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BLOW-UP RATE FOR THE SEMI-LINEAR WAVE EQUATION IN BOUNDED DOMAIN

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Abstract. In this paper, the blow-up rate of L^2-norm for the semi-linear wave equation with a power nonlinearity is obtained in the bounded domain for any $p > 1$. We also get the blow-up rate of the derivative under the condition $1 < p < 1 + \frac{4}{N-1}$ for $N \geq 2$ or $1 < p < 5$ for $N = 1$.

1. Introduction

In this paper, we study the blow-up rate for the following semi-linear wave equation

$$
\begin{cases} u_{tt} - \Delta u = |u|^{p-1}u & x \in \Omega \text{ and } t \geq 0, \\ u(t,x) = 0 & x \in \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. We assume that the initial data $u_0 \in L^{p+1}(\Omega)$ and u_1 are sufficiently smooth and bounded in Ω such that the solution $u(t,x)$ of equation (1.1) belongs to $C([0,T), H^2(\Omega)) \cap C^1([0,T), H_0^1(\Omega)) \cap C^2([0,T), L^2(\Omega))$. Here $0 < T < \infty$ is the lifespan of equation (1.1).

In the whole space $\mathbb{R}^N$, Antonini and Merle [2] and Merle and Zaag [8, 9, 10] considered the blow-up rate for the equation (1.1) in the local uniform Sobolev space, that is $H^1_{loc,u}(\mathbb{R}^N) \times L^2_{loc,u}(\mathbb{R}^N)$ which is defined as

$$
L^2_{loc,u}(\mathbb{R}^N) := \left\{ u \;\middle|\; u \in L^2_{loc}(\mathbb{R}^N) \text{ and } \|u\|_{L^2_{loc,u}} := \sup_{a \in \mathbb{R}^N} \left(\int_{|x-a|<1} |v(x)|^2 dx \right)^{1/2} < \infty \right\},
$$

$$
H^1_{loc,u}(\mathbb{R}^N) := \left\{ u \;\middle|\; u \text{ and } \partial_i u \in L^2_{loc,u}(\mathbb{R}^N), 1 \leq i \leq N \right\}.
$$

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They use the following self-similar transformation of variables:

$$w_a(s,y) = (T-t)^{\frac{2}{p-1}} u(t,x),\ y = \frac{x-a}{T-t},\ s = -\ln(T-t),$$

and (1.1) is transformed into

$$\begin{aligned} &\partial_s^2 w_a - \frac{1}{\rho} div(\rho \nabla w_a - \rho(y\cdot\nabla w_a)y) + \frac{2(p+1)}{(p-1)^2} w_a - |w_a|^{p-1} w_a \\ = &-\frac{p+3}{p-1}\partial_s w_a - 2y\cdot\nabla\partial_s w_a \end{aligned} \tag{1.2}$$

in the unit ball B and the exponent p is restricted such that $1 < p \le 1 + \frac{4}{N-1}$. Here $\rho := (1-|y|^2)^\alpha$ and $\alpha = \frac{2}{p-1} - \frac{N-1}{2}$. Via the blow-up analysis and energy estimate, they obtained the bound of w_a in $H^1(B)\times L^2(B)$, and the inverse of self-similar transformation gives

$$(T-t)^{\frac{2}{p-1}} \|u(t)\|_{L^2_{loc,u}} + (T-t)^{\frac{2}{p-1}+1}\left(\|u_t(t)\|_{L^2_{loc,u}} + \|\nabla u(t)\|_{L^2_{loc,u}}\right) \le K, \tag{1.3}$$

where K is a positive constant depending only on N, p and the initial data. By the local existence result, this blow-up rate in (1.3) is optimal, which means there exists a positive constant ε_0 such that

$$\varepsilon_0 \le (T-t)^{\frac{2}{p-1}} \|u(t)\|_{L^2_{loc,u}} + (T-t)^{\frac{2}{p-1}+1}\left(\|u_t(t)\|_{L^2_{loc,u}} + \|\nabla u(t)\|_{L^2_{loc,u}}\right).$$

In [3], Bizón, Chmaj and Tabor obtained a numerical confirmation of this result in the wider range $1 < p < 1 + \frac{4}{N-2}$. Recently, Hamza and Zaag [5] considered the case where $1 < p < 1 + \frac{4}{N-2}$ in the higher dimensions. Via the same transformation and the perturbation method, they got the blow-up rate near the blow-up graph.

Many mathematicians also studied the blow-up profile near the blow-up graph, and we referred the interested reader to Alinhac [1], Caffarelli and Friedman [4], Kichenassamy and Litman [6, 7] and the references therein.

In this paper, we consider the blow-up rate for the equation (1.1) in the bounded domain. Here the self-similar transformation loses effectiveness. So we introduce another transformation of variables:

$$s = -\ln(T-t),\ w(s,x) = (T-t)^\beta u(t,x), \tag{1.4}$$

where T is the lifespan of (1.1) and $\beta := \frac{2}{p-1}$. Then the function w satisfies the following equation

$$w_{ss} - e^{-2s}\triangle w + (2\beta+1)w_s + \beta(\beta+1)w = |w|^{p-1}w \tag{1.5}$$

with the homogeneous boundary condition. Now we state our results about the boundedness of equation (1.5).

Theorem 1.1 (Uniform bounds on solutions of (1.5))**.** *Assume $p > 1$. If w is the solution of* (1.5) *and blows up at time $T < \infty$, then for any $s \in [-\ln T + 1, \infty)$,*

$$\int_s^{s+1} \int_\Omega \left(e^{-2\tau} |\nabla w(\tau, x)|^2 + |\partial_s w(\tau, x)|^2 \right) dx d\tau \le K, \tag{1.6}$$

$$\|w(s)\|_{L^2(\Omega)} \le K, \tag{1.7}$$

where w is defined in (1.4)*, and K relies on N, p, Ω, the blow-up time T and the initial data.*

Moreover, if $1 < p < 1 + \frac{4}{N-1}$ for $N \ge 2$ or $1 < p < 5$ for $N = 1$, we obtain

$$\int_\Omega \left(e^{-2s} |\nabla w(s, x)|^2 + |\partial_s w(s, x)|^2 \right) dx \le K(1 + e^{2\gamma s}), \tag{1.8}$$

where $\gamma = \frac{N(p-1)}{2[N+3-(N-1)p]}$ for $N \ge 3$ or $\gamma = \frac{p-1}{5-p}$ for $N = 1,\ 2$.

Using the transformation (1.4), this result is rewritten in the original set of variables $u(t, x)$ as follows.

Theorem 1.2. *Assume $p > 1$. If u is a solution of* (1.1) *which blows up at time $T < \infty$, then for any $t \in [T(1 - e^{-1}), T)$ we have*

$$\int_t^T \int_\Omega \left[(T-\tau)^{2\beta+1} |\nabla u(\tau, x)|^2 + (T-\tau)^{2\beta-1} | -\beta u(\tau, x) + (T-\tau)\partial_t u(\tau, x)|^2 \right] dx d\tau \le K, \tag{1.9}$$

$$(T-t)^\beta \|u(t)\|_{L^2(\Omega)} \le K \tag{1.10}$$

for some constant K which relies on N, p, Ω, the blow-up time T and the initial data.

Moreover, we also get

$$(T-t)^{\beta+1+\gamma} \left[\|\nabla u(t)\|_{L^2} + \|\partial_t u(t)\|_{L^2} \right] \le K, \tag{1.11}$$

provided that $1 < p < 1 + \frac{4}{N-1}$ for $N \ge 2$ or $1 < p < 5$ for $N = 1$ and $t \in [T(1-e^{-1}), T) \cap [T-1, T)$. Here $\gamma = \frac{N(p-1)}{2[N+3-(N-1)p]}$ for $N \ge 3$ or $\gamma = \frac{p-1}{5-p}$ for $N = 1,\ 2$.

Remark 1.1. In the whole space $\mathbb{R}^N$, using this transformation and the method in [8], we can also get (1.9) and (1.10) but (1.11) is not satisfied. Because in the proof of (1.11) the L^{p_1}-norm dosen't control the L^{p_2}-norm in $\mathbb{R}^N$ for $p_1 > p_2 \ge 1$.

Remark 1.2. For the critical type $p = 1 + 4/(N-1)$, Claim 3.2 in Section 3 holds for $\theta = 1$, and we could not get the bound of derivative directly from the definition of energy $E(s)$ via Young inequality.

Remark 1.3. We may choose other transformations to obtain the same result as Theorem 1.2. For an example, we select

$$s = \frac{1}{T-t}, \qquad w(s,x) = (T-t)^{\beta} u(t,x),$$

and the equation (1.1) is changed to

$$s^2 w_{ss} - s^{-2}\triangle w + 2(\beta+1) s w_s + \beta(\beta+1) w = |w|^{p-1} w.$$

We define the energy function as

$$\begin{aligned} E[w](s) := \int_\Omega \Big[\frac{1}{2} s^2 |w_s(s,x)|^2 + \frac{1}{2} s^{-2} |\nabla w(s,x)|^2 \\ + \frac{1}{2}\beta(\beta+1)|w(s,x)|^2 - \frac{1}{p+1}|w(s,x)|^{p+1}\Big] dx, \end{aligned} \tag{1.12}$$

and like the proof of Theorem 1.1 we can get the same blow-up rate of (1.1).

This work is strongly inspired by the works of Merle and Zaag [8, 9, 10]. The outline of the paper is organized as follows. In Section 2, we make some blow-up analysis for equation (1.5). The proof of Theorem 1.1 is in Section 3.

2. Blow-up analysis for equation (1.5)

We define an energy function as follows.

$$\begin{aligned} E[w](s) := \int_\Omega \Big[\frac{1}{2} |w_s(s,x)|^2 + \frac{1}{2} e^{-2s} |\nabla w(s,x)|^2 \\ + \frac{1}{2}\beta(\beta+1)|w(s,x)|^2 - \frac{1}{p+1}|w(s,x)|^{p+1}\Big] dx. \end{aligned} \tag{2.1}$$

Denote that $E(s) = E[w](s)$ for simplicity. Immediately, we have the following lemma.

Lemma 2.1. *The energy function $s \mapsto E(s)$ is a decreasing function for $s \geq -\ln T$. Moreover, we have that for any $s_1, s_2 \in [-\ln T, \infty)$,*

$$E(s_2) - E(s_1) = -\int_{s_1}^{s_2}\int_\Omega \big(e^{-2\tau}|\nabla w(\tau,x)|^2 + (2\beta+1)|\partial_s w(\tau,x)|^2\big) dx d\tau. \tag{2.2}$$

Proof. Multiplying the equation (1.5) by w_s and integrating over $[s_1, s_2] \times \Omega$, we immediately get the equality (2.2), which completes the proof of Lemma 2.1. □

Using the method of Antonini and Merle [2], we get the following blow-up result for equation (1.5).

Proposition 2.1. *Assume that $p > 1$ and w is the solution of equation* (1.5). *Then w blows up in finite time provided that there exits $s_0 \in \mathbb{R}$ such that $E(s_0) < 0$.*

Proof. Arguing by contradiction, we assume that there exists a global solution w of (1.5) in Ω. Like the proof in [2], we consider the following transformation.

$$\tilde{w_\delta}(s,x) = \frac{1}{(1+\delta e^s)^\beta} w(-\ln(\delta + e^{-s}), x),\ (s,x) \in [s_0+1, \infty) \times \Omega,$$

where $\delta > 0$ is decided later. Computing clearly, we know that:

(i) $\tilde{w_\delta}$ is also satisfied the equation (1.5);

(ii) By continuity of the function $\delta \to E[\tilde{w_\delta}](s_0+1)$, we choose δ small enough such that $E[\tilde{w_\delta}](s_0+1) < 0$.

We choose one fixed $\delta > 0$, such that (i) and (ii) are satisfied. And from the definition of energy $E[\tilde{w_\delta}](s)$ we get

$$\begin{aligned} E[\tilde{w_\delta}](s) &\geq -\frac{1}{p+1}\int_\Omega |\tilde{w_\delta}(s,x)|^{p+1} dx \\ &\geq -\frac{1}{(p+1)(1+\delta e^s)^{\beta(p+1)}}\int_\Omega |w(-\ln(\delta+e^{-s}),x)|^{p+1} dx. \end{aligned} \tag{2.3}$$

Via a continuous argument and the energy equality, we can get that $u \in C([0, T-\delta], L^{p+1}(\Omega))$, so $w(-\ln(\delta + e^{-s}), \cdot)$ remains bounded in $L^{p+1}(\Omega)$. Then (2.3) yields

$$E[\tilde{w_\delta}](s) \geq -\frac{C}{(p+1)(1+\delta e^s)^{2\beta+2}},$$

which implies

$$\liminf_{s\to\infty} E[\tilde{w_\delta}](s) \geq 0.$$

But this contradicts the fact that $E[\tilde{w_\delta}](s_0+1) < 0$ and $E[\tilde{w_\delta}](s)$ descends by Lemma 2.1. This concludes the proof of Proposition 2.1. □

Remark 2.1. In [2], Theorem 2 needs the condition $1 < p < 1 + \frac{4}{N-2}$, because they obtained the inequality

$$E[\tilde{w_\delta}](s) \geq -\frac{1}{(p+1)(1+\delta e^s)^{2\beta+2-N}}\int_B |w(-\ln(\delta+e^{-s}),x)|^{p+1} dx$$

which requires $2\beta + 2 - N > 0$.

From this proposition and Lemma 2.1, we have that:

Corollary 2.1. *For all $s \geq -\ln T$, $s_2 \geq s_1 \geq -\ln T$, the following inequalities hold*:

$$0 \leq E(s) \leq E(-\log T) \leq C_0,$$

$$\int_{s_1}^{s_2}\int_\Omega \left(e^{-2\tau}|\nabla w(\tau,y)|^2 + |\partial_s w(\tau,y)|^2\right) dy d\tau \leq C_0,$$

where the constant $C_0 > 0$ depends only on the blow-up time T and the norm of initial data.

Remark 2.2. From this corollary, we get the inequality (1.6) in Theorem 1.1.

3. Uniform bounds on w

In this section, we will control the L^2- and H^1-norm of w.

Proof of Theorem 1.1. For any fixed $s \geq -\ln T + 1$, we choose $s_1 \in [s-1, s]$ and $s_2 \in [s+1, s+2]$, which will be determined later. Multiplying (1.5) by w and integrating over $[s_1, s_2] \times \Omega$, it yields

$$\begin{aligned} (3.1) \qquad & \int_{s_1}^{s_2} \int_\Omega |w|^{p+1} dx d\tau \\ = & \left[\int_\Omega w w_s \right]_{s_1}^{s_2} \\ & + \int_{s_1}^{s_2} \int_\Omega \left[-|w_s|^2 + e^{-2\tau} |\nabla w|^2 + (2\beta+1) w_s w + \beta(\beta+1)|w|^2 \right] dx d\tau. \end{aligned}$$

By the definition of $E(s)$, we get

$$\int_\Omega \left[e^{-2s} |\nabla w|^2 + \beta(\beta+1)|w|^2 \right] dx = 2E(s) - \int_\Omega \left[|w_s|^2 - \frac{2}{p+1} |w|^{p+1} \right] dx,$$

which, combining with (3.1), gives that

$$\begin{aligned} (3.2) \qquad & \frac{p-1}{p+1} \int_{s_1}^{s_2} \int_\Omega |w|^{p+1} dx d\tau \\ = & \left[\int_\Omega w w_s \right]_{s_1}^{s_2} + 2 \int_{s_1}^{s_2} E(\tau) d\tau \\ & - 2 \int_{s_1}^{s_2} \int_\Omega |w_s|^2 dx d\tau + (2\beta+1) \int_{s_1}^{s_2} \int_\Omega w_s w dx d\tau. \end{aligned}$$

Via Hölder inequality and Cauchy-Schwarz inequality, the last two terms in (3.2) yield

$$\begin{aligned} (3.3) \qquad & -2 \int_{s_1}^{s_2} \int_\Omega |w_s|^2 dx d\tau + (2\beta+1) \int_{s_1}^{s_2} \int_\Omega w_s w dx d\tau \\ \leq & -2 \int_{s_1}^{s_2} \int_\Omega |w_s|^2 dx d\tau + (2\beta+1) \\ & \left(\int_{s_1}^{s_2} \int_\Omega |w_s(\tau,x)|^2 dx d\tau \right)^{\frac{1}{2}} \left(\int_{s_1}^{s_2} \int_\Omega |w(\tau,x)|^2 dx d\tau \right)^{\frac{1}{2}} \\ \leq & C \int_{s_1}^{s_2} \int_\Omega |w(\tau,x)|^2 dx d\tau \leq C \left(\int_{s_1}^{s_2} \int_\Omega |w|^{p+1} dx d\tau \right)^{\frac{2}{p+1}} \\ \leq & \frac{p-1}{2(p+1)} \int_{s_1}^{s_2} \int_\Omega |w|^{p+1} dx d\tau + C, \end{aligned}$$

where the positive constant C relies on Ω, N and p.

Combining (3.2) with (3.3) and using Hölder inequality and Cauchy-Schwarz inequality, we get that

$$\begin{aligned}(3.4)\qquad &\int_{s_1}^{s_2}\int_\Omega |w|^{p+1}dxd\tau \\ &\le C\varepsilon\left[\|w(s_1)\|_{L^2}^2+\|w(s_2)\|_{L^2}^2\right]+\frac{C}{\varepsilon}\left[\|w_s(s_1)\|_{L^2}^2+\|w_s(s_2)\|_{L^2}^2\right]+C,\end{aligned}$$

where $\varepsilon > 0$ will be decided later.

To estimate the first term in right hand of (3.4), we give the following claim.

Claim 3.1. There exists a positive constant C, depending only on Ω, N, p and the initial data, such that

$$(3.5)\qquad \sup_{s_1\le s\le s_2}\int_\Omega |w(s,x)|^2dx \le C + C\int_{s_1}^{s_2}\int_\Omega |w|^{p+1}dxd\tau.$$

Proof of Claim 3.1. By the mean value theorem, we know there exists $s_0 \in [s_1, s_2]$ such that

$$(3.6)\quad \int_\Omega |w(s_0,x)|^2dx = \frac{1}{s_2-s_1}\int_{s_1}^{s_2}\int_\Omega |w|^2dxd\tau \le \int_{s_1}^{s_2}\int_\Omega |w|^{p+1}dxd\tau + C.$$

Here we use Hölder inequality and Cauchy-Schwarz inequality and note $1 \le s_2 - s_1 \le 3$. For any $s \in [s_1, s_2]$,

$$\begin{aligned}\int_\Omega |w(s,x)|^2dx &= \int_\Omega |w(s_0,x)|^2dx + \int_{s_0}^{s}\frac{d}{ds}\int_\Omega |w(\tau,x)|^2dxd\tau \\ &\le \int_\Omega |w(s_0,x)|^2dx + \int_{s_1}^{s_2}\int_\Omega |w(\tau,x)|^2dxd\tau \\ &\quad + \int_{s_1}^{s_2}\int_\Omega |w_s(\tau,x)|^2dxd\tau \\ &\le C\int_{s_1}^{s_2}\int_\Omega |w|^{p+1}dxd\tau + C,\end{aligned}$$

where we use (3.6) and Corollary 2.1 in the last inequality. This completes the proof of Claim 3.1. □

By mean value theorem, we choose s_1 and s_2 such that

$$\begin{aligned}\int_\Omega |w_s(s_1,x)|^2dx &= \int_{s-1}^{s}\int_\Omega |w_s(\tau,x)|^2dxd\tau \quad\text{and}\\ \int_\Omega |w_s(s_2,x)|^2dx &= \int_{s+1}^{s+2}\int_\Omega |w_s(\tau,x)|^2dxd\tau,\end{aligned}$$

which, together with (3.4), Claim 3.1 and Corollary 2.1 and choosing $\varepsilon = \frac{1}{4C^2}$, yields that

$$(3.7)\qquad \int_{s_1}^{s_2}\int_\Omega |w|^{p+1}dxd\tau \le C.$$

Immediately, via Claim 3.1, we obtain that for any $s \in [-\ln T + 1, \infty)$

$$\int_\Omega |w(s,x)|^2 dx \le C, \tag{3.8}$$

which gets the inequality (1.7) in Theorem 1.1.

To complete the proof of Theorem 1.1, we need to estimate $\int_\Omega |w(s,x)|^{p+1} dx$ under the condition $1 < p < 1 + \frac{4}{N-1}$ for $N \ge 2$ or $1 < p < 5$ for $N = 1$.

Claim 3.2. There exists a positive constant $\theta \in (0,1)$ such that

$$\int_\Omega |w(s,x)|^{p+1} dx \le C \left(\int_\Omega |\nabla w(s,x)|^2 dx \right)^\theta$$

provided $1 < p < 1 + \frac{4}{N-1}$ for $N \ge 2$ or $1 < p < 5$ for $N = 1$.

Proof of Claim 3.2. From (3.7), we know that there exists $\tilde{s} \in [s_1, s_2]$ such that

$$\int_\Omega |w(\tilde{s},x)|^{p+1} dx = \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \int_\Omega |w(\tau,x)|^{p+1} dx d\tau \le C. \tag{3.9}$$

Via Hölder inequality, (3.9) implies

$$\int_\Omega |w(\tilde{s},x)|^r dx \le C, \tag{3.10}$$

where $r := \frac{p+3}{2}$. For any $s \in [s_1, s_2]$, we have that

$$\begin{aligned} \int_\Omega |w(s,x)|^r dx &= \int_\Omega |w(\tilde{s},x)|^r dx + \int_{\tilde{s}}^s \frac{d}{ds} \int_\Omega |w(\tau,x)|^r dx d\tau \\ &\le \int_\Omega |w(\tilde{s},x)|^r dx + C \int_{s_1}^{s_2} \int_\Omega |w|^{p+1} dx d\tau \\ &\quad + C \int_{s_1}^{s_2} \int_\Omega |w_s|^2 dx d\tau, \end{aligned}$$

which, together with (3.7), (3.10) and Corollary 2.1, yields

$$\int_\Omega |w(s,x)|^r dx \le C. \tag{3.11}$$

Using Sobolev embedding inequalities, (3.11) gives

$$\begin{aligned} \int_\Omega |w(s,x)|^{p+1} dx &\le C \left(\int_\Omega |w(s,x)|^r dx \right)^{\frac{2(p+1)(1-\alpha)}{p+3}} \left(\int_\Omega |\nabla w(s,x)|^2 dx \right)^{\frac{\alpha(p+1)}{2}} \\ &\le C \left(\int_\Omega |\nabla w(s,x)|^2 dx \right)^\theta, \end{aligned}$$

where $\theta := \frac{\alpha(p+1)}{2}$ and α satisfies $\frac{1}{p+1} = \frac{2(1-\alpha)}{p+3} + \frac{\alpha(N-2)}{2N}$ for $N \ge 3$ or $\alpha = \frac{p-1}{2(p+1)}$ for $N = 1$ and 2. Calculating clearly, we know that

$$\theta = \frac{N(p-1)}{N + 6 - (N-2)p} \text{ for } N \ge 3 \text{ or } \theta = \frac{p-1}{4} \text{ for } N = 1 \text{ and } 2,$$

which, together with the assumption of p, gives that $0 < \theta < 1$. □

Using Young inequality, by Claim 3.2, we get that

$$\begin{aligned}\int_\Omega |w(s,x)|^{p+1}dx &\le C\left(\int_\Omega e^{-2s}|\nabla w(s,x)|^2dx\right)^\theta e^{2\theta s}\\ &\le \varepsilon\int_\Omega e^{-2s}|\nabla w(s,x)|^2dx + C(\varepsilon)(1-\theta)e^{\frac{2\theta s}{1-\theta}}.\end{aligned} \tag{3.12}$$

Combining (3.12) with the definition of $E(s)$ and Corollary 2.1, we get

$$\int_\Omega [|w_s(s,x)|^2 + e^{-2s}|\nabla w(s,x)|^2 + |w(s,x)|^2]dx \le C(1+e^{2\gamma s}). \tag{3.13}$$

Here $\gamma := \frac{N(p-1)}{2N+6-2(N-1)p}$ for $N \ge 3$ while $\gamma := \frac{p-1}{5-p}$ for $N = 1$ or 2. So we complete the proof of Theorem 1.1. □

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