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ON EVALUATIONS OF THE MODULAR j–INVARIANT BY MODULAR EQUATIONS OF DEGREE 2

Dae Hyun Paek† and Jinhee Yi

Abstract. We derive modular equations of degree 2 to establish explicit relations for the parameterizations for the theta functions φ and ψ. We then find specific values of the parameterizations to evaluate some new values of the modular j–invariant in terms of J_n.

1. Introduction

The invariants $J(\tau)$ and $j(\tau)$, for $\tau \in \mathbb{H} = \{\tau : \operatorname{Im}\tau > 0\}$, are defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \quad \text{and} \quad j(\tau) = 1728 J(\tau),$$

where,

$$\Delta(\tau) = g_2^3(\tau) - 27\, g_3^2(\tau)\,,$$

$$g_2(\tau) = 60 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-4},$$

and

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-6}.$$

Moreover, the function $\gamma_2(\tau)$ is defined by ([5, p. 249])

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)}\,, \tag{1.1}$$

where the principal branch is chosen. Ramanujan defined a function J_n by

$$J_n = \frac{1 - 16\alpha_n(1-\alpha_n)}{8\left(4\alpha_n(1-\alpha_n)\right)^{1/3}}, \tag{1.2}$$

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where $\alpha_n = k_n^2$ and n is a natural number. Here, as usual, in the theory of elliptic functions, let k, $0 < k < 1$, denote the modulus, then the singular modulus k_n is defined by $k_n = k(e^{-\pi\sqrt{n}})$. To identify J_n with the class invariant G_n, we first, as usual, set

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad |q|<1,$$

and let

$$\chi(q) = (-q;q^2)_\infty.$$

Furthermore, for $|ab|<1$, Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}.$$

Then, the classical theta functions φ, ψ, and f are defined by, for $|q|<1$,

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

From $(q;q)_\infty = (q;q^2)_\infty(q^2;q^2)_\infty$, it is easily seen that

$$\chi(-q) = \frac{f(-q)}{f(-q^2)}. \tag{1.3}$$

If $q_n = e^{-\pi\sqrt{n}}$ and n is a positive rational number, then the class invariant G_n is defined by

$$G_n = 2^{-1/4}q_n^{-1/24}\chi(q_n). \tag{1.4}$$

Since, by [2, p. 124],

$$\chi(q) = 2^{1/6}\left(\frac{q}{\alpha(1-\alpha)}\right)^{1/24},$$

it follows from (1.4) that

$$G_n = \left(4\alpha_n(1-\alpha_n)\right)^{-1/24}. \tag{1.5}$$

Hence, by (1.2) and (1.5), we find that

$$J_n = \frac{1}{8}G_n^8\left(1-4G_n^{-24}\right). \tag{1.6}$$

We now identify J_n with γ_2. From [5, Theorem 12.17], for $q = e^{2\pi i\tau}$,

$$\gamma_2(\tau) = 2^8 \frac{q^{2/3} f^{16}(-q^2)}{f^{16}(-q)} + \frac{f^8(-q)}{q^{1/3} f^8(-q^2)}. \tag{1.7}$$

Setting $\tau = \frac{3+\sqrt{-n}}{2}$, we deduce from (1.3), (1.4), and (1.7) that

$$\gamma_2\left(\frac{3+\sqrt{-n}}{2}\right) = -4G_n^8(1-4G_n^{-24}).$$

Hence, by (1.1) and (1.6), we have

$$J_n = -\frac{1}{32}\gamma_2\left(\frac{3+\sqrt{-n}}{2}\right) = -\frac{1}{32}\sqrt[3]{j\left(\frac{3+\sqrt{-n}}{2}\right)}. \tag{1.8}$$

Therefore, in general, the value of J_n can be obtained in terms of G_n or $j\left(\frac{3+\sqrt{-n}}{2}\right)$ for any natural number n. For 15 values of n such that $n \equiv 3 \pmod 4$, Ramanujan indicated the corresponding 15 values of J_n, although some are not given very explicitly. There are 13 cases when the class number of the order in an imaginary quadratic fields equals 1 ([5, p. 260]). In such an instance, the value of j–invariant is known to be an integer. In these cases, Ramanujan gave 7 values of J_n for $n = 3, 11, 19, 27, 43, 67$, and 163. See [1, pp. 310–311] for more details. Formula (1.6) can be used to evaluate some of these values such as $J_3 = 0$ and $J_{27} = 5 \cdot 3^{1/3}$, but in most instances the value of G_n is unavailable. For the rest of cases of degree 1, the values of J_n can be evaluated by using the relation (1.8) and the corresponding j–invariant given in [5, p. 261]. It is also known that there are 29 cases when the degree of $j\left(\frac{3+\sqrt{-n}}{2}\right)$ equals 2. Ramanujan dealt with 6 of these: the values of J_n for $n = 35, 51, 75, 91, 99$, and 115, even though he did not record the value of J_{99} explicitly. See also [1, pp. 311–312] for more details.

As an instance of using the values of J_n, Ramanujan further defined a function t_n by, for $q = e^{-\pi\sqrt{n}}$,

$$t_n = \sqrt{3}\, q^{1/18} \frac{f(q^{1/3}) f(q^3)}{f^2(q)}$$

and asserted that

$$t_n = \left(2\sqrt{64J_n^2 - 24J_n + 9} - (16J_n - 3)\right)^{1/6}.$$

Ramanujan then considered the polynomials $p_n(t)$ satisfied by t_n for $n = 11, 35, 59, 83$, and 107. These polynomials are extremely simple, whereas the corresponding polynomials of the same degrees satisfied by J_n are more complicated. Refer [4] to see that if n is square free, $n \equiv 11 \pmod{24}$, and the class number of the Hilbert class field is odd, then t_n and J_n satisfy irreducible polynomials of the same degree.

Since modular equations are crucial in this study on evaluations of the modular j–invariant in terms of J_n, we now give a definition of a modular equation.

Let a, b, and c be arbitrary complex numbers except that c cannot be a non-positive integer. Then, for $|z| < 1$, the Gaussian or ordinary hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, $$

where $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for each positive integer n.

The complete elliptic integral of the first kind $K(k)$ is defined by

$$ K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \frac{\pi}{2}\, {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right) = \frac{\pi}{2} \varphi^2\left(e^{-\pi \frac{K'}{K}}\right), \tag{1.9} $$

where $0 < k < 1$, $K' = K(k')$, and $k' = \sqrt{1 - k^2}$. The number k is called the modulus of K and k' is called the complementary modulus. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively, where $0 < k < 1$ and $0 < l < 1$. Suppose that

$$ \frac{L'}{L} = n \frac{K'}{K} \tag{1.10} $$

holds for some positive integer n. Then a relation between k and l induced by (1.10) is called a modular equation of degree n. If we set

$$ q = e^{-\pi \frac{K'}{K}} \quad \text{and} \quad q' = e^{-\pi \frac{L'}{L}}, \tag{1.11} $$

then (1.10) is equivalent to the relation $q^n = q'$. Hence a modular equation can be viewed as an identity involving theta functions at the arguments q and q^n. Set $\alpha = k^2$ and $\beta = l^2$, then we say that β has degree n over α.

We now turn to evaluations of J_n by using a parametrization $r_{k,n}$ for the theta function f. In [6], $r_{k,n}$ is defined by, for any positive real numbers k and n,

$$ r_{k,n} = \frac{f(-q)}{k^{1/4} q^{(k-1)/24} f(-q^k)}, $$

where $q = e^{-2\pi\sqrt{n/k}}$. For convenience, we write r_n instead of $r_{2,n}$. Then, G_n can be written in terms of r_n ([6, Theorem 2.2.3])

$$ G_n = \frac{r_{2n}}{2^{1/4} r_{n/2}}. $$

Hence (1.6) can also be written in terms of r_n

$$ J_n = \frac{1}{32} \left(\frac{r_{2n}}{r_{n/2}}\right)^8 \left(1 - 2^8 \left(\frac{r_{n/2}}{r_{2n}}\right)^{24}\right). \tag{1.12} $$

Using the formula (1.12), the explicit values of J_1, J_2, …, J_{10} were evaluated in [6]. In particular,

$$J_2 = \frac{5}{16}\left(-19+13\sqrt{2}\right),$$

$$J_4 = \frac{3}{32}(-724+513\sqrt{2}),$$

$$J_8 = \frac{(1+\sqrt{2})^3\left(4+\sqrt{2+10\sqrt{2}}\right)^3 - 256}{32(1+\sqrt{2})^2\left(4+\sqrt{2+10\sqrt{2}}\right)^2}.$$

Meanwhile the relation (1.6) can also be written in terms of parametrizations $h'_{k,n}$ and $l'_{k,n}$ for the theta function φ and ψ, respectively. In [7, 8], $h'_{k,n}$ and $l'_{k,n}$ are defined by, for any positive real numbers k and n,

$$h'_{k,n} = \frac{\varphi(-q)}{k^{1/4}\varphi(-q^k)},$$

where $q = e^{-2\pi\sqrt{n/k}}$ and

$$l'_{k,n} = \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)},$$

where $q = e^{-\pi\sqrt{n/k}}$. For convenience, we write h'_n and l'_n instead of $h'_{2,n}$ and $l'_{2,n}$, respectively, throughout this paper. Then it follows from [8, Theorem 6.3] that

$$l'_{2n} = \frac{r_{2n}^2}{r_{n/2}} \quad \text{and} \quad h'_{2n} = \frac{r_{n/2}^2}{r_{2n}}.$$

Hence (1.12) can be written in the alternative form

$$J_n = \frac{1}{32}\left(\frac{l'_{2n}}{h'_{n/2}}\right)^{8/3}\left(1-2^8\left(\frac{h'_{n/2}}{l'_{2n}}\right)^8\right). \tag{1.13}$$

In addition, in [7], $h_{k,n}$ is defined by, for any positive real numbers k and n,

$$h_{k,n} = \frac{\varphi(q)}{k^{1/4}\varphi(q^k)},$$

where $q = e^{-\pi\sqrt{n/k}}$. For convenience, we also write h_n in stead of $h_{2,n}$ throughout this paper.

Note that specific values of h_n will play crucial roles in evaluating the corresponding values of h'_n and l'_n later on. This study is motivated by the values of J_2, J_4, and J_8 obtained from (1.12). Some values of the modular j–invariant, as mentioned before, were used to evaluate the values of J_n for $n = 11, 19, 43, 67,$ and 163. Hence, instead of evaluating some values of the modular j–invariant to find the corresponding values of J_n, we first employ (1.13) to evaluate the values of J_n, and then find the corresponding values of $j\left(\frac{3+\sqrt{-n}}{2}\right)$ by the relation (1.8), in the case of when n is of the form $n = 2^{2m-1}$ or $n = 2^{2m}$ for every positive integer m. Note that our results contain the values of J_2, J_4, and J_8 as special cases. In order to do so, we first derive modular equations of degree 2 for the theta functions φ and ψ. We then find explicit relations for the corresponding parameterizations, evaluate some numerical values of h_n, h'_n, and l'_n, and evaluate some new values of J_n so that we have the corresponding values of the modular j–invariant.

2. Preliminary results

In this section, we introduce fundamental theta function identities that will play key roles in deriving modular equations of degree 2. Let k be the modulus as in (1.9). Set $x = k^2$ and also set

$$k^2 = x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{2.1}$$

Then

$$\varphi^2(q) = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; x\right) = z, \tag{2.2}$$

where

$$q = e^{-y} = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}\right) = \exp\left(-\pi \frac{K(k')}{K(k)}\right). \tag{2.3}$$

Lemma 2.1 ([3, Theorem 5.4.1]). *If x, q, and z are related by* (2.1), (2.2), *and* (2.3), *then*

(i) $\varphi(q) = \sqrt{z}$,
(ii) $\varphi(-q) = \sqrt{z}\,(1-x)^{1/4}$,
(iii) $\varphi(q^2) = \sqrt{z}\sqrt{\frac{1+\sqrt{1-x}}{2}}$,
(iv) $\varphi(-q^2) = \sqrt{z}\,(1-x)^{1/8}$,
(v) $\varphi(q^4) = \frac{1}{2}\sqrt{z}\left(1 + (1-x)^{1/4}\right)$.

Lemma 2.2 ([3, Theorem 5.4.2]). *If x, q, and z are related by* (2.1), (2.2), *and* (2.3), *then*

(i) $\psi(q) = \sqrt{\frac{1}{2}z}\left(\frac{x}{q}\right)^{1/8}$,
(ii) $\psi(q^2) = \frac{1}{2}\sqrt{z}\left(\frac{x}{q}\right)^{1/4}$.

3. Modular equations of degree 2

In this section, we derive modular equations of degree 2 and establish some explicit relations for h_n, h'_n, and l'_n for some positive real number n by using these modular equations.

Theorem 3.1. *If* $P = \frac{\varphi(q)}{\varphi(q^2)}$ *and* $Q = \frac{\varphi(q^2)}{\varphi(q^4)}$, *then*

$$PQ + \frac{2}{PQ} = \frac{Q}{P} + 2\,. \tag{3.1}$$

Proof. By Lemma 2.1(i), (iii), and (v),

$$P = \sqrt{\frac{2}{1+\sqrt{1-\alpha}}} \quad \text{and} \quad Q = \frac{\sqrt{2}\sqrt{1+\sqrt{1-\alpha}}}{1+(1-\alpha)^{1/4}}.$$

Combine and rearrange the last two equalities in terms of P and Q to complete the proof. □

For a different proof of Theorem 3.1, see [7, Theorem 4.2]. Now, by the definition of h_n, we have:

Corollary 3.2. *For every positive real number n, we have*

$$\sqrt{2}\left(h_n h_{4n} + \frac{1}{h_n h_{4n}}\right) = \frac{h_{4n}}{h_n} + 2\,. \tag{3.2}$$

Note that (3.2) is the same equation as in [7, Theorem 4.6].

Theorem 3.3. *If* $P = \frac{\varphi(q)}{\varphi(q^2)}$ *and* $Q = \frac{\varphi(-q)}{\varphi(-q^2)}$, *then*

$$P^2(Q^4+1) = 2. \tag{3.3}$$

Proof. By Lemma 2.1(i)–(iv),

$$P = \sqrt{\frac{2}{1+\sqrt{1-\alpha}}} \quad \text{and} \quad Q = (1-\alpha)^{1/8}.$$

Combining and rearranging the last two equalities in terms of P and Q, we complete the proof. □

By the definitions of h_n and h'_n, we have:

Corollary 3.4. *For every positive real number n, we have*

$$h_n^2(2h'^4_{n/4} + 1) = \sqrt{2}\,. \tag{3.4}$$

Theorem 3.5. *If* $P = \frac{\varphi(q)}{\varphi(q^2)}$ *and* $Q = \frac{\psi(q)}{q^{1/8}\psi(q^2)}$, *then*

$$4P^4 = (P^2-1)Q^8. \tag{3.5}$$

Proof. By Lemma 2.1(i) and (iii) and Lemma 2.2,

$$P = \sqrt{\frac{2}{1+\sqrt{1-\alpha}}} \quad \text{and} \quad Q = \frac{\sqrt{2}}{\alpha^{1/8}}.$$

Combining and rearranging the last two equalities in terms of P and Q, we have the required result. □

By the definitions of h_n and l'_n, we have:

Corollary 3.6. *For every positive real number n, we have*

$$2\,h_n^4 = (\sqrt{2}\,h_n^2 - 1)\,l_n'^8\,. \tag{3.6}$$

4. Specific values of h_n, h'_n, and l'_n

We are now in position to evaluate specific values of h_n, h'_n, and l'_n for some positive real number n by using the explicit relations established in Section 3. To begin with, we show how to evaluate the values of $h_{2^{2m}}$ and $h_{2^{2m+1}}$ for every positive integer m. We only state the instances when $m = 1, 2, 3$, and 4.

Theorem 4.1. *We have*

(i) $h_4 = 1 + \sqrt{2} - \sqrt{1+\sqrt{2}}\,,$

(ii) $h_8 = \frac{\sqrt{2+\sqrt{2}}}{1+2^{1/4}}\,,$

(iii) $h_{16} = \frac{1+\sqrt{1+\sqrt{2}}}{\sqrt{1+\sqrt{2}}+\left(2(1+\sqrt{2}\,)\right)^{1/4}}\,,$

(iv) $h_{32} = \frac{2^{1/4}+\sqrt{2}}{2^{5/8}+\sqrt{1+\sqrt{2}}}\,,$

(v) $h_{64} = \frac{2^{3/4}(1+\sqrt{2}\,)^{1/4}+\sqrt{2(1+\sqrt{2}\,)}}{1+2^{7/8}(1+\sqrt{2}\,)^{3/8}+\sqrt{1+\sqrt{2}}}\,,$

(vi) $h_{128} = \frac{2^{7/8}+\sqrt{2+\sqrt{2}}}{1+2^{1/4}+2^{11/16}(2+\sqrt{2}\,)^{1/4}}\,,$

(vii) $h_{256} = \frac{\sqrt{2}}{a+\sqrt{\sqrt{2}-a^2}}\,,$

(viii) $h_{512} = \frac{\sqrt{2}}{b+\sqrt{\sqrt{2}-b^2}}\,,$

where

$$a = \frac{2^{3/4}(1+\sqrt{2}\,)^{1/4} + \sqrt{2(1+\sqrt{2}\,)}}{1 + 2^{7/8}(1+\sqrt{2}\,)^{3/8} + \sqrt{1+\sqrt{2}}}$$

and

$$b = \frac{2^{7/8} + \sqrt{2+\sqrt{2}}}{1 + 2^{1/4} + 2^{11/16}(2+\sqrt{2}\,)^{1/4}}\,.$$

Proof. For (i), letting $n = 1$ in (3.2) and putting the value of $h_1 = 1$ from [7, Theorem 2.2], we find that

$$\sqrt{2}\left(h_4 + \frac{1}{h_4}\right) = h_4 + 2\,.$$

Solving the last equation for h_4 and using the fact that $h_4 < 1$, we complete the proof.

For (ii), letting $n = 2$ in (3.2), putting the value of $h_2 = \sqrt{2\sqrt{2}-2}$ from [7, Theorem 4.7(i)], we find that

$$(-4 + 3\sqrt{2}\,)\,h_8^2 - 4\sqrt{-1+\sqrt{2}}\;h_8 + 2 = 0. \tag{4.1}$$

Solving (4.1) for h_8, and using the fact that $h_8 < 1$, we complete the proof.

For (iii), letting $n = 4$ in (3.2), putting the value of h_4 from the result of (i), solving for h_{16}, and using the fact that $h_{16} < 1$, we complete the proof.

For (iv)–(viii), repeat the same argument as in the proof of (iii) to complete the proof. □

See [7, Theorem 4.7] for different proofs of Theorem 4.1(i) and (ii). Hence $h_{2^{2m}}$ and $h_{2^{2m+1}}$ for $m = 5, 6, 7, \ldots$ can be evaluated as in the proof of Theorem 4.1. We next evaluate the values of $h'_{2^{2m-2}}$ and $h'_{2^{2m-1}}$ for every positive integer m. We only state the instances when $m = 1, 2, 3$, and 4.

Theorem 4.2. *Let a and b be as in Theorem 4.1. Then we have*

(i) $h'_1 = \left(\frac{-1+\sqrt{2}}{2}\right)^{1/8}$,

(ii) $h'_2 = \frac{2^{1/16}}{(1+\sqrt{2})^{1/4}}$,

(iii) $h'_4 = \frac{\left(2(1+\sqrt{2})\right)^{1/16}}{\left(\sqrt{2}+\sqrt{1+\sqrt{2}}\right)^{1/4}}$,

(iv) $h'_8 = \frac{2^{5/32}(1+\sqrt{2})^{1/8}}{\sqrt{1+2^{1/4}}}$,

(v) $h'_{16} = \left(\frac{1}{\sqrt{2}\,a^2} - \frac{1}{2}\right)^{1/4}$,

(vi) $h'_{32} = \left(\frac{1}{\sqrt{2}\,b^2} - \frac{1}{2}\right)^{1/4}$,

(vii) $h'_{64} = \left(\frac{a^2(\sqrt{2}-a^2)}{2}\right)^{1/8}$,

(viii) $h'_{128} = \left(\frac{b^2(\sqrt{2}-b^2)}{2}\right)^{1/8}$.

Proof. For (i), letting $n = 4$ in (3.4) and putting the value of h_4 from Theorem 4.1(i), we find that

$$\left(1 + \sqrt{2} - \sqrt{1 + \sqrt{2}}\right)^2 (2h_1'^4 + 1) = \sqrt{2}. \tag{4.2}$$

Solving (4.2) for h'_1 and using the fact that $h'_1 > 0$, we complete the proof.

For (ii)–(viii), use (3.4) and Theorem 4.1(ii)–(viii), respectively, to repeat the same argument as in the proof of (i). □

Next we show how to evaluate the values of $l'_{2^{2m}}$ and $l'_{2^{2m+1}}$ for every positive integer m. We only state the instances when $m = 1, 2, 3$, and 4.

Theorem 4.3. *Let a and b be as in Theorem 4.1. Then we have*

(i) $l'_4 = \left(2(1 + \sqrt{2})\right)^{1/4}$,

(ii) $l'_8 = \sqrt{2 + \sqrt{2}}$,

(iii) $l'_{16} = 1 + \sqrt{1 + \sqrt{2}}$,

(iv) $l'_{32} = \sqrt{4 + 3\sqrt{2} + 2\sqrt{8 + 6\sqrt{2}}}$,

(v) $l'_{64} = \left(\frac{2a^4}{\sqrt{2}\,a^2-1}\right)^{1/8}$,

(vi) $l'_{128} = \left(\frac{2b^4}{\sqrt{2}\,b^2-1}\right)^{1/8}$,

(vii) $l'_{256} = \left(\frac{2}{\sqrt{2}\,a^2-1}\right)^{1/4}$,

(viii) $l'_{512} = \left(\frac{2}{\sqrt{2}\,b^2-1}\right)^{1/4}$.

Proof. For (i), letting $n = 4$ in (3.6) and putting the value of h_4 from Theorem 4.1(i), we find that

$$\left(\sqrt{2}\left(1+\sqrt{2}-\sqrt{1+\sqrt{2}}\right)^2-1\right) l_4'^8 = 2\left(1+\sqrt{2}-\sqrt{1+\sqrt{2}}\right)^4. \tag{4.3}$$

Solving (4.3) for l'_4 and using the fact that $l'_4 > 0$, we complete the proof.

For (ii)–(viii), use (3.6) and Theorem 4.1(ii)–(viii), respectively, to repeat the same argument as in the proof of (i). □

See [8, Theorem 4.10] for different proofs of Theorem 4.3(i), (ii), and (iii).

5. Evaluations of J_n

We now turn to evaluations of the modular j–invariant in terms of J_n in the case of when n is of the form $n = 2^{2m-1}$ or $n = 2^{2m}$ for every positive integer m. We only show the instances when $m = 1, 2, 3$, and 4. Note that we have some new values of J_n such as J_{16}, J_{32}, J_{64}, J_{128}, and J_{256}.

Theorem 5.1. *Let a and b be as in Theorem 4.1. Then we have*

(i) $J_2 = \frac{5}{16}\left(-19+13\sqrt{2}\right)$,

(ii) $J_4 = \frac{3}{32}(-724+513\sqrt{2})$,

(iii) $J_8 = \frac{5\left(83+58\sqrt{2}+2\sqrt{2(1423+1073\sqrt{2})}\right)}{16\left(8(5+4\sqrt{2})+\sqrt{2(1817+1285\sqrt{2})}\right)}$,

(iv) $J_{16} = \frac{3(1+\sqrt{2})\left(731+540\sqrt{2}-120\sqrt{60+43\sqrt{2}}\right)}{2\cdot 2^{3/4}(2+2^{3/4})^4}$,

(v) $J_{32} = \frac{a^8-32(\sqrt{2}-a^2)^2(\sqrt{2}\,a^2-1)}{16a^{16/3}(\sqrt{2}-a^2)^{2/3}(\sqrt{2}\,a^2-1)^{1/3}}$,

(vi) $J_{64} = \frac{b^8-32(\sqrt{2}-b^2)^2(\sqrt{2}\,b^2-1)}{16b^{16/3}(\sqrt{2}-b^2)^{2/3}(\sqrt{2}\,b^2-1)^{1/3}}$,

(vii) $J_{128} = \frac{1-32a^2(\sqrt{2}-a^2)(\sqrt{2}\,a^2-1)^2}{16a^{2/3}(\sqrt{2}-a^2)^{1/3}(\sqrt{2}\,a^2-1)^{2/3}}$,

(viii) $J_{256} = \frac{1-32b^2(\sqrt{2}-b^2)(\sqrt{2}\,b^2-1)^2}{16b^{2/3}(\sqrt{2}-b^2)^{1/3}(\sqrt{2}\,b^2-1)^{2/3}}$.

Proof. For (i), letting $n = 2$ in (1.13) and putting the values of h'_1 from Theorem 4.2(i) and l'_4 from Theorem 4.3(i), we complete the proof.

For (ii)–(viii), repeat the same argument as in the proof of (i) by using Theorems 4.2 and 4.3. □

See [6, Theorem 7.2.2] for alternative proofs of Theorem 5.1(i), (ii), and (iii). Observe that $J_{2^{2m}}$ and $J_{2^{2m+1}}$ for $m = 5, 6, 7, \ldots$ can be evaluated by repeating the same argument as in the proof of Theorem 5.1.

Corollary 5.2. *The value of* $j\left(\frac{3+\sqrt{-n}}{2}\right)$ *can be evaluated in the case of when* n *is of the form* $n = 2^{2m-1}$ *or* $n = 2^{2m}$ *for every positive integer* m*.*

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Dae Hyun Paek
Department of Mathematics Education
Busan National University of Education
Busan 611-736, Korea
E-mail address: paek@bnue.ac.kr

Jinhee Yi
Department of Mathematics and Computer Science
Korea Science Academy of KAIST
Busan 614-822, Korea
E-mail address: jhyi100@kaist.ac.kr