Bull. Korean Math. Soc. **52** (2015), No. 1, pp. 201–213
http://dx.doi.org/10.4134/BKMS.2015.52.1.201

LIGHTLIKE HYPERSURFACES OF AN INDEFINITE KAEHLER MANIFOLD WITH A QUARTER-SYMMETRIC METRIC CONNECTION

Dae Ho Jin

Abstract. In this paper, we study lightlike hypersurfaces of an indefinite Kaehler manifold with a quarter-symmetric metric connection. We prove several classification theorems for such a lightlike hypersurface.

1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *quarter-symmetric connection* if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X, Y) = \pi(Y)JX - \pi(X)JY, \tag{1.1}$$

for any vector fields X and Y on $\bar{M}$, where J is a $(1,1)$-type tensor field and π is a 1-form associated with a non-vanishing smooth vector field ζ, which is called the *torsion vector field* of $\bar{M}$, by $\pi(X) = \bar{g}(X, \zeta)$. Moreover, if $\bar{\nabla}$ satisfies $\bar{\nabla}\bar{g} = 0$, then it is called a *quarter-symmetric metric connection.*

Quarter-symmetric metric connection was introduced by K. Yano and T. Imai [15], and then it have been studied by S. C. Rastogi [13, 14], D. Kamilya and U. C. De [8], R. S. Mishra and S. N. Pandey [9], S. Golab [7] and others. On the other hand, N. Pušić [12], and J. Nikić and Pušić [10] studied quarter-symmetric metric connections on Kaehler manifold.

The theory of lightlike hypersurfaces is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by Duggal and Bejancu [3] and later studied by many authors (see recent results in two books [4, 6]). Although now we have lightlike version of a large variety of Riemannian submanifolds, the geometry of lightlike hypersurfaces of semi-Riemannian manifolds with quarter-symmetric metric connections is hardly known.

In this paper, we study lightlike hypersurfaces of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, J)$ with a quarter-symmetric metric connection, in which the tensor

Received December 16, 2013; Revised February 27, 2014.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C40, 53C50.
Key words and phrases. quarter-symmetric connection, metric connection, lightlike hypersurface.

field J of (1.1) is identical with the indefinite almost complex structure J of $\bar{M}$. We prove several classification theorems for such a lightlike hypersurface.

2. Lightlike hypersurfaces

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be a $2n$-dimensional indefinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v\,(0 < v < n)$ and J is an indefinite almost complex structure on $\bar{M}$ satisfying

$$J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X, Y), \quad (\bar{\nabla}_X J)Y = 0 \tag{2.1}$$

for any vector fields X and Y of $\bar{M}$ [3].

Let (M, g) be a lightlike hypersurface of $\bar{M}$. It is well known that the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM, of rank 1. A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is non-degenerate distribution on M, which is called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{orth} S(TM), \tag{2.2}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M, by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M and by $(-.-)_i$ the i-th equation of the equations $(-.-)$. We use same notations for any others. Due to [3], it is known that, for any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$, respectively. Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.3}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.2). From (2.2) and (2.3), the local Gauss and Weingartan formulas of M and $S(TM)$ are given, respectively, by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.4}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$
$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.6}$$
$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.7}$$

for any X, $Y \in \Gamma(TM)$, where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$, respectively, B and C are the local second fundamental forms on TM and $S(TM)$, respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$, respectively and τ is a 1-form on TM.

The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.8}$$

for any $X,\,Y,\,Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM).$$

But the connection ∇^* is metric. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of $S(TM)$ and satisfies

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.9}$$

The above second fundamental forms are related to their shape operators by

$$g(A^*_\xi X, Y) = B(X,Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.10}$$

$$g(A_N X, PY) = C(X,PY), \quad \bar{g}(A_N X, N) = 0. \tag{2.11}$$

Definition. A lightlike hypersurface M of $\bar{M}$ is said to be

(1) *totally umbilical* [3] if there is a smooth function β on any coordinate neighborhood $\mathcal{U}$ in M such that $A^*_\xi X = \beta PX$, or equivalently,

$$B(X,Y) = \beta g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.12}$$

(2) *screen totally umbilical* [3] if there exists a smooth function γ on $\mathcal{U}$ such that $A_N X = \gamma PX$, or equivalently,

$$C(X,PY) = \gamma g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.13}$$

In case $\gamma = 0$ ($\gamma \neq 0$) on $\mathcal{U}$, we say that M is *screen totally geodesic* (*proper screen totally umbilical*).

(3) *screen conformal* [1] if there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X,PY) = \varphi B(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.14}$$

3. Quarter-symmetric metric connections

Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ admitting a quarter-symmetric metric connection. For a lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$, $S(TM)$ splits as follows [3]:

If ξ and N are local sections of $TM^\perp$ and $tr(TM)$, respectively, we have

$$\bar{g}(J\xi,\xi) = \bar{g}(J\xi,N) = \bar{g}(JN,\xi) = \bar{g}(JN,N) = 0, \quad \bar{g}(J\xi,JN) = 1. \tag{3.1}$$

These equations show that $J\xi$ and JN belong to $S(TM)$. Thus $J(TM^\perp)$ and $J(tr(TM))$ are distributions on M, of rank 1 such that $TM^\perp \cap J(TM^\perp) = \{0\}$ and $TM^\perp \cap J(tr(TM)) = \{0\}$. Hence $J(TM^\perp) \oplus J(tr(TM))$ is a vector subbundle of $S(TM)$, of rank 2. Then there exists a non-degenerate almost complex distribution D_o on M with respect to J, *i.e.*, $J(D_o) = D_o$, such that

$$TM = TM^\perp \oplus_{orth} \{J(TM^\perp) \oplus J(tr(TM)) \oplus_{orth} D_o\}.$$

Consider the 2-lightlike almost complex distribution D such that

$$D = \{TM^{\perp} \oplus_{orth} J(TM^{\perp})\} \oplus_{orth} D_o, \quad TM = D \oplus J(tr(TM)) \tag{3.2}$$

and the local lightlike vector fields U and V such that

$$U = -JN, \qquad V = -J\xi. \tag{3.3}$$

Denote by S the projection morphism of TM on D with respect to the decomposition $(3.2)_2$. Then any vector field X on M is expressed as follow:

$$X = SX + u(X)U,$$

where u and v are 1-forms locally defined on M by

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad \forall\, X \in \Gamma(TM). \tag{3.4}$$

Using (3.3), the action JX of X by J is expressed as follow:

$$JX = FX + u(X)N, \tag{3.5}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$.

Using (1.1), (2.4) and (3.5), we show that

$$T(X, Y) = \pi(Y)FX - \pi(X)FY, \tag{3.6}$$

$$B(X, Y) - B(Y, X) = \pi(Y)u(X) - \pi(X)u(Y), \tag{3.7}$$

for all X, $Y \in \Gamma(TM)$, where T is the torsion tensor with respect to ∇. From (2.8) and (3.6), we show that ∇ is a quarter-symmetric non-metric connection of M. In the entire discussion of this article, we shall assume that the torsion vector field ζ of $\bar{M}$ to be unit spacelike, without loss of generality. We set $b = \pi(\xi)$. Replacing X by ξ to (3.7) and using (2.9), we have

$$B(\xi, X) = -bu(X), \quad \forall\, X \in \Gamma(TM). \tag{3.8}$$

From this, (2.10) and the fact that $S(TM)$ is non-degenerate, we have

$$A^{*}_{\xi}\xi = -bV. \tag{3.9}$$

Applying $\bar{\nabla}_X$ to (3.3) and (3.4) by turns, and using (2.1), (2.4), (2.5), (2.7), (2.9), (2.10), (2.11) (3.3), (3.4) and (3.5), we have

$$\nabla_X U = F(A_N X) + \tau(X)U, \tag{3.10}$$

$$\nabla_X V = F(A^{*}_{\xi} X) - \tau(X)V, \tag{3.11}$$

$$(\nabla_X F)(Y) = u(Y)A_N X - B(X, Y)U, \tag{3.12}$$

$$B(X, U) = C(X, V), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.13}$$

Example 1. Let $(\mathbf{R}^6_2, \bar{g})$ be a 6-dimensional semi-Euclidean space of index 2 with signature $(-, -, +, +, +, +)$ of the canonical basis $(\partial_0, \ldots, \partial_5)$. Consider a Monge hypersurface M of $\mathbf{R}^6_2$ given by

$$x_0 = u_1 + u_2 + u_3 \quad \text{and} \quad x_i = u_i\, (1 \le i \le 5).$$

Then the tangent bundle TM of M is spanned by

$$\left\{\partial_{u_1} = \partial_0 + \partial_1,\ \partial_{u_2} = \partial_0 + \partial_2,\ \partial_{u_3} = \partial_0 + \partial_3,\ \partial_{u_4} = \partial_4,\ \partial_{u_5} = \partial_5\right\}.$$

It is easy to check that M is a lightlike hypersurface of $(\mathbf{R}^6_2, \bar{g})$ such that the normal bundle $TM^\perp$ is spanned by

$$\xi = \partial_0 - \partial_1 + \partial_2 + \partial_3.$$

Let $E = \partial_0 - \partial_1$, then $g(E, E) = -2$ and $g(\xi, E) = -2$. Then the lightlike transversal vector bundle is given by

$$tr(TM) = Span\{N = -\frac{1}{4}(\partial_0 - \partial_1 - \partial_2 - \partial_3)\}.$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = \partial_0 + \partial_1,\ W_2 = \partial_2 - \partial_3,\ W_3 = \partial_4,\ W_4 = \partial_5\}.$$

Since $\mathbf{R}^6_2$ has complex structure J, we see that $J\xi = W_1 - W_2 \in \Gamma(S(TM))$, $JN = -\frac{1}{4}\{W_1 + W_2\} \in \Gamma(S(TM))$, $JW_3 = W_4$ and $JW_4 = -W_3$. Thus the almost complex distribution D_o is given by $D_o = Span\{W_3, W_4\}$.

Theorem 3.1. *There exist no lightlike hypersurfaces of an indefinite Kaehler manifold admitting a quarter-symmetric metric connection such that the local second fundamental form B of M is symmetric.*

Proof. Assume that B is symmetric. From (3.7), we have

$$\pi(X)u(Y) = \pi(Y)u(X)$$

for all $X, Y \in \Gamma(TM)$. Replacing Y by U to this, we have

$$\pi(X) = \pi(U)u(X).$$

Taking $X = \xi$ and $X = V$ by turns, we get $b = 0$, *i.e.*, the torsion vector field ζ is tangent to M, and $\pi(V) = 0$, respectively. As ζ is tangent to M, we have

$$u(\zeta) = g(\zeta, V) = \pi(V) = 0.$$

Taking $Y = \zeta$ to $\pi(Y)u(X) = \pi(X)u(Y)$, we get $u(X) = u(\zeta)\pi(X) = 0$ for all $X \in \Gamma(TM)$. It is a contradiction to $u(U) = 1$. Thus there exist no lightlike hypersurfaces of an indefinite Kaehler manifold admitting a quarter-symmetric metric connection such that B is symmetric. □

Assume that M is totally umbilical. Then B is symmetric. Thus we have:

Corollary 3.2. *There exist no totally umbilical lightlike hypersurfaces of an indefinite Kaehler manifold admitting a quarter-symmetric metric connection.*

Theorem 3.3. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ admitting a quarter-symmetric metric connection. If M is either screen totally umbilical or screen conformal, then $b = 0$ and ζ is tangent to M.*

Proof. Assume that M is screen totally umbilical. Replacing X by U to (3.8) and using (2.13) and (3.13), we have

$$-b = B(\xi, U) = C(\xi, V) = \gamma g(\xi, V) = 0.$$

Assume that M is screen conformal. From (3.8), we have $B(\xi, U) = -b$ and $B(\xi, V) = 0$. From these two equations, (2.14) and (3.13), we have

$$-b = B(\xi, U) = C(\xi, V) = \varphi B(\xi, V) = 0.$$

In the above two cases, we get $b = 0$. It follow that ζ is tangent to M. □

Theorem 3.4. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ admitting a quarter-symmetric metric connection. If V and U are parallel with respect to ∇, then M is screen totally geodesic, τ vanishes and ζ is tangent to M. Moreover, M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where $\mathcal{C}_\xi$ is a null curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$.*

Proof. If V is parallel with respect ∇, then, from (3.5) and (3.11), we have

$$J(A^*_\xi X) - u(A^*_\xi X)N - \tau(X)V = 0, \quad \forall\, X \in \Gamma(TM).$$

Applying J to this equation and using (2.1) and (3.3), we obtain

$$A^*_\xi X - u(A^*_\xi X)U + \tau(X)\xi = 0, \quad \forall\, X \in \Gamma(TM).$$

Taking the scalar product with N, we get $\tau = 0$. Consequently, we have

$$A^*_\xi X = u(A^*_\xi X)U, \quad \forall\, X \in \Gamma(TM).$$

Taking the scalar product with U to this and using (3.13), we have

$$u(A_N X) = v(A^*_\xi X) = g(A^*_\xi X,\, U) = u(A^*_\xi X)g(U, U) = 0.$$

If U is parallel with respect to ∇, then, from (3.5) and (3.10), we have

$$J(A_N X) - u(A_N X)N + \tau(X)U = 0, \quad \forall\, X \in \Gamma(TM).$$

Applying J to this equation and using (2.1) and (3.3), we obtain

$$A_N X - u(A_N X)U + \tau(X)N = 0, \quad \forall\, X \in \Gamma(TM).$$

Taking the scalar product with ξ to this equation, we get $\tau = 0$ and

$$A_N X = u(A_N X)U, \qquad \forall\, X \in \Gamma(TM).$$

In case V and U are parallel with respect to ∇. From the above two equations $u(A_N X) = 0$ and $A_N X = u(A_N X)U$, we obtain $A_N = 0$. Thus M is screen totally geodesic. By Theorem 3.3, the torsion vector field ζ is tangent to M. As $C = 0$ and $b = 0$, from (2.6), (2.7) and (3.9), we see that $S(TM)$ and $TM^\perp$ are auto-parallel distributions such that $TM = TM^\perp \oplus_{orth} S(TM)$. By the decomposition theorem of de Rham [2], M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where $\mathcal{C}_\xi$ is a null curve tangent to $TM^\perp$ and M^* is a leaf of the screen distribution $S(TM)$. □

Theorem 3.5. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ admitting a quarter-symmetric metric connection. If F is parallel with respect to the connection ∇, then D and $J(tr(TM))$ are parallel distributions on M. Moreover, M is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of D.*

Proof. In general, by using (2.1), (2.8), (2.9), (3.5) and (3.11), we derive

$$g(\nabla_X \xi, V) = -B(X, V),\ g(\nabla_X V, V) = 0,\ g(\nabla_X Z, V) = B(X, FZ) \tag{3.14}$$

for all $X \in \Gamma(TM)$ and $Z \in \Gamma(D_o)$. If F is parallel with respect to ∇, then, from (3.12), we have $B(X, Y)U = u(Y)A_{\scriptscriptstyle N}X$, *i.e.*, we get

$$B(X, Y) = u(Y)u(A_{\scriptscriptstyle N}X), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Taking $Y = V$ and $Z \in \Gamma(D_o)$ to this equation by turns, we have $B(X, V) = 0$ and $B(X, Z) = 0$ for all $X \in \Gamma(TM)$, respectively. It follow from (3.14) that

$$\nabla_X Y \in \Gamma(D), \quad \forall\, X \in \Gamma(TM),\ \ \forall\, Y \in \Gamma(D),$$

due to $FZ \in \Gamma(D_o)$. Thus D is a parallel distribution on M.

Taking $Y = U$ to $B(X, Y)U = u(Y)A_{\scriptscriptstyle N}X$, we get

$$A_{\scriptscriptstyle N}X = B(X, U)U, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Applying F to this relation and using the fact that $FU = 0$, we get

$$F(A_{\scriptscriptstyle N}X) = B(X, U)FU = 0, \quad \forall\, X \in \Gamma(TM).$$

Thus, from (3.10), we obtain

$$\nabla_X U \in \Gamma(J(tr(TM))), \quad \forall\, X \in \Gamma(TM),$$

and $J(tr(TM))$ is also a parallel distribution on M.

As D and $J(tr(TM))$ are parallel distributions and $TM = D \oplus J(tr(TM))$. By the decomposition theorem [2], M is locally a product manifold $\mathcal{C}_u \times M^\sharp$, where $\mathcal{C}_u$ is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of D. □

Theorem 3.6. *There exist no screen conformal lightlike hypersurfaces of an indefinite Kaehler manifold $\bar{M}$ with a quart-symmetric metric connection such that at least one of the objects V, U and F is parallel with respect to ∇.*

Proof. In the proof of Theorem 3.4, if V is parallel, then $\tau = 0$, $u(A_{\scriptscriptstyle N}X) = 0$ and $A^*_\xi X = u(A^*_\xi X)U$ for any $X \in \Gamma(TM)$. Using the second equation of the above relations and the fact that $A_{\scriptscriptstyle N} = \varphi A^*_\xi$, we have

$$u(A^*_\xi X) = \varphi^{-1} u(A_{\scriptscriptstyle N}X) = 0,\ \forall\, X \in \Gamma(TM).$$

From this and the fact that $A^*_\xi X = u(A^*_\xi X)U$ for all $X \in \Gamma(TM)$, we have $A^*_\xi = 0$. It is a contradiction to Corollary 3.2.

If U is parallel, then $\tau = 0$ and $A_{\scriptscriptstyle N}X = u(A_{\scriptscriptstyle N}X)U$ for any $X \in \Gamma(TM)$. From the last equation, we have $v(A_{\scriptscriptstyle N}X) = 0$ for any $X \in \Gamma(TM)$. Using (3.13) and the fact that $A_{\scriptscriptstyle N} = \varphi\, A^*_\xi$, we have

$$u(A_{\scriptscriptstyle N}X) = v(A^*_\xi X) = \varphi^{-1} v(A_{\scriptscriptstyle N}X) = 0,\ \forall\, X \in \Gamma(TM).$$

From this and $A_{\scriptscriptstyle N}X = u(A_{\scriptscriptstyle N}X)U$, we have $A_{\scriptscriptstyle N} = 0$. As M is screen conformal, it follow that $A^*_\xi = 0$. It is also a contradiction to Corollary 3.2.

If F is parallel, then we have $B(X,Y) = u(Y)u(A_N X)$ and $B(X,V) = 0$ for all $X,\, Y \in \Gamma(TM)$. Thus

$$u(A_N X) = \varphi\, u(A^*_\xi X) = \varphi\, B(X,V) = 0, \quad \forall\, X \in \Gamma(TM).$$

From this and $B(X,Y) = u(Y)u(A_N X)$ we have $B = 0$. It is also a contradiction to Corollary 3.2. □

As $\{U,\, V\}$ is a basis of $J(TM^\perp) \oplus J(tr(TM))$, the vector fields

$$\mu = U - \varphi V, \qquad \nu = U + \varphi V \tag{3.15}$$

form an orthogonal basis of $J(TM^\perp) \oplus J(tr(TM))$.

Theorem 3.7. *Let M be a screen conformal lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a quart-symmetric metric connection. Then μ is parallel with respect to ∇ if and only if τ vanishes and φ is a constant.*

Proof. From (3.10), (3.11) and the linearity of F, we have

$$\nabla_X \mu = \tau(X)\nu - (X\varphi)V, \quad \forall\, X \in \Gamma(TM),$$

due to $A_N = \varphi\, A^*_\xi$. Thus we see that μ is parallel if and only if

$$\tau(X)U - \{X\varphi - \varphi\tau(X)\}V = 0, \quad \forall\, X \in \Gamma(TM).$$

Taking the scalar product with V and U in turns, we get our assertion. □

Let $\mathcal{G}(\mu) = Span\{\mu\}$ and $\mathcal{S}(\mu) = TM^\perp \oplus_{orth} D_o \oplus_{orth} Span\{\nu\}$. Then $\mathcal{S}(\mu)$ is a complementary vector subbundle to $\mathcal{G}(\mu)$ in TM such that

$$TM = \mathcal{G}(\mu) \oplus_{orth} \mathcal{S}(\mu).$$

From (2.14), (3.13) and $(3.15)_1$, we show that

$$B(X,\mu) = 0, \quad \forall\, X \in \Gamma(TM). \tag{3.16}$$

Theorem 3.8. *Let M be a screen conformal lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a quart-symmetric metric connection. If μ is parallel with respect to ∇, then M is locally a product manifold $\mathcal{C}_\mu \times M^\flat$, where $\mathcal{C}_\mu$ is a non-null geodesic tangent to $\mathcal{G}(\mu)$ and $M^\flat$ is a leaf of $\mathcal{S}(\mu)$.*

Proof. For any $X \in \Gamma(TM)$ and $Y \in \Gamma(D_o)$, we get

$$\begin{aligned} g(\nabla_X Y,\, \mu) &= g(\bar{\nabla}_X Y,\, \mu) = -g(Y,\, \nabla_X \mu) = 0, \\ g(\nabla_X \xi,\, \mu) &= -g(\xi,\, \bar{\nabla}_X \mu) = -B(X,\mu) = 0, \\ g(\nabla_X \nu,\, \mu) &= -g(\nu,\, \nabla_X \mu) = 0. \end{aligned}$$

Thus $\mathcal{S}(\mu)$ is a parallel distribution on M. As μ is parallel with respect to ∇, $\mathcal{G}(\mu)$ is also parallel distribution on M such that $TM = \mathcal{G}(\mu) \oplus_{orth} \mathcal{S}(\mu)$. By the decomposition theorem [2], M is locally a product manifold $\mathcal{C}_\mu \times M^\flat$, where $\mathcal{C}_\mu$ is a non-null geodesic tangent to $\mathcal{G}(\mu)$ and $M^\flat$ is a leaf of $\mathcal{S}(\mu)$. □

Theorem 3.9. *There exist no screen conformal lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a quart-symmetric metric connection such that the vector field ν is parallel with respect to ∇.*

Proof. If M is screen conformal, then, from (3.10) and (3.11), we have

$$\nabla_X \nu = 2F(A_{_N}X) + \tau(X)U + \{X\varphi - \varphi\tau(X)\}V, \quad \forall\, X \in \Gamma(TM).$$

As $g(F(A_{_N}X), V) = g(F(A_{_N}X), U) = 0$, we show that ν is parallel if and only if $\tau = 0$ on M, φ is a constant and $F(A_{_N}X) = 0$. Therefore, by using (3.10), (3.11) and the fact that $A_{_N} = \varphi A^*_\xi$, we show that U and V are parallel with respect to ∇. Thus, by Theorem 3.6, we have our assertion. □

4. Indefinite complex space forms

Denote by $\bar{R}$, R and R^* the curvature tensors of the quarter-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten formulas, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned}
(4.1)\quad \bar{g}(\bar{R}(X,Y)Z,\, PW) \;=\;& g(R(X,Y)Z,\, PW) \\
&+ B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW),\\
(4.2)\quad \bar{g}(\bar{R}(X,Y)Z,\, \xi) \;=\;& (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\
&+ \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\
&- \pi(X)B(FY,Z) + \pi(Y)B(FX,Z),\\
(4.3)\quad \bar{g}(\bar{R}(X,Y)Z,\, N) =\;& \bar{g}(R(X,Y)Z,\, N),\\
(4.4)\quad g(R(X,Y)PZ,\, N) \;=\;& (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\
&- \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\
&- \pi(X)C(FY,PZ) + \pi(Y)C(FX,PZ),
\end{aligned}$$

for any $X,\, Y,\, Z,\, W \in \Gamma(TM)$.

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\begin{aligned}
(4.5)\quad \bar{R}(X,Y)Z \;=\;& \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX \\
&- \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\}, \quad \forall\, X,Y,Z \in \Gamma(T\bar{M}).
\end{aligned}$$

Theorem 4.1. *Let M be a screen conformal lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ admitting a quarter-symmetric metric connection. Then $c = 0$, and the conformal factor φ satisfies the differential equation*

$$\xi\varphi - \varphi\tau(\xi) = 0.$$

Proof. Substituting (4.5) into (4.2), for all $X,\, Y,\, Z \in \Gamma(TM)$, we have

$$\begin{aligned}
(4.6)\quad & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\
&= B(X,Z)\tau(Y) - B(Y,Z)\tau(X) + B(FY,Z)\pi(X) - B(FX,Z)\pi(Y)
\end{aligned}$$

$$+ \ \frac{c}{4}\{u(X)\bar{g}(JY,Z) - u(Y)\bar{g}(JX,Z) + 2u(Z)\bar{g}(X,JY)\}.$$

As M is screen conformal, we get $b = 0$, *i.e.*, ζ is tangent to M by Theorem 3.3. Applying ∇_X to $C(Y,PZ) = \varphi B(Y,PZ)$, we have

$$(\nabla_X C)(Y,PZ) = (X\varphi)B(Y,PZ) + \varphi(\nabla_X B)(Y,PZ)$$

for all $X,Y,Z \in \Gamma(TM)$. Substituting this into (4.4) and using (4.6), we get

$$\begin{aligned} & g(R(X,Y)PZ,\, N) \\ &= \{X\varphi - \varphi\tau(X)\}B(Y,PZ) - \{Y\varphi - \varphi\tau(Y)\}B(X,PZ) \\ &\quad + \ \frac{c}{4}\varphi\{u(X)\bar{g}(JY,PZ) - u(Y)\bar{g}(JX,PZ) + 2u(PZ)\bar{g}(X,JY)\}. \end{aligned}$$

Substituting this equation and (4.5) into (4.3) with $Z = PZ$, we have

$$\begin{aligned} & \frac{c}{4}\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y) \\ &\quad + \ v(X)\bar{g}(JY,PZ) - v(Y)\bar{g}(JX,PZ) + 2v(PZ)\bar{g}(X,JY)\} \\ &= \{X\varphi - \varphi\tau(X)\}B(Y,PZ) - \{Y\varphi - \varphi\tau(Y)\}B(X,PZ) \\ &\quad + \ \frac{c}{4}\varphi\{u(X)\bar{g}(JY,PZ) - u(Y)\bar{g}(JX,PZ) + 2u(PZ)\bar{g}(X,JY)\} \end{aligned}$$

for all $X,Y,Z \in \Gamma(TM)$. Replacing Y by ξ and using (3.8), we have

$$\begin{aligned} & \{\xi\varphi - \varphi\tau(\xi)\}B(X,PY) \\ &= \frac{c}{4}\{g(X,PY) + v(X)u(PY) + 2u(X)v(PY) - 3\varphi u(X)u(PY)\}. \end{aligned} \tag{4.7}$$

Let $\mu = U - \varphi V$. From (2.14) and (3.13), we show that

$$B(X,\mu) = 0, \quad \forall\, X \in \Gamma(TM). \tag{4.8}$$

Replacing PY by μ to (4.7) and using (3.4) and (4.8), we have

$$\frac{c}{2}\{v(X) - 3\varphi u(X)\} = 0, \quad \forall\, X \in \Gamma(TM).$$

Taking $X = V$ to this equation and using (3.4), we obtain $c = 0$. Therefore, from (4.7) we have $\{\xi\varphi - \varphi\tau(\xi)\}B(X,PY) = 0$. Using Corollary 3.2, we get $\xi\varphi - \varphi\tau(\xi) = 0$. Thus we have our theorem. □

Theorem 4.2. *Let M be a screen totally umbilical lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ admitting a quarter-symmetric metric connection. Then $c = 0$, and the second fundamental form B of M becomes*

$$B(X,Y) = \alpha g(X,Y) - \pi(X)u(Y), \quad \forall\, X,\, Y \in \Gamma(TM),$$

where α is a smooth function given by $\alpha = \pi(V)$.

Proof. As M is screen totally umbilical, we show that $b = 0$, *i.e*, ζ is tangent to M by Theorem 3.3. Applying ∇_Z to (2.13) and using (2.7), we obtain

$$(\nabla_X C)(Y,PZ) = (X\gamma)g(Y,PZ) + \gamma B(X,PZ)\eta(Y)$$

for all $X, Y, Z \in \Gamma(TM)$. Substituting this equation into (4.4), we have

$$\begin{aligned} & g(R(X,Y)PZ, N) \\ &= \{X\gamma - \gamma\tau(X)\}g(Y,PZ) - \{Y\gamma - \gamma\tau(Y)\}g(X,PZ) \\ &\quad + \gamma\{B(X,PZ)\eta(Y) - B(Y,PZ)\eta(X) \\ &\quad + g(FX,PZ)\pi(Y) - g(FY,PZ)\pi(X)\}. \end{aligned}$$

Substituting this equation and (4.5) into (4.3) with $Z = PZ$, we have

$$\begin{aligned} & \frac{c}{4}\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y) \\ &\quad + v(X)\bar{g}(JY,PZ) - v(Y)\bar{g}(JX,PZ) + 2v(PZ)\bar{g}(X,JY)\} \\ &= \{X\gamma - \gamma\tau(X)\}g(Y,PZ) - \{Y\gamma - \gamma\tau(Y)\}g(X,PZ) \\ &\quad + \gamma\{B(X,PZ)\eta(Y) - B(Y,PZ)\eta(X) \\ &\quad + \bar{g}(JX,PZ)\pi(Y) - \bar{g}(JY,PZ)\pi(X)\}. \end{aligned}$$

Replacing Y by ξ and using (3.3), (3.4) and (3.8), we have

$$\begin{aligned} \gamma B(X,PY) &= \{\xi\gamma - \gamma\tau(\xi) - \frac{c}{4}\}g(X,PY) \\ &\quad - \frac{c}{4}\{v(X)u(PY) + 2u(X)v(PY)\} - \gamma\pi(X)u(PY) \end{aligned} \tag{4.9}$$

for all $X, Y \in \Gamma(TM)$. Taking $X = U$ and $PY = V$ to (4.9), we have

$$\gamma B(U,V) = \xi\gamma - \gamma\tau(\xi) - \frac{3}{4}c.$$

In case $\gamma = 0$, we have $c = 0$. In case $\gamma \neq 0$. Taking $X = V$ and $Y = PU$ to (4.9) and using (3.4), we have

$$\gamma B(V,U) = \xi\gamma - \gamma\tau(\xi) - \frac{2}{4}c - \gamma\pi(V). \tag{4.10}$$

Substituting the last two equation into (3.7), we get $c = 0$.

From (2.13) and (3.13), we obtain

$$B(X,U) = \gamma u(X), \quad \forall\, X \in \Gamma(TM).$$

Replacing X by V to this, we have $B(V,U) = 0$. From this and (4.10), we get

$$\gamma\pi(V) = \xi\gamma - \gamma\tau(\xi).$$

Substituting this into (4.9) and using (2.9) and the fact $u(\xi) = 0$, we have

$$B(X,Y) = \pi(V)g(X,Y) - \pi(X)u(Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

As $\alpha = \pi(V)$, we have our theorem. □

The induced Ricci type tensor $R^{(0,\,2)}$ of M is defined by

$$R^{(0,\,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}$$

for any $X,\, Y \in \Gamma(TM)$. Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$. Using this quasi-orthonormal frame field, for any $X, Y \in \Gamma(TM)$, we obtain

$$R^{(0,\,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, g(R(W_a, X)Y,\, W_a) + \bar{g}(R(\xi, X)Y,\, N), \tag{4.11}$$

where $\epsilon_a = g(W_a, W_a)$ is the sign of W_a. In general, the induced Ricci type tensor $R^{(0,\,2)}$, defined by the method of the geometry of the non-degenerate submanifolds [11], is not symmetric [4, 5]. Therefore $R^{(0,\,2)}$ has no geometric or physical meaning similar to the Ricci curvature of the non-degenerate submanifolds and it is just a tensor quantity. Hence we need the following definition: A tensor field $R^{(0,\,2)}$ on M is called its *induced Ricci tensor* of M if it is symmetric. A symmetric $R^{(0,\,2)}$ tensor will be denoted by Ric.

In case $c = b = 0$. (4.1) and (4.3) are reduced, respectively, to

$$g(R(X,Y)Z,\, PW) = B(Y,Z)C(X,PW) - B(X,Z)C(Y,PW), \tag{4.12}$$

$$\bar{g}(R(X,Y)Z,\, N) = 0, \quad \forall\, X,\, Y,\, Z,\, W \in \Gamma(TM). \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.11) and using (3.7), we have

$$\begin{aligned} R^{(0,\,2)}(X,Y) &= B(X,Y)tr\, A_{_N} - g(A^*_\xi Y, A_{_N} X) \\ &\quad + \pi(A_{_N} X)u(Y) - u(A_{_N} X)\pi(Y),\ \forall\, X, Y \in \Gamma(TM). \end{aligned} \tag{4.14}$$

Remark 4.3. From the last equation, we show that if M is screen totally geodesic, then M is Ricci flat.

Theorem 4.4. *There exist no proper screen totally umbilical lightlike hypersurfaces of an indefinite almost complex space form admitting a quarter-symmetric metric connection such that the Ricci type tensor $R^{(0,\,2)}$ of M is symmetric.*

Proof. Using (2.13) and (3.7), we show that $tr\, A_{_N} = m\gamma$ and

$$\begin{aligned} g(A^*_\xi Y, A_{_N} X) &= C(X, A^*_\xi Y) = \gamma g(X, A^*_\xi Y) = \gamma B(Y, X) \\ &= \gamma\{B(X,Y) - \pi(Y)u(X) + \pi(X)u(Y)\} \end{aligned}$$

for all $X,\, Y \in \Gamma(TM)$. Substituting these equations into (4.14) and using the fact that $\pi(\xi) = 0 = u(\xi)$, we obtain

$$R^{(0,\,2)}(X,Y) = \gamma(m-1)B(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{4.15}$$

This result implies that $R^{(0,\,2)}$ is symmetric if and only if B is symmetric. Thus, by Theorem 3.1, we have our theorem. □

References

[1] C. Atindogbe and K. L. Duggal, *Conformal screen on lightlike hypersurfaces*, Int. J. Pure Appl. Math. **11** (2004), no. 4, 421–442.

[2] G. de Rham, *Sur la reductibilité d'un espace de Riemannian*, Comment. Math. Helv. **26** (1952), 328–344.

[3] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.

[4] K. L. Duggal and D. H. Jin, *Null curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.

[5] ———, *A classification of Einstein lightlike hypersurfaces of a Lorentzian space form*, J. Geom. Phys. **60** (2010), no. 12, 1881–1889.

[6] K. L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Frontiers in Mathematics, Birkhäuser, 2010.

[7] S. Golab, *On semi-symmetric and quarter-symmetric connections*, Tensor (N.S.) **29** (1975), 249–254.

[8] D. Kamilya and U. C. De, *Some properties of a Ricci quarter-symmetric metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math. **26** (1995), no. 1, 29–34.

[9] R. S. Mishra and S. N. Pandey, *On quarter-symmetric F-connections*, Tensor (N.S.) **34** (1980), no. 1, 1–7.

[10] J. Nikić and N. Pušić, *A remakable class of natural metric quarter-symmetric connection on a hyperbolic Kaehler space*, Conference "Applied Differential Geometry: General Relativity"-Workshop "Global Analysis, Differential Geometry, Lie Algebras", 96–101, BSG Proc., 11, Geom. Balkan Press, Bucharest, 2004.

[11] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983.

[12] N. Pušić, *On quarter-symmetric metric connections on a hyperbolic Kaehler space*, Publ. de l' Inst. Math. **73(87)** (2003), 73–80.

[13] S. C. Rastogi, *On quarter-symmetric metric connections*, C. R. Acad Bulgare Sci. **31** (1978), no. 7, 811–814.

[14] ———, *On quarter-symmetric metric connections*, Tensor (N.S.) **44** (1987), no. 2, 133–141.

[15] K. Yano and T. Imai, *Quarter-symmwtric metric connection and their curvature tensors*, Tensor (N.S.) **38** (1982), 13–18.

Department of Mathematics
Dongguk University
Kyongju 780-714, Korea
E-mail address: jindh@dongguk.ac.kr