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EIGENVALUE INEQUALITIES OF THE SCHRÖDINGER-TYPE OPERATOR ON BOUNDED DOMAINS IN STRICTLY PSEUDOCONVEX CR MANIFOLDS

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Abstract. In this paper, we study the eigenvalue problem of Schrödinger-type operator on bounded domains in strictly pseudoconvex CR manifolds and obtain some universal inequalities for lower order eigenvalues. Moreover, we will give some generalized Reilly-type inequalities of the first nonzero eigenvalue of the sub-Laplacian on a compact strictly pseudoconvex CR manifold without boundary.

1. Introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Let Δ be the Laplacian acting on functions on M and consider the following eigenvalue problem for the Laplacian

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is known that this eigenvalue problem has a discrete spectrum

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots,$$

where each eigenvalue is repeated with its finite multiplicity.

In 1955, Payne, Pólya and Weinberger showed that for any open bounded domain $\mathbb{R}^2$ the bound $\frac{\lambda_2}{\lambda_1} \leq 3$ holds [21, 22]. Based on exact calculations for simple domains they also conjectured that

$$\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2(\mathbb{S}^1)}{\lambda_1(\mathbb{S}^1)} = \frac{j_{1,1}^2}{j_{0,1}^2} \approx 2.539, \tag{1.2}$$

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where, $\mathbb{S}^1 \subset \mathbb{R}^2$ is a circular disk, and $j_{n,m}$ denotes the m^{th} positive zero of the Bessel function $j_n(x)$. This conjecture and the corresponding inequalities in n-dimensions were proven in 1991 by Ashbaugh and Benguria [2, 3, 4].

Furthermore, when $M = \mathbb{R}^n, \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, Ashbaugh and Benguria [5] in 1993 proved

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n\left(1 + \frac{4}{n}\right). \tag{1.3}$$

In 2008, when M is one of complex projective spaces, unit spheres, and compact complex submanifolds of a complex projective space, by making use of the orthogonalization of Gram and Schmidt (QR-factorization theorem), Sun, Cheng and Yang [24] gave some universal inequalities similar to (1.3). For more results, we refer to [10, 11, 15, 16, 17, 25], etc.

Recently, many mathematicians (cf. [7, 8, 14, 18], etc.) study the eigenvalue problem of the sub-Laplacian Δ_b of the pseudo-Hermitian structure on a strictly pseudoconvex CR manifold. We know that the sub-Laplacian Δ_b is a hypoelliptic operator which has a discrete spectrum when M is a closed manifold [19]. In [1], Aribi and El Soufi consider a Dirichlet problem of Schrödinger-type operator on a bounded domain Ω in a strictly pseudoconvex CR manifold which is given by

$$\begin{cases} (-\Delta_b + V)\, u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where V is a nonnegative continuous function. They get some universal bounds for eigenvalues of the eigenvalue problem (1.4). In this paper, we will give some universal inequalities for lower order eigenvalues of the eigenvalue problem (1.4).

Assume that M^n is a compact Riemannian manifold immersed into Euclidean space $\mathbb{R}^m$. In [23], Reilly obtained the well-known estimate for the first nonzero eigenvalue μ_1 of the Laplacian as follows

$$\mu_1 \leq \frac{n}{Vol(M)} \int_M |\mathbf{H}|^2 dv,$$

where $\mathbf{H}$ is the mean curvature vector field of the immersion M^n in $\mathbb{R}^m$, with equality if and only if M is a round sphere in $\mathbb{R}^m$. For the first nonzero eigenvalue of the sub-Laplacian Δ_b on a compact strictly pseudoconvex CR manifold without boundary, Aribi and El Soufi [1] gave some Reilly-type inequalities. In this paper, for the first nonzero eigenvalue of the sub-Laplacian on a compact strictly pseudoconvex CR manifold without boundary, we will give some generalized Reilly-type inequalities.

2. Preliminaries

In this section, we firstly introduce some basic notions. For more details, we refer to [1, 6, 9, 12, 14, 18], etc.

A CR manifold is a smooth and oriented manifold M of real dimension $2n+1$, with a fixed n-dimensional complex subbundle $\mathcal{H}$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ satisfying $\mathcal{H} \cap \overline{\mathcal{H}} = 0$ and $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. If we let $H(M) = \mathrm{Re}\left(\mathcal{H} \oplus \overline{\mathcal{H}}\right)$, the real sub-bundle $H(M)$ is equipped with a formally integrable almost complex structure J such that, $\forall X, Y \in H(M)$

$$([JX, Y] + [X, JY]) \in H(M)$$

and the Nijenhuis tensor

$$N^J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0.$$

Let θ be a *pseudohermitian structure* on M, i.e., θ is a differential 1-form such that $H(M) = Ker\theta$. In other words, the hermitian bilinear form which is usually called *Levi form*

$$2G_\theta(X, Y) = -d\theta(JX, Y)$$

is non-degenerate. The integrability of J implies that G_θ is symmetric and J-invariant. The vector field ξ dual to θ with respect to G_θ satisfying $\theta(\xi) = 1$ and $\xi \neg d\theta = 0$ is called the *Reeb vector field*. Let g_θ be the *Webster metric* such that

$$g_\theta(X, Y) = G_\theta(X, Y), \;\; g_\theta(X, \xi) = 0, \;\; g_\theta(\xi, \xi) = 1$$

for any $X, Y \in H(M)$. A CR manifold (M, θ) is called *strictly pseudoconvex* if Levi form G_θ is a positive definite for a pseudohermitian structure θ. If a CR manifold (M, θ) is strictly pseudoconvex, then (M, g_θ) is a Riemannian manifold .

In all the sequel, a pair (M, θ) will be called a strictly pseudoconvex CR manifold if M is a strictly pseudoconvex CR manifold endowed with a compatible pseudo-Hermitian structure θ with positive definite Levi form. The pseudo-Hermitian structure θ is then a contact form which induces on M the following volume form

$$v_\theta = \frac{1}{2^n n!} \theta \wedge d\theta^n,$$

and the Riemannian volume form associated to g_θ coincides with v_θ. Let (M, θ) be a strictly pseudoconvex CR manifold, the *sub-Laplacian* is

$$\Delta_b f = \mathrm{div}(\nabla f), f \in C^2(M),$$

where $\mathrm{div}(X)$ is the divergence of the vector field X (with respect to the Riemannian metric g_θ) and $\nabla f = \pi_H(\widetilde{\nabla} f)$ is the *horizontal gradient*. Precisely, $\widetilde{\nabla} f$ is the ordinary gradient (i.e., $g_\theta(\widetilde{\nabla} f, X) = X(f)$ for any $X \in TM$) and $\pi_H : TM \to H(M)$ is the projection associated to the direct sum decomposition $TM = H(M) \oplus \mathbb{R}\xi$. Similarly, the *horizontal Hessian* $\nabla^2 f = \pi_H(\widetilde{\nabla}^2 f)$ can be defined. Let ∇ be the *Tanaka-Webster connection* of (M, θ), i.e., the unique linear connection on M obeying to i) $H(M)$ is ∇-parallel, ii) $\nabla g_\theta = 0$, $\nabla J = 0$, iii) the torsion T_∇ of ∇ satisfies $\forall X, Y \in H(M)$,

$$T_\nabla(X, Y) = -\theta([X, Y])\xi, \text{ and } T_\nabla(\xi, JX) = -JT_\nabla(\xi, X) \subset H(M).$$

Given a local G_θ-orthonormal frame $X_1, \dots, X_{2n}$ of $H(M)$, one has

$$\Delta_b u = \sum_{i=1}^{2n} \left(X_i^2 \cdot u - \nabla_{X_i} X_i \cdot u \right) = \sum_{i=1}^{2n} \langle \nabla_{X_i}(\nabla u), X_i \rangle_{G_\theta} .$$

As is known, the sub-Laplacian is a sub-elliptic operator of order $\frac{1}{2}$. This means that for any smooth compactly support function f and positive constant C, the following inequality holds

$$\|u\|_{\frac{1}{2}}^2 \leq C \left(\|(\Delta_b u, u)\| + \|u\| \right),$$

where $\|\cdot\|_{\frac{1}{2}}, \|\cdot\|$ stand for the Sobolev $\left(\frac{1}{2}\right)$-norm and the L^2-norm respectively. More generally, Δ_b satisfies a priori estimate

$$\|u\|_{s+1}^2 \leq C_s \left(\|(\Delta_b u)\|_s^2 + \|u\|^2 \right), \ C_s > 0, \ s \geq 0.$$

In the following, we will introduce a lemma which plays a key role in the proof of the main results of this paper.

Lemma 2.1. *Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n+1$, and let Ω be a bounded domain in M with smooth boundary. Let λ_i be the i^{th} eigenvalue of the following eigenvalue problem*

$$(-\Delta_b + V)u = \lambda u \ \text{in} \ \Omega, \ u = 0 \ \text{on} \ \partial\Omega,$$

and u_i be the orthonormal eigenfunction corresponding to λ_i, that is,

$$\begin{cases} (-\Delta_b + V)u_i = \lambda_i u_i, & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial\Omega, \\ \int_\Omega u_i u_j = \delta_{ij}, & \forall \ i, j = 1, 2, \dots, \end{cases}$$

if $g_i \in C^4(\Omega) \cap C^3(\partial\Omega)$ satisfies $\int_\Omega g_i u_1 u_j = 0$ for $j = 2, \dots, i$, then for any positive integer i, we have

$$(\lambda_{i+1} - \lambda_1) \int_\Omega u_1^2 \langle \nabla g_i, \nabla g_i \rangle_{G_\theta} \leq \int_\Omega \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta} \right)^2 . \tag{2.1}$$

Proof. Defining

$$\psi_i = (g_i - a_i) u_1, \tag{2.2}$$

where $a_i = \int_\Omega g_i u_1^2$, we have $\int_\Omega \psi_i u_1 = 0$. Noticing $\int_\Omega g_i u_1 u_j = 0$, we infer

$$\int_\Omega \psi_i u_j = 0 \text{ for any } j \leq i, \ \text{and } \psi_i\big|_\Omega = 0.$$

From the Rayleigh-Ritz inequality, we have

$$\begin{aligned} \lambda_{i+1} \int_\Omega \psi_i^2 &\leq \int_\Omega \psi_i (-\Delta_b + V) \psi_i \\ &= \int_\Omega \psi_i (\lambda_1 g_i u_1 - (u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta})) \end{aligned}$$

$$= \lambda_1 \int_\Omega \psi_i^2 - \int_\Omega \psi_i \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right),$$

which implies

$$(2.3) \qquad (\lambda_{i+1} - \lambda_1) \int_\Omega \psi_i^2 \le - \int_\Omega \psi_i \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right) =: \omega_i.$$

Using integration by parts, we have

$$(2.4) \qquad \begin{aligned} \omega_i = & - \int_\Omega \psi_i \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right) \\ = & - \int_\Omega g_i u_1 \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right) \\ & + \int_\Omega a_i u_1 \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right) \\ = & \int_\Omega u_1^2 \langle \nabla g_i, \nabla g_i \rangle_{G_\theta}. \end{aligned}$$

On the other hand, by the Schwarz inequality and (2.3), we have

$$\begin{aligned} (\lambda_{i+1} - \lambda_1)\omega_i^2 &= (\lambda_{i+1} - \lambda_1) \left\{ \int_\Omega \psi_i \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right) \right\}^2 \\ &\le (\lambda_{i+1} - \lambda_1) \int_\Omega \psi_i^2 \int_\Omega \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right)^2 \\ &\le \omega_i \int_\Omega \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right)^2, \end{aligned}$$

which implies

$$(2.5) \qquad (\lambda_{i+1} - \lambda_1)\omega_i \le \int_\Omega \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right)^2.$$

By (2.4) and (2.5), we have

$$(\lambda_{i+1} - \lambda_1) \int_\Omega u_1^2 \langle \nabla g_i, \nabla g_i \rangle_{G_\theta} \le \int_\Omega \left(u_1 \Delta_b g_i + 2 \langle \nabla g_i, \nabla u_1 \rangle_{G_\theta}\right)^2,$$

and this completes the proof of Lemma 2.1. □

3. Universal inequalities of the Schrödinger-type operator on a bounded domain in strictly pseudoconvex CR manifolds

In this section, we will give the proofs of our main results. Firstly, we introduce some results of the semi-isometric map (for more details, we refer to [1, 9, 13, 14]).

3.1. Some results of semi-isometric map

Let ∇^{g_θ} be the Levi-Civita connection of the Riemannian manifold (M, g_θ), which is related to the Tanaka-Webster connection ∇ by the following identities $\forall X, Y \in H(M)$, $\nabla_X Y = (\nabla^{g_\theta}_X Y)^H$ and, moreover,

$$\nabla^{g_\theta}_\xi X - \nabla_\xi X = \frac{1}{2}JX,\ \nabla^{g_\theta}_X \xi - \nabla_X \xi = \nabla^{g_\theta}_X \xi = (\frac{1}{2}J + \tau)X,$$
$$\nabla^{g_\theta}_X Y - \nabla_X Y = -\left\langle (\frac{1}{2}J + \tau)X, Y \right\rangle_{g_\theta} \xi \text{ and } \nabla^{g_\theta}_\xi \xi = \nabla_\xi \xi = 0,$$

where $\tau : H(M) \to H(M)$ is the traceless symmetric (1,1)-tensor defined by $\tau X = T_\nabla(\xi, X) = \nabla_\xi X - [\xi, X]$. Notice that $\tau = 0$ if and only if ξ is a Killing vector field with respect to the metric g_θ (and then the metric g_θ is a Sasakian metric on M). Known as Greenleaf's formula [14]:

$$\Delta_b = \Delta_{g_\theta} - \xi^2,$$

where Δ_{g_θ} is the Laplace-Beltrami operator of (M, g_θ).

Let (M, θ) be a strictly pseudoconvex CR manifold of real dimension $2n+1$ and let (N, h) be a Riemannian manifold. The energy density of a smooth map $f : (M, \theta) \to (N, h)$ with respect to horizontal directions is defined at a point $x \in M$ by

$$e_b(f)_x = \frac{1}{2} trace_{G_\theta}(\pi H f^* h)_x = \frac{1}{2} \sum_{i=1}^{2n} |df(X_i)|_h^2,$$

where $X_1, \ldots, X_{2n}$ is a local G_θ-orthonormal frame of $H(M)$. According to [9], the first variation of the energy functional

$$E_b(f) = \int_M e_b(f) v_\theta$$

is determined by the vector, which we will call 'Levi tension' of f

$$H_b(f) = trace_{G_\theta} \beta_f,$$

where β_f is the vector valued 2-form on H given by

$$\beta_f(X, Y) = \nabla^f_X df(Y) - df(\nabla_X Y).$$

Here, $\nabla^f$is the connection induced on the bundle $f^{-1}TN$ by the Levi-Civita connection of (N, h), and ∇ is the Tanaka-Webster connection of (M, θ). Therefore,

$$H_b(f) = \sum_{i=1}^{2n} \nabla^f_{X_i} df(X_i) - df(\nabla_{X_i} X_i).$$

Mappings with $H_b(f) = 0$ are called *pseudo-harmonic* [9]. In the case where (N, h) is the standard Euclidean space $\mathbb{R}^m$, it is clear that

$$H_b(f) = (\Delta_b f_1, \ldots, \Delta_b f_m). \tag{3.1}$$

Since $\nabla^{g_\theta}_X Y - \nabla_X Y = -\left\langle (\frac{1}{2}J+\tau)X, Y\right\rangle_{g_\theta} \xi$ for every pair (X,Y) of horizontal vector fields, one has

$$\beta_f(X,Y) = B_f(X,Y) + \left\langle (\frac{1}{2}J+\tau)X, Y\right\rangle_{g_\theta} df(\xi)$$

and

$$H_b(f) = H(f) - B_f(\xi,\xi) = H(f) - \nabla^f_\xi df(\xi),$$

where $B_f(X,Y) = \nabla^f_X df(Y) - df(\nabla^f_X Y)$ and $H(f) = trace_{g_\theta} B_f$ is the *tension vector field* (see [13]). In the particular case where f is an isometric immersion from (M, g_θ) to (N,h), B_f coincides with the second fundamental form of f and $H(f)$ coincides with its mean curvature vector.

For the natural inclusion $j : S^{2n+1} \to C^{n+1}$ of S^{2n+1}, the form β_j is given by $\beta_j(X,Y) = \langle X,Y\rangle_{C_{n+1}} \overrightarrow{x} + \langle JX,Y\rangle_{C_{n+1}} J\overrightarrow{x}$, where $\overrightarrow{x}$ is the position vector field (here $\nu(\overrightarrow{x}) = -\overrightarrow{x}$ and $\xi(\overrightarrow{x}) = 2J(\overrightarrow{x})$). Thus,

$$H_b(j) = -2n\overrightarrow{x}. \tag{3.2}$$

In the sequel, we will focus on maps $f : (M,\theta) \to (N,h)$ that preserve lengths in the horizontal directions as well as the orthogonality between $H(M)$ and ξ, that is, $\forall X \in H(M)$,

$$|df(X)|_h = |X|_{G_\theta} \text{ and } \langle df(X), df(\xi)\rangle_h = 0,$$

which also amount to $f^*h = g_\theta + (\mu-1)\theta^2$ for some nonnegative function μ on M. For convenience, such a map will be termed *semi-isometric*. Notice that the dimension of the target manifold N should be at least $2n$. When the dimension of N is $2n$, then a semi-isometric map $f : (M,\theta) \to (N,h)$ is nothing but a Riemannian submersion satisfying $df(\xi) = 0$. Important examples are given by the standard projection from the Heisenberg group $\mathbb{H}^n$ to $\mathbb{R}^{2n}$ and the Hopf fibration $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$. In the following, we introduce two lemmas which are given by Aribi and El Soufi [1].

Lemma 3.1 ([1])**.** *Let (M,θ) be a strictly pseudoconvex CR manifold and let (N,h) be a Riemannian manifold. If $f : (M,\theta) \to (N,h)$ is a C^2 semi-isometric map, then the form β_f takes its values in the orthogonal complement of $df(H)$. In particular, the vector $H_b(f)$ is orthogonal to $df(H)$.*

Lemma 3.2 ([1])**.** *If $f : (M,\theta) \to (N,h)$ is a Riemannian submersion from a strictly pseudoconvex CR manifold (M,θ) to a Riemannian manifold (N,h) with $df(\xi) = 0$, then $\beta_f = 0$ and $H_b(f) = 0$.*

3.2. Main results

When $f : (M,\theta) \to \mathbb{R}^m$ is a semi-isometric map, by Lemma 2.1 and the results of the semi-isometric map, we can obtain:

Theorem 3.1. *Let Ω be a bounded domain in a strictly pseudoconvex CR manifold (M,θ) of real dimension $2n+1$, and let $f : (M,\theta) \to \mathbb{R}^m (m \geq 2n)$*

be a C^4 semi-isometric map. Let λ_i be the i^{th} eigenvalue of the eigenvalue problem

$$\begin{cases} (-\Delta_b + V)u = \lambda u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{3.3}$$

Then, we have

$$\sum_{i=1}^{n} \lambda_{i+1} \leq \frac{2n+1}{2}\lambda_1 + \frac{1}{2}\int_\Omega \left(|H(f)|^2_{\mathbb{R}^m} - 4V\right) u_1^2, \tag{3.4}$$

where $H(f)$ is the Levi tension vector field of f. If $D_0 = \sup_\Omega\{|H(f)|^2_{\mathbb{R}^m} - 4V\}$, we have

$$\sum_{i=1}^{n} \lambda_{i+1} \leq \frac{2n+1}{2}\lambda_1 + \frac{1}{2}D_0. \tag{3.5}$$

Proof. Let $f : (M, \theta) \to \mathbb{R}^m$ be a semi-isometric map and let $f_\alpha, \alpha = 1, \ldots, m$ be its Euclidean components. Now, since f preserves the Levi-form, then $\forall p \in M$, we have a G_θ-orthonormal frame field $\{e_i\}_{i=1}^{2n}$ of $H_p(M)$ such that

$$\begin{aligned} \sum_{\alpha=1}^{m} |\nabla f_\alpha|^2_{G_\theta} &= \sum_{\alpha=1}^{m}\sum_{i=1}^{2n} \langle \nabla f, e_i\rangle^2_{G_\theta} = \sum_{i=1}^{2n}\sum_{\alpha=1}^{m} \langle \nabla f, e_i\rangle^2_{G_\theta} \\ &= \sum_{i=1}^{2n} |df(e_i)|^2_{\mathbb{R}^m} = \sum_{i=1}^{2n} |e_i|^2_{G_\theta} = 2n. \end{aligned} \tag{3.6}$$

Using the isometry property of f with respect to horizontal directions, we get

$$\begin{aligned} \sum_{\alpha=1}^{m} \langle \nabla f_\alpha, \nabla u_i\rangle^2_{G_\theta} &= \sum_{\alpha=1}^{m} \langle \nabla f_\alpha, \nabla u_i\rangle^2_{G_\theta} = \sum_{\alpha=1}^{m} |df_\alpha(\nabla u_i)|^2_{\mathbb{R}^m} \\ &= |df_\alpha(\nabla u_i)|^2_{\mathbb{R}^m} = |\nabla u_i|^2_{G_\theta}. \end{aligned} \tag{3.7}$$

Using (3.1) we have

$$\sum_{\alpha=1}^{m} (\Delta_b f_\alpha)^2 = |H(f)|^2_{\mathbb{R}^m}. \tag{3.8}$$

Denote by $\{E_\alpha\}_{\alpha=1}^{m}$ the standard basis of $\mathbb{R}^m$ and using Lemma 3.1, we have

$$\begin{aligned} &\sum_{\alpha=1}^{m} \Delta_b f_\alpha \left\langle \nabla f_\alpha, \nabla u_i^2\right\rangle_{G_\theta} \\ &= \left\langle \sum_{\alpha=1}^{m} \Delta_b f_\alpha E_\alpha, \sum_{\alpha=1}^{m} \left\langle \nabla f_\alpha, \nabla u_i^2\right\rangle_{G_\theta} E_\alpha \right\rangle_{\mathbb{R}^m} = \left\langle H(f), df(\nabla u_i^2)\right\rangle_{\mathbb{R}^m} = 0. \end{aligned} \tag{3.9}$$

Define a $m \times m$-matrix B as follows

$$B := (b_{\alpha\beta}),$$

where $b_{\alpha\beta} = \int_\Omega f_\alpha u_1 u_{\beta+1}$. Using the orthogonalization of Gram and Schmidt, we know that there exist an upper triangle matrix $R = (R_{\alpha\beta})$ and an orthogonal matrix $Q = (q_{\alpha\beta})$ such that $R = QB$, i.e.,

$$R_{\alpha\beta} = \sum_{\gamma=1}^{2n} q_{\alpha\gamma} b_{\gamma\beta} = \int_\Omega \sum_{\gamma=1}^{2n} q_{\alpha\gamma} f_\gamma u_1 u_{\beta+1} = 0, \ \alpha < \beta. \tag{3.10}$$

Defining $g_\alpha = \sum_{\gamma=1}^{2n} q_{\alpha\gamma} f_\gamma$, we have $\int_\Omega g_\alpha u_1 u_\beta = 0$, where $\alpha = 1, \dots, 2n$ and $\alpha < \beta$. So, we infer from (2.1) that

$$\begin{aligned}(\lambda_{\alpha+1} - \lambda_1) &\int_\Omega u_1^2 \left\langle \nabla g_\alpha, \nabla g_\alpha \right\rangle_{G_\theta} \\ \leq &\int_\Omega \left(u_1 \Delta_b g_\alpha + 2 \left\langle \nabla g_\alpha, \nabla u_1 \right\rangle_{G_\theta} \right)^2 .\end{aligned} \tag{3.11}$$

Since $g_\alpha = \sum_{\gamma=1}^{2n} q_{\alpha\gamma} f_\gamma$ and Q is an orthogonal matrix, by (3.6)-(3.9), we have

$$\sum_{\alpha=1}^{m} |\nabla g_\alpha|^2_{G_\theta} = 2n, \qquad \sum_{\alpha=1}^{m} \left\langle \nabla g_\alpha, \nabla u_1 \right\rangle^2_{G_\theta} = |\nabla u_1|^2_{G_\theta};$$

$$\sum_{i=\alpha}^{m} (\Delta_b g_\alpha)^2 = |H(f)|^2_{\mathbb{R}^m}, \qquad \sum_{i=1}^{m} \Delta_b g_\alpha \left\langle \nabla g_\alpha, \nabla u_1^2 \right\rangle_{G_\theta} = 0.$$

Hence, we can get

$$\begin{aligned}&\sum_{\alpha=1}^{m} \int_\Omega \left(u_1 \Delta_b g_\alpha + 2 \left\langle \nabla g_\alpha, \nabla u_1 \right\rangle_{G_\theta} \right)^2 \\ &= \sum_{\alpha=1}^{m} \int_\Omega \left(u_1^2 (\Delta_b g_\alpha)^2 + 4 u_1 \Delta_b g_\alpha \left\langle \nabla g_\alpha, \nabla u_1 \right\rangle_{G_\theta} + 4 \left\langle \nabla g_\alpha, \nabla u_1 \right\rangle^2_{G_\theta} \right) \\ &= \int_\Omega \left(|H(f)|^2_{\mathbb{R}^m} u_1^2 + 4 |\nabla u_1|^2_{G_\theta} \right) \\ &= \lambda_1 + \int_\Omega \left(|H(f)|^2_{\mathbb{R}^m} - 4V \right) u_1^2.\end{aligned} \tag{3.12}$$

On the other hand, since f preserves the Levi-form, then $\forall p \in M$, by a transformation of G_θ-orthonormal frame if necessary, for any α, we have

$$|\nabla g_\alpha|^2_{G_\theta} \leq 1.$$

From $\sum_{\alpha=1}^{m} |\nabla g_\alpha|^2_{G_\theta} = 2n$ and $m \geq 2n$, we infer that

$$\begin{aligned}&\sum_{\alpha=1}^{m} (\lambda_{\alpha+1} - \lambda_1) |\nabla g_\alpha|^2_{G_\theta} \\ &\geq \sum_{i=1}^{n} (\lambda_{i+1} - \lambda_1) |\nabla g_i|^2_{G_\theta} + (\lambda_{n+1} - \lambda_1) \sum_{A=n+1}^{2n} |\nabla g_A|^2_{G_\theta}\end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}(\lambda_{i+1}-\lambda_1)|\nabla g_i|^2_{G_\theta} + (\lambda_{n+1}-\lambda_1)\left(2n - \sum_{j=1}^{n}|\nabla g_j|^2_{G_\theta}\right)\\
&= \sum_{i=1}^{n}(\lambda_{i+1}-\lambda_1)|\nabla g_i|^2_{G_\theta} + (\lambda_{n+1}-\lambda_1)\sum_{j=1}^{n}\left(2 - |\nabla g_j|^2_{G_\theta}\right)\\
&\geq \sum_{i=1}^{n}(\lambda_{i+1}-\lambda_1)|\nabla g_i|^2_{G_\theta} + \sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1)\left(2 - |\nabla g_j|^2_{G_\theta}\right)\\
&= 2\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1).
\end{aligned}
$$

For (3.11), summing over α from 1 to m, and using (3.12)-(3.13), we have

$$
2\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_1) \leq \lambda_1 + \int_\Omega \left(|H(f)|^2_{\mathbb{R}^m} - 4V\right) u_1^2.
$$

This completes the proof of Theorem 3.1. □

Applying this result to the standard CR sphere whose standard embedding $j : S^{2n+1} \to C^{n+1}$ satisfies $|H_b(j)|^2_{\mathbb{C}^{n+1}} = 4n^2$ (see (3.2)), by Theorem 3.1, we get the following.

Corollary 3.1. *Let Ω be a domain in the standard CR sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. If λ_i is the i^{th} eigenvalue of the eigenvalue problem* (3.3), *then we have*

$$
\sum_{i=1}^{n}\lambda_{i+1} \leq \frac{2n+1}{2}\lambda_1 + 2\left(n^2 - V_0\right), \tag{3.14}
$$

where $V_0 = \min_{x\in\Omega} V(x)$.

If the standard projection $f : \mathbb{H}^n \to \mathbb{R}^{2n}$ is a semi-isometric with zero Levi-tension, that is $H(f) = 0$, by Theorem 3.1, we get the following.

Corollary 3.2. *Let Ω be a domain in the Heisenberg group $\mathbb{H}^n$. If λ_i is the i^{th} eigenvalue of the eigenvalue problem* (3.3), *then we have*

$$
\sum_{i=1}^{n}\lambda_{i+1} \leq \frac{2n+1}{2}\lambda_1 - 2V_0, \tag{3.15}
$$

where $V_0 = \min_{x\in\Omega} V(x)$.

Theorem 3.2. *Let Ω be a bounded domain in a strictly pseudoconvex CR manifold (M,θ) of real dimension $2n+1$ and let $f : (M,\theta) \to N$ be a Riemannian submersion over a complete Riemannian manifold of dimension $2n$ such that $df(\xi) = 0$. Let λ_i be the i^{th} eigenvalue of the eigenvalue problem*

$$
\begin{cases}
(-\Delta_b + V)u = \lambda u, & in\ \Omega,\\
u = 0, & on\ \partial\Omega.
\end{cases}
$$

Then we have

$$\sum_{i=1}^{n}\lambda_{i+1} \le \frac{2n+1}{2}\lambda_1 + \frac{1}{2}\int_\Omega \left(|H(\phi)|^2 - 4V\right) u_1^2, \tag{3.16}$$

where $H(\phi)$ is the mean curvature vector of ϕ. Let $\Phi := \{\phi \,|\, \phi$ is an isometric immersion from N into a Euclidean space $\mathbb{R}^m\}$ and $D_1 = \inf_{\phi\in\Phi}\sup_\Omega\{|H(\phi)|^2 - 4V\}$. Then we have

$$\sum_{i=1}^{n}\lambda_{i+1} \le \frac{2n+1}{2}\lambda_1 + \frac{1}{2}D_1. \tag{3.17}$$

Proof. By the Nash's theorem [20], we know that each complete Riemannian manifold can be isometrically immersed into a Euclidean space. Let $\phi : N \to \mathbb{R}^m$ be an isometric immersion, it is straightforward to check that the map $\widehat{f} = \phi \circ f : (M,\theta) \to \mathbb{R}^m$ is semi-isometric and that, $\forall X, Y \in H$,

$$\beta_{\widehat{f}}(X,Y) = d\phi(\beta_f(X,Y)) + B_\phi(df(X), df(Y)), \tag{3.18}$$

where B_ϕ is the second fundamental form of ϕ. Now, by the assumptions on f, the differential of f induces, $\forall p \in M$, an isometry between $H_p(M)$ and $T_{f(p)}N$. Thus, if $\{X_1, \dots, X_{2n}\}$ is a local orthonormal frame field of $H(M)$, then $\{df(X_1), \dots, df(X_{2n})\}$ is also an orthonormal frame field of TN. This leads to the equality

$$H_b(\widehat{f}) = H(\phi).$$

Since $\phi : N \to \mathbb{R}^m$ is an isometric immersion, we know that $H(\phi)$ is the mean curvature vector of ϕ. Hence, by Theorem 1.1, we have (3.16). This completes the proof of Theorem 3.2. □

If N is the minimal submanifold of $\mathbb{R}^m$, we know that $H(\phi) = 0$. Then, by Theorem 3.2, we get the following.

Corollary 3.3. *Under the assumption of Theorem 1.2, if N is a minimal submanifold of the Euclidean space $\mathbb{R}^m$, we have*

$$\sum_{i=1}^{n}\lambda_{i+1} \le \frac{2n+1}{2}\lambda_1 - 2V_1, \tag{3.19}$$

where $V_1 = \min_{x\in\Omega} V(x)$.

If N is a unit sphere $\mathbb{S}^{2n}$, we know that $H(j) = 4n^2$. Then, by Theorem 3.2, we get the following.

Corollary 3.4. *Under the assumption of Theorem 3.2, if N is a unit sphere $\mathbb{S}^{2n}$, we have*

$$\sum_{i=1}^{n}\lambda_{i+1} \le \frac{2n+1}{2}\lambda_1 + 2\left(n^2 - V_1\right), \tag{3.20}$$

where $V_1 = \min_{x\in\Omega} V(x)$.

Corollary 3.5. *Under the assumption of Theorem 3.2, if N is a projective space $\mathbb{F}P^m$ of real dimension $2n$, we have*

$$\sum_{i=1}^{n} \lambda_{i+1} \leq \frac{2n+1}{2}\lambda_1 + 2\left(n(2n+d_{\mathbb{F}}) - V_1\right), \tag{3.21}$$

where $d_{\mathbb{F}} = \dim_{\mathbb{R}}\mathbb{F}$ and $V_1 = \min_{x\in\Omega} V(x)$.

Proof. We denote by $\mathbb{F}P^m$ the m-dimensional real projective space if $\mathbb{F} = \mathbb{R}$, the complex projective space of real dimension $2m$ if $\mathbb{F} = \mathbb{C}$, and the quaternionic projective space of real dimension $4m$ if $\mathbb{F} = \mathcal{Q}$. The manifold $\mathbb{F}P^m$ carries a natural metric so that the Hopf fibration $\pi : \mathbb{S}^{d_F(m+1)-1} \subset \mathbb{F}^{m+1} \to \mathbb{C}P^m$ is a Riemannian fibration, where

$$d_F = \dim_{\mathbb{R}}\mathbb{F} = \begin{cases} 1, & \text{if } \mathbb{F} = \mathbb{R} \\ 2, & \text{if } \mathbb{F} = \mathbb{C} \\ 4, & \text{if } \mathbb{F} = \mathcal{Q}. \end{cases}$$

Let $\mathcal{H}_{m+1}(\mathbb{F}) = \{A \in \mathcal{M}_{m+1}(\mathbb{F}) | A^* := {}^t\overline{A} = A\}$ be the vector space of $(m+1) \times (m+1)$ Hermitian matrices with coefficients in $\mathbb{F}$, which we endow with the inner product

$$\langle A, B\rangle = \frac{1}{2} trace(AB).$$

The map $\psi : \mathbb{S}^{d_F(m+1)-1} \subset \mathbb{F}^{m+1} \to \mathcal{H}_{m+1}(\mathbb{F})$ given by

$$\psi = \begin{pmatrix} |z_0|^2 & z_0\overline{z_1} & \cdots & z_0\overline{z_m} \\ z_1\overline{z_0} & |z_1|^2 & \cdots & z_1\overline{z_m} \\ \cdots & \cdots & \cdots & \cdots \\ z_m\overline{z_0} & z_m\overline{z_1} & \cdots & |z_m|^2 \end{pmatrix}$$

induces through the Hopf fibration an isometric embedding ϕ from $\mathbb{F}P^m$ into $\mathcal{H}_{m+1}(\mathbb{F})$. Moreover, $\phi(\mathbb{F}P^m)$ is a minimal submanifold of the hypersphere $\mathbb{S}(\frac{I}{m+1}, \sqrt{\frac{m}{2(m+1)}})$ of $\mathcal{H}_{m+1}(\mathbb{F})$ of radius $\sqrt{\frac{m}{2(m+1)}}$ centered at $\frac{I}{m+1}$. One deduces that the mean curvature $H(\phi)$ satisfies

$$|H(\phi)|^2 = 2m(m+1)d_{\mathbb{F}}^2.$$

Therefore, by $m = \frac{2n}{d_{\mathbb{F}}}$, Theorem 3.2 leads to (3.21). This completes the proof of Corollary 3.5. □

It is well known that the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is the space $\mathbb{R}^{2n+1}$ equipped with the non-commutative group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(\langle x', y\rangle_{\mathbb{R}^n} - \langle x, y'\rangle_{\mathbb{R}^n})),$$

where $x, y, x', y' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}$, and $\langle , \rangle_{\mathbb{R}^n}$ denotes the inner product in $\mathbb{R}^n$. The Lie algebra $\mathcal{H}^n$ of $\mathbb{H}^n$ has a basis formed by the following the vector fields

$$X_p = \frac{\partial}{\partial x_p} - \frac{y_p}{2}\frac{\partial}{\partial t},\ Y_p = \frac{\partial}{\partial y_p} - \frac{x_p}{2}\frac{\partial}{\partial t},\ T = \frac{\partial}{\partial t},\ p = 1, 2, \dots, n.$$

We note that the only nontrivial commutators are $[Y_p, X_q] = -T\delta_{pq}$. The Heisenberg group is also an important example of the strictly pseudoconvex CR manifolds. When a strictly pseudoconvex CR manifold (M, θ) is a semi-isometric map into a Heisenberg group, we have:

Theorem 3.3. *Let Ω be a bounded domain in a strictly pseudoconvex CR manifold (M, θ) of real dimension $2n+1$ and $\mathbb{H}^m$ be a Heisenberg group. Let $f : (M, \theta) \to \mathbb{H}^m$ be a C^4 semi-isometric map satisfying $df(H(M)) \subseteq H(\mathbb{H}^m)$. Let λ_i be the i^{th} eigenvalue of the eigenvalue problem*

$$\begin{cases} (-\Delta_b + V)u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We have

$$\sum_{i=1}^{n} \lambda_{i+1} \leq \frac{2n+1}{2}\lambda_1 + \frac{1}{2}\int_\Omega \left(|H(f)|^2_{\mathbb{H}^m} - 4V\right) u_1^2, \tag{3.22}$$

where $H(f)$ is a horizontal vector field of f. If $D_2 = \sup_\Omega\{|H(f)|^2_{\mathbb{H}^m} - 4V\}$, we have

$$\sum_{i=1}^{n} \lambda_{i+1} \leq \frac{2n+1}{2}\lambda_1 + \frac{1}{2}D_2. \tag{3.23}$$

Proof. Let

$$\begin{aligned} f :\quad & (M, \theta) \to \mathbb{H}^m \simeq \mathbb{C}^m \\ & x \to f(x) = (F_1(x), \dots, F_m(x), \alpha(x)), \end{aligned}$$

and let

$$\varphi_i(x) = \mathrm{Re}(F_j(x)),\ \psi_j(x) = \mathrm{Im}(F_j(x)).$$

By Proposition 5.1 in [1], we have

$$\sum_{j=1}^{m} \left(|\nabla\varphi_j|^2_{G_\theta} + |\nabla\psi_j|^2_{G_\theta}\right) = \frac{n}{2}, \tag{3.24}$$

$$\sum_{j=1}^{m} \left(\langle\nabla\varphi_j, \nabla u_i\rangle^2_{G_\theta}\right) + \sum_{j=1}^{m} \left(\langle\nabla\psi_j, \nabla u_i\rangle^2_{G_\theta}\right) = \frac{1}{4}|\nabla u_i|^2_{G_\theta}, \tag{3.25}$$

$$\sum_{j=1}^{m} \left((\Delta_b\varphi_j)^2 + (\Delta_b\psi_j)^2\right) = \frac{1}{4}|H_b(f)|^2_{\mathbb{H}^m}, \tag{3.26}$$

$$\sum_{j=1}^{m} \left(\Delta_b\varphi_j \left\langle\nabla\varphi_j, \nabla u_i^2\right\rangle_{G_\theta} + \Delta_b\varphi_j \left\langle\nabla\psi_j, \nabla u_i^2\right\rangle_{G_\theta}\right) = 0. \tag{3.27}$$

By a similar computation as in the proof of Theorem 3.1, we can get (3.22). □

4. Reilly-type inequalities of the sub-Laplacian on compact strictly pseudoconvex CR manifolds

In this section, we will give some generalized Reilly-type inequalities for eigenvalues of the sub-Laplacian on a compact strictly pseudoconvex CR manifold without boundary. Actually, we have the following.

Theorem 4.1. *Let (M,θ) be a compact strictly pseudoconvex CR manifold of real dimension $2n+1$ without boundary, and let λ_i be the i^{th} eigenvalue of the closed eigenvalue problem*

$$\Delta_b u = -\lambda u \quad in\ M.$$

If $f:(M,\theta)\to\mathbb{R}^m (m\geq 2n)$ is a C^2 semi-isometric map, we can get

$$\sum_{i=1}^{n}\lambda_{i+1}\leq\frac{\int_M |H_b(f)|^2_{\mathbb{R}^m}}{2V(M,\theta)}, \tag{4.1}$$

where the equality holds if and only if $f(M)$ is contained in a sphere $\mathbb{S}^{m-1}(r)$ of $r=\sqrt{2n}/\lambda_2^{\frac{1}{2}}$ and f is a pseudo-harmonic map from (M,θ) to the sphere $\mathbb{S}^{m-1}(r)$.

Proof. By Theorem 3.1, we have

$$\sum_{i=1}^{n}\lambda_{i+1}\leq\frac{2n+1}{2}\lambda_1+\frac{1}{2}\int_\Omega\left(|H(f)|^2_{\mathbb{R}^m}-4V\right)u_1^2. \tag{4.2}$$

Since (M,θ) is a compact manifold without boundary, we have $\lambda_1=0$ and $u_1^2=\frac{1}{V(M,\theta)}$. Together with (4.2), we can get

$$\sum_{i=1}^{n}\lambda_{i+1}\leq\frac{\int_M |H_b(f)|^2_{\mathbb{R}^m}}{2V(M,\theta)}. \tag{4.3}$$

By Corollary 6.1 in [1], we know that

$$\lambda_2\leq\frac{\int_M |H_b(f)|^2_{\mathbb{R}^m}}{2nV(M,\theta)}, \tag{4.4}$$

where the equality holds if and only if $f(M)$ is contained in a sphere $\mathbb{S}^{m-1}(r)$ of $r=\sqrt{2n}/\lambda_2^{\frac{1}{2}}$ and f is a pseudo-harmonic map from (M,θ) to the sphere $\mathbb{S}^{m-1}(r)$.

From (4.3) and (4.4), we know that

$$\lambda_2\leq\frac{1}{n}\sum_{i=1}^{n}\lambda_{i+1}\leq\frac{\int_M |H_b(f)|^2_{\mathbb{R}^m}}{2nV(M,\theta)}. \tag{4.5}$$

Thus, if equality holds in (4.4), then equality holds in (4.3). On the other hand, if the equality holds in (4.3), then the equality holds in (3.13). Thus, we can get

$$\lambda_2 = \lambda_3 = \cdots = \lambda_{n+1},$$

which implies the equality holds in (4.5). Hence, Theorem 4.1 is true. □

Theorem 4.2. *Let (M, θ) be a compact strictly pseudoconvex CR manifold of real dimension $2n+1$ without boundary and $\mathbb{H}^m$ be a Heisenberg group. Let λ_i be the i^{th} eigenvalue of the closed eigenvalue problem*

$$\Delta_b u = -\lambda u \quad \text{in } M.$$

If $f : (M, \theta) \to \mathbb{H}^m (m \geq 2n)$ is a C^2 semi-isometric map, we can get

$$\sum_{i=1}^{n} \lambda_{i+1} \leq \frac{\int_M |H_b(f)|^2_{\mathbb{H}^m}}{2V(M, \theta)}, \tag{4.6}$$

where the equality holds if and only if $f(M)$ is contained in a product $\mathbb{S}^{2m-1}(r) \times \mathbb{R} \subset \mathbb{H}^m$ with $r = \sqrt{2n}/\lambda_2^{\frac{1}{2}}$ and $\varphi \circ f$ is a pseudo-harmonic map from (M, θ) to the sphere $\mathbb{S}^{m-1}(r)$, with $\varphi : \mathbb{H}^m \to \mathbb{R}^{2m}$ the standard projection.

Proof. Let $\varphi : \mathbb{H}^m \to \mathbb{R}^{2m}$ be the standard projection. Since f is a semi-isometric map and $df(H(M)) \subseteq H(\mathbb{H}^m)$, we know that $\hat{f} = f \circ \phi : (M, \theta) \to \mathbb{R}^{2m}$ is a semi-isometric map. By Proposition 5.1 in [1], we have

$$|H_b(\hat{f})|^2_{\mathbb{R}^{2m}} = \frac{1}{4}|H_b(f)|^2_{\mathbb{H}^m}, \text{ and } E_b(\hat{f}) = \frac{1}{4}E_b(f).$$

Hence, together with Theorem 4.1, we can obtain Theorem 4.2. □

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