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OSCILLATION CRITERIA FOR DIFFERENCE EQUATIONS WITH SEVERAL OSCILLATING COEFFICIENTS

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Abstract. This paper presents a new sufficient condition for the oscillation of all solutions of difference equations with several deviating arguments and oscillating coefficients. Corresponding difference equations of both retarded and advanced type are studied. Examples illustrating the results are also given.

1. Introduction

Let $m \in \mathbb{N}$. Consider the retarded (delayed) difference equation of the form

$$(\mathrm{E_R}) \qquad \Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0,$$

where, for all $i \in \{1, \dots, m\}$, $p_i : \mathbb{N}_0 \to \mathbb{R}$, $\tau_i : \mathbb{N}_0 \to \mathbb{Z}$ such that

$$(1.1) \qquad \tau_i(n) \le n - 1, \quad n \in \mathbb{N}_0 \quad \text{and} \quad \lim_{n\to\infty} \tau_i(n) = \infty$$

and the (dual) advanced difference equation of the form

$$(\mathrm{E_A}) \qquad \nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) = 0, \quad n \in \mathbb{N},$$

where, for all $i \in \{1, \dots, m\}$, $p_i : \mathbb{N} \to \mathbb{R}$, $\sigma_i : \mathbb{N} \to \mathbb{N}$ such that

$$(1.2) \qquad \sigma_i(n) \ge n + 1, \quad n \in \mathbb{N}, \quad 1 \le i \le m.$$

Here, as usual, Δ denotes the forward difference operator and ∇ denotes the backward difference operator defined by

$$\Delta x(n) = x(n+1) - x(n) \quad \text{and} \quad \nabla x(n) = x(n) - x(n-1), \quad n \in \mathbb{Z}.$$

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Strong interest in $(\mathrm{E_R})$ is motivated by the fact that it represents a discrete analogue of the differential equation (see [4] and the references cited therein)

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \geq 0, \tag{1.3}$$

where, for all $i \in \{1, \dots, m\}$, $p_i : [0, \infty) \to \mathbb{R}$ is oscillating and continuous and $\tau_i : [0, \infty) \to \mathbb{R}$ is continuous such that

$$\tau_i(t) \leq t, \quad t \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty, \tag{1.4}$$

while $(\mathrm{E_A})$ represents a discrete analogue of the advanced differential equation (see [4] and the references cited therein)

$$x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) = 0, \quad t \geq 1, \tag{1.5}$$

where, for all $i \in \{1, \dots, m\}$, $p_i : [1, \infty) \to \mathbb{R}$ is oscillating and continuous and $\sigma_i : [1, \infty) \to \mathbb{R}$ is continuous such that

$$\sigma_i(t) \geq t, \quad t \geq 1. \tag{1.6}$$

By a *solution* of $(\mathrm{E_R})$ we mean a sequence of real numbers $\{x(n)\}_{n\geq -w}$ which satisfies $(\mathrm{E_R})$ for all $n \in \mathbb{N}_0$. Here,

$$w = -\min_{\substack{n\in\mathbb{N}_0\\ 1\leq i\leq m}} \tau_i(n).$$

It is clear that, for each choice of real numbers c_{-w}, c_{-w+1}, …, c_{-1}, c_0, there exists a unique solution $\{x(n)\}_{n\geq -w}$ of $(\mathrm{E_R})$ which satisfies the initial conditions $x(-w) = c_{-w}$, $x(-w+1) = c_{-w+1}$, …, $x(-1) = c_{-1}$, $x(0) = c_0$. By a *solution* of the advanced difference equation $(\mathrm{E_A})$ we mean a sequence of real numbers $\{x(n)\}_{n\in\mathbb{N}_0}$ which satisfies $(\mathrm{E_A})$ for all $n \in \mathbb{N}$.

A solution $\{x(n)\}_{n\geq -w}$ (or $\{x(n)\}_{n\in\mathbb{N}_0}$) of $(\mathrm{E_R})$ (or $(\mathrm{E_A})$) is called *oscillatory* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*. In the last few decades, oscillatory behavior of all solutions of difference equations has been extensively studied when the coefficients $p_i(n)$ are nonnegative. However, for the general case when $p_i(n)$ are allowed to oscillate, it is difficult to study oscillation of $(\mathrm{E_R})$ and $(\mathrm{E_A})$. Therefore, the results on oscillation of difference and differential equations with oscillating coefficients are relatively scarce. Thus, only a small number of papers is dealing with this case. See, for example, [2, 3, 7–9, 11–16] and the references cited therein. For the general theory of difference equations, the reader is referred to the monographs [1,6,10].

For (1.3) with $p_i(t) \geq 0$ for all $i \in \{1, \dots, m\}$, Grammatikopoulos, Koplatadze and Stavroulakis [5] established the following theorem.

Theorem 1.1 (See [5, Theorems 2.5 and 2.6]). *Assume that τ_i are increasing for all $i \in \{1, \dots, m\}$,*

$$\int_0^\infty |p_i(s) - p_j(s)| \mathrm{d}s < \infty, \quad 1 \le i, j \le m$$

and

$$\liminf_{t\to\infty} \int_{\tau_i(t)}^t p_i(s)\mathrm{d}s = \beta_i, \quad 1 \le i \le m.$$

If, moreover

$$\min\left\{\sum_{i=1}^m \frac{e^{\beta_i \lambda}}{\lambda} : \lambda \in (0,\infty)\right\} > 1 \quad \text{or} \quad \sum_{i=1}^m \beta_i > \frac{1}{e},$$

then all solutions of (1.3) *oscillate.*

For (1.3) and (1.5) with $p_i(t) \ge 0$ for all $i \in \{1, \dots, m\}$, Fukagai and Kusano [4] established the following theorems.

Theorem 1.2 (See [4, Theorem 1′ (i)]). *Assume* (1.4) *and that there exists a continuous nondecreasing $\tilde{\tau}$ such that $\tau_i(t) \le \tilde{\tau}(t) \le t$ for $t \ge 0$, $1 \le i \le m$. If*

$$\liminf_{t\to\infty} \int_{\tilde{\tau}(t)}^t \sum_{i=1}^m p_i(s)\mathrm{d}s > \frac{1}{e},$$

then all solutions of (1.3) *oscillate.*

Theorem 1.3 (See [4, Theorem 1′ (ii)]). *Assume* (1.6) *and that there exists a continuous nondecreasing $\tilde{\sigma}$ such that $t \le \tilde{\sigma}(t) \le \sigma_i(t)$ for $t \ge 0$, $1 \le i \le m$. If*

$$\liminf_{t\to\infty} \int_t^{\tilde{\sigma}(t)} \sum_{i=1}^m p_i(s)\mathrm{d}s > \frac{1}{e},$$

then all solutions of (1.5) *oscillate.*

In the same paper [4], the authors also studied the oscillating coefficients case and established the following theorems.

Theorem 1.4 (See [4, Theorem 3′ (i)]). *Assume* (1.4) *and that there exists a continuous nondecreasing $\tilde{\tau}$ such that $\tau_i(t) \le \tilde{\tau}(t) \le t$ for $t \ge 0$, $1 \le i \le m$. Suppose moreover that there exists $\{t(n)\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t(n) = \infty$,*

$$[\tilde{\tau}^n(t(n)), t(n)] \quad \text{are disjoint for all} \quad n \in \mathbb{N},$$

and

$$p_i(t) \ge 0 \quad \text{for all} \quad t \in \bigcup_{n\in\mathbb{N}} [\tilde{\tau}^n(t(n)), t(n)], \quad 1 \le i \le m.$$

If there exists a constant c such that

$$\int_{\tilde{\tau}(t)}^t \sum_{i=1}^m p_i(s)\mathrm{d}s > c > \frac{1}{e} \quad \text{for all} \quad t \in \bigcup_{n\in\mathbb{N}} \left[\tilde{\tau}^{n-1}(t(n)), t(n)\right],$$

then all solutions of (1.3) *oscillate.*

Theorem 1.5 (See [4, Theorem 3′ (ii)]). *Assume* (1.6) *and that there exists a continuous nondecreasing $\tilde{\sigma}$ such that $t \le \tilde{\sigma}(t) \le \sigma_i(t)$ for $t \ge 0$, $1 \le i \le m$. Suppose moreover that there exists $\{t(n)\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t(n) = \infty$,*

$$[t(n), \tilde{\sigma}^n(t(n))] \quad \text{are disjoint for all} \quad n \in \mathbb{N},$$

and

$$p_i(t) \ge 0 \quad \text{for all} \quad t \in \bigcup_{n\in\mathbb{N}} [t(n), \tilde{\sigma}^n(t(n))], \quad 1 \le i \le m.$$

If there exists a constant c such that

$$\int_t^{\tilde{\sigma}(t)} \sum_{i=1}^m p_i(s)\mathrm{d}s > c > \frac{1}{e} \quad \text{for all} \quad t \in \bigcup_{n\in\mathbb{N}} \left[t(n), \tilde{\sigma}^{n-1}(t(n))\right],$$

then all solutions of (1.5) *oscillate.*

For $(\mathrm{E_R})$ and $(\mathrm{E_A})$ with $p_i(n) \ge 0$ for all $i \in \{1, \ldots, m\}$, Chatzarakis, Pinelas and Stavroulakis [3] established the following results.

Theorem 1.6 (See [3, Theorems 2.1 and 2.2]). *Assume* (1.1) *and that τ_i is increasing for all $i \in \{1, \ldots, m\}$. If*

$$\limsup_{n\to\infty} \sum_{i=1}^m \sum_{j=\tau(n)}^{n} p_i(j) > 1,$$

where $\tau(n) = \max_{1\le i\le m} \tau_i(n)$, $n \in \mathbb{N}_0$, or

$$\limsup_{n\to\infty} \sum_{i=1}^m p_i(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{i=1}^m \sum_{j=\tau_i(n)}^{n-1} p_i(j) > \frac{1}{e}, \tag{1.7}$$

then all solutions of $(\mathrm{E_R})$ *oscillate.*

Theorem 1.7 (See [3, Theorems 3.1 and 3.2]). *Assume* (1.2) *and that σ_i is increasing for all $i \in \{1, \ldots, m\}$. If*

$$\limsup_{n\to\infty} \sum_{i=1}^m \sum_{j=n}^{\sigma(n)} p_i(j) > 1,$$

where $\sigma(n) = \min_{1\le i\le m} \sigma_i(n)$, $n \in \mathbb{N}$, or

$$\limsup_{n\to\infty} \sum_{i=1}^m p_i(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{i=1}^m \sum_{j=n+1}^{\sigma_i(n)} p_i(j) > \frac{1}{e}, \tag{1.8}$$

then all solutions of $(\mathrm{E_A})$ *oscillate.*

For equations $(\mathrm{E_R})$ and $(\mathrm{E_A})$ with oscillating coefficients, very recently, Bohner, Chatzarakis and Stavroulakis [2] established the following results.

Theorem 1.8 (See [2, Theorem 2.4]). *Assume* (1.1) *and that, for all* $i \in \{1, \ldots, m\}$, τ_i *is increasing and there exists* $n_i : \mathbb{N} \to \mathbb{N}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$ *and*

(1.9)
$$p_k(n) \geq 0 \quad \textit{for all} \quad n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m,$$

where

$$\tau(n) = \max_{1\leq i\leq m} \tau_i(n), \quad n \in \mathbb{N}_0. \tag{1.10}$$

If, moreover

$$\limsup_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)} p_i(q) > 1, \tag{1.11}$$

where $n(j) = \min\{n_i(j) : 1 \leq i \leq m\}$, *then all solutions of* $(\mathrm{E_R})$ *oscillate.*

Theorem 1.9 (See [2, Theorem 3.4]). *Assume* (1.2) *and that, for all* $i \in \{1, \ldots, m\}$, σ_i *is increasing and there exists* $n_i : \mathbb{N} \to \mathbb{N}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$ *and*

(1.12)
$$p_k(n) \geq 0 \quad \textit{for all} \quad n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma(\sigma(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m,$$

where

$$\sigma(n) = \min_{1\leq i\leq m} \sigma_i(n), \quad n \in \mathbb{N}. \tag{1.13}$$

If, moreover

$$\limsup_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n(j)}^{\sigma(n(j))} p_i(q) > 1, \tag{1.14}$$

where $n(j) = \max\{n_i(j) : 1 \leq i \leq m\}$, *then all solutions of* $(\mathrm{E_A})$ *oscillate.*

An interesting question then arises whether there exist the analogues of (1.7) and (1.8) for $(\mathrm{E_R})$ and $(\mathrm{E_A})$ in the case of oscillating coefficients. In the present paper, optimal conditions for the oscillation of all solutions of $(\mathrm{E_R})$ and $(\mathrm{E_A})$ are established and a positive answer to the above question is given. Examples illustrating the main results are also given.

2. Retarded equations

In this section, we present a new sufficient condition for the oscillation of all solutions of $(\mathrm{E_R})$ under the assumption that the sequences τ_i are increasing for all $i \in \{1, \ldots, m\}$.

Theorem 2.1. *Assume* (1.1) *and that, for all* $i \in \{1, \ldots, m\}$, τ_i *is increasing and there exists* $n_i : \mathbb{N} \to \mathbb{N}$ *with* $\lim_{j\to\infty} n_i(j) = \infty$,

(2.1)
$$p_k(n) \geq 0 \quad \text{for all} \quad n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau_i(\tau_i(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m,$$

and

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \quad \text{for all} \quad n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau_i(\tau_i(n_i(j))), n_i(j)] \cap \mathbb{N} \right\}. \tag{2.2}$$

If, moreover

$$\liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau_i(n_i(j))}^{n_i(j)-1} p_i(q) > \frac{1}{e}, \tag{2.3}$$

then all solutions of $(\mathrm{E_R})$ *oscillate.*

Proof. Assume, for the sake of contradiction, that $\{x(n)\}_{n\geq -w}$ is an eventually positive solution of $(\mathrm{E_R})$. Then, in view of (2.1), it is clear that there exists $j_0 \in \mathbb{N}$ such that

$$p_k(n) \geq 0 \quad \text{for all} \quad n \in \bigcap_{i=1}^{m} [\tau_i(\tau_i(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}, \quad 1 \leq k \leq m, \tag{2.4}$$

$$x(\tau_k(n)) > 0 \quad \text{for all} \quad n \in \bigcap_{i=1}^{m} [\tau_i(\tau_i(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}, \quad 1 \leq k \leq m, \tag{2.5}$$

and

$$\sum_{i=1}^{m} \sum_{q=\tau_i(n_i(j_0))}^{n_i(j_0)-1} p_i(q) > \frac{1}{e} + \varepsilon_0 \tag{2.6}$$

for some $\varepsilon_0 > 0$. In view of (2.4) and (2.5), $(\mathrm{E_R})$ gives

$$x(n+1) - x(n) = -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leq 0$$

for all $n \in \bigcap_{i=1}^{m} [\tau_i(\tau_i(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}$. This guarantees that the sequence x is decreasing on $\bigcap_{i=1}^{m} [\tau_i(\tau_i(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}$. Set

$$b_i(n_i(j_0)) = \left(\frac{n_i(j_0) - \tau_i(n_i(j_0))}{n_i(j_0) - \tau_i(n_i(j_0)) + 1} \right)^{n_i(j_0)-\tau_i(n_i(j_0))+1}, \quad 1 \leq i \leq m. \tag{2.7}$$

Clearly

$$\frac{1}{4} \leq b_i(n_i(j_0)) \leq \frac{1}{e}, \quad 1 \leq i \leq m. \tag{2.8}$$

Let

$$d = 1 + e\varepsilon_0. \tag{2.9}$$

Combining (2.6), (2.8) and (2.9), we obtain

$$\begin{aligned}
\sum_{i=1}^{m} \sum_{q=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \frac{p_i(q)}{b_i(n_i(j_0))} &\geq \sum_{i=1}^{m} \sum_{q=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \frac{p_i(q)}{1/e} \\
&= e \sum_{i=1}^{m} \sum_{q=\tau_i(n_i(j_0))}^{n_i(j_0)-1} p_i(q) > e\left(\frac{1}{e} + \varepsilon_0\right) = d > 1,
\end{aligned}$$

i.e.,

$$\sum_{i=1}^{m} \sum_{q=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \frac{p_i(q)}{b_i(n(j_0))} > d > 1. \tag{2.10}$$

Since $\{x(n)\}_{n\geq -w}$ is decreasing on $\bigcap_{i=1}^{m} [\tau_i(\tau_i(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}$, clearly

$$\frac{x(\tau_i(n_i(j_0)))}{x(n_i(j_0))} \geq 1, \quad 1 \leq i \leq m. \tag{2.11}$$

By $(\mathrm{E_R})$, we have

$$\frac{x(n_i(j_0)+1)}{x(n_i(j_0))} = 1 - \sum_{i=1}^{m} p_i(n_i(j_0)) \frac{x(\tau_i(n_i(j_0)))}{x(n_i(j_0))}. \tag{2.12}$$

On the other hand,

$$\begin{aligned}
\frac{x(n_i(j_0))}{x(\tau_i(n_i(j_0)))} &= \prod_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \frac{x(\lambda+1)}{x(\lambda)} \\
&= \prod_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \left(1 - \sum_{i=1}^{m} p_i(\lambda) \frac{x(\tau_i(\lambda))}{x(\lambda)}\right) \\
&\leq \prod_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \left(1 - \sum_{i=1}^{m} p_i(\lambda)\right),
\end{aligned}$$

i.e.,

$$\frac{x(n_i(j_0))}{x(\tau_i(n_i(j_0)))} \leq \prod_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \left(1 - \sum_{i=1}^{m} p_i(\lambda)\right). \tag{2.13}$$

By using (2.13) and the well-known arithmetic-geometric mean inequality, we find

$$\frac{x(n_i(j_0))}{x(\tau_i(n_i(j_0)))} \leq \prod_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \left(1 - \sum_{i=1}^{m} p_i(\lambda)\right)$$

$$\leq \left[1 - \frac{1}{n_i(j_0) - \tau_i(n_i(j_0))} \sum_{i=1}^{m} \sum_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} p_i(\lambda)\right]^{n_i(j_0)-\tau_i(n_i(j_0))},$$

i.e.,

$$\begin{aligned} &\frac{x(\tau_i(n_i(j_0)))}{x(n_i(j_0))} \\ &\geq \left[1 - \frac{1}{n_i(j_0) - \tau_i(n_i(j_0))} \sum_{i=1}^{m} \sum_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} p_i(\lambda)\right]^{-(n_i(j_0)-\tau_i(n_i(j_0)))}. \end{aligned} \tag{2.14}$$

In view of

$$y(1-y)^\rho \leq \frac{\rho^\rho}{(1+\rho)^{1+\rho}} \quad \text{for all} \quad y \in (0,1) \quad \text{and} \quad \rho \in \mathbb{N},$$

inequality (2.14) gives

$$\begin{aligned} &\frac{x(\tau_i(n_i(j_0)))}{x(n_i(j_0))} \\ &\geq \sum_{i=1}^{m} \sum_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} p_i(\lambda) \left(\frac{n_i(j_0) - \tau_i(n_i(j_0)) + 1}{n_i(j_0) - \tau_i(n_i(j_0))}\right)^{n_i(j_0)-\tau_i(n_i(j_0))+1}. \end{aligned} \tag{2.15}$$

Combining (2.15), (2.7) and (2.10), we obtain

$$\frac{x(\tau_i(n_i(j_0)))}{x(n_i(j_0))} \geq \sum_{i=1}^{m} \sum_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \frac{p_i(\lambda)}{b_i(n_i(j_0))} > d. \tag{2.16}$$

Similarly,

$$\begin{aligned} \frac{x(n_i(j_0))}{x(\tau_i(n_i(j_0)))} &= \prod_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \frac{x(\lambda+1)}{x(\lambda)} \\ &= \prod_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \left(1 - \sum_{i=1}^{m} p_i(\lambda) \frac{x(\tau_i(\lambda))}{x(\lambda)}\right) \\ &\leq \prod_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \left(1 - d \sum_{i=1}^{m} p_i(\lambda)\right) \\ &\leq \left[1 - \frac{d}{n_i(j_0) - \tau_i(n_i(j_0))} \sum_{i=1}^{m} \sum_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} p_i(\lambda)\right]^{n_i(j_0)-\tau_i(n_i(j_0))}. \end{aligned}$$

Therefore

$$\frac{x(\tau_i(n_i(j_0)))}{x(n_i(j_0))} \geq d \sum_{i=1}^{m} \sum_{\lambda=\tau_i(n_i(j_0))}^{n_i(j_0)-1} \frac{p_i(\lambda)}{b_i(n_i(j_0))} > d^2.$$

Applying this procedure k times, we obtain

$$\frac{x(\tau_i(n_i(j_0)))}{x(n_i(j_0))} > d^k. \tag{2.17}$$

On the other hand, since (2.2) holds, there exists a subsequence of integers $\theta(n_i(j_0))$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{m} p_i(\theta(n_i(j_0))) \geq c > 0.$$

By (2.12), we have

$$0 < \frac{x(n_i(j_0)+1)}{x(n_i(j_0))} = 1 - \sum_{i=1}^{m} p_i(n_i(j_0)) \frac{x(\tau_i(n_i(j_0)))}{x(n_i(j_0))},$$

i.e.,

$$\frac{x(\tau(n_i(j_0)))}{x(n_i(j_0))} \sum_{i=1}^{m} p_i(n_i(j_0)) < 1,$$

where τ is defined by (1.10). Thus

$$\frac{x(\tau(\theta(n_i(j_0))))}{x(\theta(n_i(j_0)))} \sum_{i=1}^{m} p_i(\theta(n_i(j_0))) < 1,$$

i.e.,

$$\frac{x(\tau(\theta(n_i(j_0))))}{x(\theta(n_i(j_0)))} < \frac{1}{\sum_{i=1}^{m} p_i(\theta(n_i(j_0)))} \leq \frac{1}{c} < \infty,$$

i.e., $\liminf_{n\to\infty} \frac{x(\tau(n(j_0)))}{x(n(j_0))}$ exists. This contradicts (2.17). □

A slight modification in the proof of Theorem 2.1 leads to the following result about retarded difference inequalities.

Theorem 2.2. *Assume that all conditions of Theorem 2.1 hold. Then*

(i) *the difference inequality*

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leq 0, \quad n \in \mathbb{N}_0$$

has no eventually positive solutions;

(ii) *the difference inequality*

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \geq 0, \quad n \in \mathbb{N}_0$$

has no eventually negative solutions.

3. Advanced equations

Oscillation of all solutions of (E_A) is described by the following result. Note that the proof is an easy modification of the proof of Theorem 2.1 and hence is omitted.

Theorem 3.1. *Assume* (1.2) *and that, for all* $i \in \{1, \ldots, m\}$, σ_i *is increasing and there exists* $n_i : \mathbb{N} \to \mathbb{N}$ *with* $\lim_{j\to\infty} n_i(j) = \infty$,

(3.1)
$$p_k(n) \geq 0 \quad \textit{for all} \quad n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma_i(\sigma_i(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m,$$

and

(3.2)
$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \quad \textit{for all} \quad n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma_i(\sigma_i(n_i(j)))] \cap \mathbb{N} \right\}.$$

If, moreover

$$\liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n_i(j)+1}^{\sigma_i(n_i(j))} p_i(q) > \frac{1}{e}, \tag{3.3}$$

then all solutions of (E_A) *oscillate.*

A slight modification in the proof of Theorem 3.1 leads to the following result about advanced difference inequalities.

Theorem 3.2. *Assume that all conditions of Theorem 3.1 hold. Then*

(i) *the difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \geq 0, \quad n \in \mathbb{N}$$

has no eventually positive solutions;

(ii) *the difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \leq 0, \quad n \in \mathbb{N}$$

has no eventually negative solutions.

4. Examples

The significance of the results is illustrated in the following examples. It is also demonstrated that the oscillation conditions of Theorems 2.1 and 1.8 and of Theorems 3.1 and 1.9 are independent.

Example 4.1. Consider the retarded difference equation

$$\Delta x(n) + p_1(n)x(n-2) + p_2(n)x(n-3) = 0, \quad n \in \mathbb{N}_0, \tag{4.1}$$

where

$$p_1(n) = \frac{1}{2\left(2+\sqrt{3}\right)} \cdot \cos\frac{n\pi}{6} \quad \text{and} \quad p_2(n) = \frac{1}{2\left(2+\sqrt{3}\right)} \cdot \sin\frac{n\pi}{6}, \quad n \in \mathbb{N}_0.$$

Here, $\tau_1(n) = n-2$ and $\tau_2(n) = n-3$. Observe that for

$$n_1(j) = 12j + 15, \quad j \in \mathbb{N},$$

we have $p_1(n) \geq 0$ for all $n \in A$, where

$$A = \bigcup_{j\in\mathbb{N}} [\tau_1(\tau_1(n_1(j))), n_1(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [12j+11, 12j+15] \cap \mathbb{N}.$$

Also, for

$$n_2(j) = 12j + 18, \quad j \in \mathbb{N},$$

we have $p_2(n) \geq 0$ for all $n \in B$, where

$$B = \bigcup_{j\in\mathbb{N}} [\tau_2(\tau_2(n_2(j))), n_2(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [12j+12, 12j+18] \cap \mathbb{N}.$$

Therefore

$$p_1(n) \geq 0 \quad \text{and} \quad p_2(n) \geq 0 \quad \text{for all} \quad n \in A \cap B = \bigcup_{j\in\mathbb{N}} [12j+12, 12j+15] \cap \mathbb{N}.$$

Now, for all $n \in A \cap B$, we have

$$\limsup_{n\to\infty} \sum_{i=1}^{2} p_i(n) = \frac{1+\sqrt{3}}{4(2+\sqrt{3})} > 0$$

and

$$\begin{aligned}
\liminf_{j\to\infty} \sum_{i=1}^{2} \sum_{q=\tau_i(n_i(j))}^{n_i(j)-1} p_i(q) &= \liminf_{j\to\infty} \left[\sum_{q=12j+13}^{12j+14} p_1(q) + \sum_{q=12j+15}^{12j+17} p_2(q) \right] \\
&= \frac{1}{2\left(2+\sqrt{3}\right)} \cdot \left(\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\right) \\
&\quad + \frac{1}{2\left(2+\sqrt{3}\right)} \cdot \left(\sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6}\right) \\
&= \frac{1}{2\left(2+\sqrt{3}\right)} \cdot \left(2+\sqrt{3}\right) = \frac{1}{2} > \frac{1}{e},
\end{aligned}$$

i.e., (2.2) and (2.3) of Theorem 2.1 are satisfied, and therefore all solutions of (4.1) oscillate.

On the other hand, by (1.10), it is obvious that $\tau(n) = n-2$. Also,

$$n(j) = \min\{n_i(j) : \ 1 \leq i \leq 2\} = 12j + 15, \quad j \in \mathbb{N}.$$

Observe that $p_1(n) \geq 0$ for all $n \in A' = A$ and $p_2(n) \geq 0$ for all $n \in B'$, where

$$B' = \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_2(j))), n_2(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [12j+14, 12j+18] \cap \mathbb{N}.$$

Therefore

$$p_1(n) \geq 0 \quad \text{and} \quad p_2(n) > 0 \quad \text{for all} \quad n \in A' \cap B' = \bigcup_{j\in\mathbb{N}} [12j+14, 12j+15] \cap \mathbb{N}.$$

Now,

$$\begin{aligned}
\limsup_{j\to\infty} \sum_{i=1}^{2} \sum_{q=\tau(n(j))}^{n(j)} p_i(q) &= \limsup_{j\to\infty} \left[\sum_{q=12j+13}^{12j+15} p_1(q) + \sum_{q=12j+13}^{12j+15} p_2(q) \right] \\
&= \frac{1}{2\left(2+\sqrt{3}\right)} \cdot \left(\cos\frac{\pi}{6} + \cos\frac{\pi}{3} + \cos\frac{\pi}{2} \right) \\
&\quad + \frac{1}{2\left(2+\sqrt{3}\right)} \cdot \left(\sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} \right) \\
&= \frac{1}{2\left(2+\sqrt{3}\right)} \cdot \left(2+\sqrt{3}\right) = \frac{1}{2} < 1,
\end{aligned}$$

i.e., (1.11) of Theorem 1.8 is not satisfied.

Example 4.2. Consider the advanced difference equation

$$\nabla x(n) - p_1(n)x(n+1) - p_2(n)x(n+3) = 0, \quad n \in \mathbb{N}, \tag{4.2}$$

where

$$p_1(n) = \frac{1}{5+\sqrt{3}} \cdot \cos\frac{n\pi}{6} \quad \text{and} \quad p_2(n) = \frac{1}{5+\sqrt{3}} \cdot \sin\frac{n\pi}{6}, \quad n \in \mathbb{N}.$$

Here, $\sigma_1(n) = n+1$ and $\sigma_2(n) = n+3$. Observe that for

$$n_1(j) = 12j+11, \quad j \in \mathbb{N},$$

we have $p_1(n) > 0$ for all $n \in A$, where

$$A = \bigcup_{j\in\mathbb{N}} [n_1(j), \sigma_1(\sigma_1(n_1(j)))] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [12j+11, 12j+13] \cap \mathbb{N}.$$

Also, for

$$n_2(j) = 12j+12, \quad j \in \mathbb{N},$$

we have $p_2(n) \geq 0$ for all $n \in B$, where

$$B = \bigcup_{j\in\mathbb{N}} [n_2(j), \sigma_2(\sigma_2(n_2(j)))] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [12j+12, 12j+18] \cap \mathbb{N}.$$

Therefore

$$p_1(n) > 0 \quad \text{and} \quad p_2(n) \geq 0 \quad \text{for all} \quad n \in A \cap B = \bigcup_{j\in\mathbb{N}} [12j+12, 12j+13] \cap \mathbb{N}.$$

Now, for all $n \in A \cap B$, we have

$$\limsup_{n\to\infty} \sum_{i=1}^{2} p_i(n) = \frac{1+\sqrt{3}}{2(5+\sqrt{3})} > 0$$

and

$$\begin{aligned}
&\liminf_{j\to\infty} \sum_{i=1}^{2} \sum_{q=n_i(j)+1}^{\sigma_i(n_i(j))} p_i(q) \\
&= \liminf_{j\to\infty} \left[\sum_{q=12j+12}^{12j+12} p_1(q) + \sum_{q=12j+13}^{12j+15} p_2(q) \right] \\
&= \frac{1}{5+\sqrt{3}} \cdot \left(\cos 0 + \sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} \right) = \frac{1}{2} > \frac{1}{e},
\end{aligned}$$

i.e., (3.2) and (3.3) of Theorem 3.1 are satisfied, and therefore all solutions of (4.2) oscillate.

On the other hand, by (1.13), it is obvious that $\sigma(n) = n + 1$. Also,

$$n(j) = \max\{n_i(j) :\ 1 \le i \le 2\} = 12j + 12, \quad j \in \mathbb{N}.$$

Observe that $p_1(n) > 0$ for all $n \in A' = A$ and $p_2(n) \ge 0$ for all $n \in B'$, where

$$B' = \bigcup_{j\in\mathbb{N}} [n_2(j), \sigma(\sigma(n_2(j)))] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [12j+12, 12j+14] \cap \mathbb{N}.$$

Therefore

$$p_1(n) > 0 \quad \text{and} \quad p_2(n) \ge 0 \quad \text{for all} \quad n \in A' \cap B' = \bigcup_{j\in\mathbb{N}} [12j+12, 12j+13] \cap \mathbb{N}.$$

Now,

$$\begin{aligned}
\limsup_{j\to\infty} \sum_{i=1}^{2} \sum_{q=n(j)}^{\sigma(n(j))} p_i(q) &= \limsup_{j\to\infty} \left[\sum_{q=12j+12}^{12j+13} p_1(q) + \sum_{q=12j+12}^{12j+13} p_2(q) \right] \\
&= \frac{1}{5+\sqrt{3}} \cdot \left(\cos 0 + \cos\frac{\pi}{6} + \sin 0 + \sin\frac{\pi}{6} \right) \\
&= \frac{3+\sqrt{3}}{2(5+\sqrt{3})} \approx 0.3515 < 1,
\end{aligned}$$

i.e., (1.14) of Theorem 1.9 is not satisfied.

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