the Taylor's formula of $f(x)$ is given by

$$f(x) = ax^2 + f_3(x), \tag{3.1}$$

where $2a = f''(0)$ and $f_3(x)$ is an $O(|x|^3)$ function. From $\kappa(A_1) = f''(0) > 0$, we see that a is positive.

For any point $A_2(t, f(t))$ with sufficiently small t, we denote by $P = A$ the point on X such that the chord A_1A_2 is parallel to the tangent of X at $P = A$. Then we have $A = (g(t), f(g(t)))$, for a function $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ which satisfies $|g(t)| < |t|$ and

$$tf'(g(t)) = f(t). \tag{3.2}$$

Since $g(t)$ tends to 0 as $t \to 0$, we may assume that $g(0) = 0$.

Then we have

$$B_1 = (g(t) - \frac{tf(g(t))}{f(t)}, 0), B = (t - \frac{f(t)}{f'(t)}, 0) \tag{3.3}$$

and

$$B_2 = (t + \frac{tf(g(t)) - f(t)g(t)}{tf'(t) - f(t)}, f(t) + \frac{tf(g(t)) - f(t)g(t)}{tf'(t) - f(t)}f'(t)). \tag{3.4}$$

If we let h the distance from $P = A$ to the chord A_1A_2, then by the definition of $V_P(h)$ we have

$$V_P(h) = \epsilon\frac{f(t)}{2}\{t - \frac{f(t)}{f'(t)}\}, \tag{3.5}$$

where $\epsilon = 1$ if $t > 0$ and $\epsilon = -1$ otherwise.

We now prove the following lemma, which is useful in the proof of Theorem 1.2.

Lemma 3.1. *Suppose that $S_P(h) = \frac{2}{3}V_P(h)$. Then the function $f(t)$ satisfies*

$$2f(t)^2f''(t) = f'(t)^2\{tf'(t) - f(t)\}. \tag{3.6}$$

Proof. Note that

$$V_P(h) = S_P(h) + \epsilon \int_0^t f(x)dx - \frac{\epsilon}{2}\frac{f(t)^2}{f'(t)}, \tag{3.7}$$

where $\epsilon = 1$ if $t > 0$ and $\epsilon = -1$ otherwise.

By the assumption $S_P(h) = \frac{2}{3}V_P(h)$, we get from (3.7)

$$2V_P(h) = 6\epsilon \int_0^t f(x)dx - 3\epsilon \frac{f(t)^2}{f'(t)}. \tag{3.8}$$

After substituting $V_P(h)$ in (3.5) into (3.8), let us differentiate (3.8) with respect to t. Then we get (3.6). This completes the proof. □

Now, it follows from Lemma 3.1 that

$$2\frac{f''(t)}{f'(t)^2} = \frac{tf'(t) - f(t)}{f(t)^2}. \tag{3.9}$$

Hence we obtain

$$2\left(\frac{1}{f'(t)}\right)' = \left(\frac{t}{f(t)}\right)', \tag{3.10}$$

which implies that

$$\frac{2}{f'(t)} = \frac{t}{f(t)} + a, \tag{3.11}$$

where a is a constant.

After replacing t by x, for $y = f(x)$ we get a differential equation:

$$2ydx - (x + ay)dy = 0. \tag{3.12}$$

Hence, using a standard method of differential equations, we see that for some positive constant b, $y = f(x)$ satisfies

$$(x - ay)^2 = 2by. \tag{3.13}$$

Thus we have

$$f(x) = \begin{cases} \frac{1}{a^2}\{ax + b - \sqrt{2abx + b^2}\}, & \text{if } a \neq 0, \\ \frac{x^2}{2b}, & \text{if } a = 0. \end{cases} \tag{3.14}$$

Note that

$$f(0) = f'(0) = 0,\ f''(0) = \frac{1}{b} \quad \text{and} \quad f'''(0) = -\frac{3a}{b^2} \quad \text{or} \quad 0. \tag{3.15}$$

It follows from (3.13) that the curve X around an arbitrary point A_1 is an open part of the parabola defined by

$$x^2 - 2axy + a^2y^2 - 2by = 0. \tag{3.16}$$

Finally using (3.15), in the same manner as in [7], we can show that the curve X is globally an open part of a parabola. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

In this section, we use the main result of [7] (Theorem 3 in [7]) and Lemma 2.3 in Section 2 in order to prove Theorem 1.3.

It is obvious that any open part of parabolas satisfy 1) and 2) in Theorem 1.3.

Conversely, suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies $U_P(h) = \lambda(P)T_P(h)$ for all $P \in X$ and sufficiently small $h > 0$. Then, it follows from Lemma 2.1 that $\lambda(P) = \frac{1}{2}$.

First, we note the following which can be easily shown.

Lemma 4.1. *For a point $P \in X$ and a sufficiently small $h > 0$, the following are equivalent.*

1) $U_P(h) = \frac{1}{2}T_P(h)$,
2) $T_P(h) = \frac{1}{2}V_P(h)$,
3) $\ell_P(h) = \frac{1}{2}L_P(h)$.

Next, using Lemma 2.3 we get the following.

Lemma 4.2. *Suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies $U_P(h) = \frac{1}{2}T_P(h)$ for all $P \in X$ and sufficiently small $h > 0$. Then for all $P \in X$ and sufficiently small $h > 0$ we have*

$$L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}. \tag{4.1}$$

Proof. Together with 3) of Lemma 4.1, Lemma 2.3 shows

$$h\frac{d}{dh}L_P(h) = \frac{1}{2}L_P(h), \tag{4.2}$$

which yields for some constant $C = C(P)$

$$L_P(h) = C\sqrt{h}. \tag{4.3}$$

Thus, Lemma 2.1 completes the proof. $\square$

Finally, we prove Theorem 1.3 as follows.

Since $S'_P(h) = L_P(h)$ ([7]) and $S_P(0) = 0$, by integrating we get from (4.1)

$$S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}h\sqrt{h}. \tag{4.4}$$

Noting $2T_P(h) = hL_P(h)$, one gets from (4.1) and (4.4) that

$$S_P(h) = \frac{4}{3}T_P(h). \tag{4.5}$$

Theorem 3 of [7] states that (4.5) implies X is an open part of a parabola, completing the proof of Theorem 1.3.

5. Proof of Theorem 1.4

In this section, in order to prove Theorem 1.4 we use the main result of [7] (Theorem 3 in [7]) and Lemma 2.3 in Section 2.

It is trivial to show that any open part of parabolas satisfy 1) and 2) in Theorem 1.4.

Conversely, suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies $S_P(h) = \lambda(P)W_P(h)$ for all $P \in X$ and sufficiently small $h > 0$. Then, it follows from Lemmas 2.1 and 2.2 that $\lambda(P) = \frac{8}{9}$.

First, using Lemma 2.3 we get the following.

Lemma 5.1. *Suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies $S_P(h) = \frac{8}{9}W_P(h)$ for all $P \in X$ and sufficiently small $h > 0$. Then for all $P \in X$ and sufficiently small $h > 0$ we have*

$$L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}. \tag{5.1}$$

Proof. Note that

$$W_P(h) = \frac{1}{2}\{L_P(h) + \ell_P(h)\}h. \tag{5.2}$$

Hence, together with Lemma 2.3 the assumption shows that

$$9S_P(h) = 8L_P(h)h - 4L'_P(h)h^2. \tag{5.3}$$

By differentiating (5.3) with respect to h and using $S'_P(h) = L_P(h)$, we get

$$4L''_P(h)h^2 + L_P(h) = 0, \tag{5.4}$$

which is a second order Euler equation. Its general solutions are given by

$$L_P(h) = C_1\sqrt{h} + C_2\sqrt{h}\ln h, \tag{5.5}$$

where $C_1 = C_1(P)$ and $C_2 = C_2(P)$ are constant.

It follows from Lemma 2.1 that

$$C_1(P) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}} \quad \text{and} \quad C_2(P) = 0. \tag{5.6}$$

This completes the proof. □

Finally, the argument following Lemma 4.2 completes the proof of Theorem 1.4.

6. Corollaries

In this section, we give some corollaries.

Suppose that X is a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies for all $P \in X$ and sufficiently small $h > 0$

$$S_P(h) = \lambda(P)V_P(h)^{\mu(P)}, \tag{6.1}$$

where $\lambda(P)$ and $\mu(P)$ are some functions. Using Lemmas 2.1 and 2.2, by letting $h \to 0$ we see that

$$\lim_{h\to 0} V_P(h)^{\mu(P)-1} = \frac{2}{3\lambda(P)}. \tag{6.2}$$

Since $V_P(h)$ tends to zero as $h \to 0$, (6.2) shows that $\mu(P) = 1$. Hence we also obtain from (6.1) that $\lambda(P) = 2/3$. Thus, from Theorem 1.2 we get:

Corollary 6.1. *Suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then, the following are equivalent.*

1) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies $S_P(h) = \lambda(P)V_P(h)^{\mu(P)}$, where $\lambda(P)$ and $\mu(P)$ are some functions.*
2) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies $S_P(h) = \frac{2}{3}V_P(h)$.*
3) *X is an open part of a parabola.*

The similar argument as in the proof of Corollary 6.1 shows the following.

Corollary 6.2. *Suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then, the following are equivalent.*

1) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies $U_P(h) = \lambda(P)T_P(h)^{\mu(P)}$, where $\lambda(P)$ and $\mu(P)$ are some functions.*
2) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies $U_P(h) = \frac{1}{2}T_P(h)$.*
3) *X is an open part of a parabola.*

Corollary 6.3. *Suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then, the following are equivalent.*

1) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies $S_P(h) = \lambda(P)W_P(h)^{\mu(P)}$, where $\lambda(P)$ and $\mu(P)$ are some functions.*
2) *For all $P \in X$ and sufficiently small $h > 0$, X satisfies $S_P(h) = \frac{8}{9}W_P(h)$.*
3) *X is an open part of a parabola.*

References

[1] W. A. Day, *Inequalities for areas associated with conics*, Amer. Math. Monthly **98** (1991), no. 1, 36–39.

[2] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.

[3] D.-S. Kim and S. H. Kang, *A characterization of conic sections*, Honam Math. J. **33** (2011), no. 3, 335–340.

[4] D.-S. Kim and Y. H. Kim, *A characterization of ellipses*, Amer. Math. Monthly **114** (2007), no. 1, 66–70.

[5] ———, *Some characterizations of spheres and elliptic paraboloids*, Linear Algebra Appl. **437** (2012), no. 1, 113–120.

[6] ———, *Some characterizations of spheres and elliptic paraboloids II*, Linear Algebra Appl. **438** (2013), no. 3, 1356–1364.

[7] ———, *On the Archimedean characterization of parabolas*, Bull. Korean Math. Soc. **50** (2013), no. 6, 2103–2114.

[8] D.-S. Kim, Y. H. Kim, and J. H. Park, *Some characterizations of parabolas*, Kyungpook Math. J. **53** (2013), no. 1, 99–104.

[9] D.-S. Kim and K.-C. Shim, *Area of triangles associated with a curve*, Bull. Korean Math. Soc. To appear. arXiv:1401.4751.
[10] J. Krawczyk, *On areas associated with a curve*, Zesz. Nauk. Uniw.Opol. Mat. **29** (1995), 97–101.
[11] B. Richmond and T. Richmond, *How to recognize a parabola*, Amer. Math. Monthly **116** (2009), no. 10, 910–922.
[12] S. Stein, *Archimedes*, What did he do besides cry Eureka? Mathematical Association of America, Washington, DC, 1999.

Dong-Soo Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea
E-mail address: dosokim@chonnam.ac.kr

Wonyong Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea
E-mail address: yong4625@naver.com

Young Ho Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail address: yhkim@knu.ac.kr

Dae Heui Park
Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea
E-mail address: dhpark3331@chonnam.ac.kr