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SHADOWABLE CHAIN COMPONENTS AND HYPERBOLICITY

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Abstract. We show that C^1-generically, the shadowable chain component of a C^1-vector field containing a hyperbolic periodic orbit is hyperbolic if it is locally maximal.

1. Introduction

Chain components and homoclinic classes are central objects in the theory of differentiable dynamical systems since they are natural candidates to replace the Smale's hyperbolic basic sets in non-hyperbolic theory of dynamical systems. Many recent papers have explored their hyperbolic-like properties such as dominated splitting, partial hyperbolicity, etc. For instance, Sakai [9] proved that if the chain component $C_f(p)$ of a diffeomorphism f containing a hyperbolic periodic point p is C^1-robustly shadowable and the $C_f(p)$-germ of f is expansive, then $C_f(p)$ is hyperbolic. Moreover Wen *et al.* [11] claimed that the assumption of the $C_f(p)$-germ expansivity of f can be dropped in the above result to show the hyperbolicity of the C^1-robustly shadowable chain component $C_f(p)$. However, it is still open whether the above results can be extended to the case of vector fields. More precisely, the first open problem can be formally stated as follows.

Open Problem 1. *If the chain component $C_X(\gamma)$ of a C^1-vector field X containing a hyperbolic periodic orbit γ is C^1-robustly shadowable, then is it hyperbolic?*

Here, we say that $C_X(\gamma)$ is C^1-*robustly shadowable* if there exists a neighborhood $\mathcal{U}(X)$ of X such that for any $Y \in \mathcal{U}(X)$, $C_Y(\gamma_Y)$ is shadowable, where γ_Y is the continuation of γ. In relation to the above open problem, very recently, Lee *et al.* [6] proved that if the chain component $C_X(\gamma)$ does not contain a singularity, then the answer of Open Problem 1 is "yes".

In generic view point, as far as we know, there are no interesting results on the hyperbolicity of shadowable chain components. Here we suggest two open

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problems in relation to the hyperbolicity of chain components in generic sense. A subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ is called *residual* if it contains a countable intersection of open and dense subsets of $\mathfrak{X}^1(M)$. We will say that a property holds *generically* if there exists a residual subset $\mathcal{R}$ such that any $X \in \mathcal{R}$ has that property.

Open Problem 2. *If the chain component $C_f(p)$ of a diffeomorphism f containing a hyperbolic periodic point p is C^1-generically shadowable, then is it hyperbolic?*

Open Problem 3. *If the chain component $C_X(\gamma)$ of a C^1-vector field X containing a hyperbolic periodic orbit γ is C^1-generically shadowable, then is it hyperbolic?*

In this paper, we claim that the answer of Open Problem 3 is "yes" if $C_X(\gamma)$ is locally maximal. More precisely, we prove that C^1-generically, any shadowable chain component of a C^1-vector field containing a hyperbolic periodic orbit is hyperbolic if it is locally maximal. Note that the chain components may contain singularities.

Let us pass to the main definitions and results. Let M be a compact C^∞ Riemannian manifold without boundary. Denote by $\mathfrak{X}^1(M)$ the set of C^1-vector fields on M endowed with the C^1-topology. Note that every $X \in \mathfrak{X}^1(M)$ generates a C^1-flow $X_t : M \times \mathbb{R} \to M$. We say that a point p in M is *periodic* if there is $T > 0$ such that $X_T(p) = p$, and the set of periodic points of $X \in \mathfrak{X}^1(M)$ will be denoted by $P(X)$. A point $p \in M$ is said to be a *singulairity* of X if $X(p) = 0$, and the set of singularities of X will be denoted by $Sing(X)$. Denote by $Crit(X) = P(X) \cup Sing(X)$, and the elements of $Crit(X)$ are called *critical elements* of X.

Let d be the distance induced from the Riemannian structure on M. A sequence $\{(x_i, t_i) : x_i \in M,\ t_i \geq 1,\ i \in \mathbb{Z}\}$ in $M \times \mathbb{R}$ is called a *δ-pseudo orbit* (or δ-chain) of X if

$$d(X_{t_i}(x_i), x_{i+1}) < \delta$$

for any $i \in \mathbb{Z}$. We say that a compact X_t-invariant set $\Lambda \subset M$ is *shadowable* for X if for any $\epsilon > 0$, there is $\delta > 0$ satisfying the following property: given any δ-pseudo orbit $\{(x_i, t_i) : t_i \geq 1,\ i \in \mathbb{Z}\}$ in Λ, there are a point $y \in M$ and $h \in Rep(\mathbb{R})$ such that

$$d(X_{h(t)}(y), X_{t-T_i}(x_i)) < \epsilon, \quad T_i \leq t < T_{i+1},$$

where $T_i = t_0 + t_1 + \cdots + t_i$ for $i > 0$, $T_i = 0$ for $i = 0$, and $T_i = -(t_{-1} + t_{-2} + \cdots + t_i)$ for $i < 0$; $Rep(\mathbb{R})$ denotes the set of increasing homeomorphisms (reparametrization) $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$.

A point $x \in M$ is said to be *chain recurrent* if for any $\delta > 0$, there exists a δ-pseudo orbit $\{(x_i, t_i) : t_i \geq 1\}_{i=1}^n$ such that $x_0 = x$ and $d(X_{t_{n-1}}(x_{n-1}), x_0) < \delta$.

The set of chain recurrent points of X is called the *chain recurrent set* of X, and will be denoted by $CR(X)$. For any $x, y \in M$, we say that $x \sim y$, if for any $\delta > 0$ there are a δ-pseudo orbit $\{(x_i, t_i) : t_i \geq 1\}_{i=1}^n$ such that $x_0 = x$

and $d(X_{t_{n-1}}(x_{n-1}), y) < \delta$, and a δ-pseudo orbit $\{(x'_i, t'_i) : t'_i \geq 1\}_{i=1}^m$ such that $x'_0 = y$ and $d(X_{t'_{m-1}}(x'_{m-1}), x) < \delta$.

It is easy to see that $\sim$ gives an equivalence relation on the set $CR(X)$. Every equivalence class of $\sim$ is called a *chain component* of X.

A compact invariant set Λ of X is called *hyperbolic* if there are constants $C > 0$ and $\lambda > 0$ such that the tangent flow $DX_t : T_\Lambda M \to T_\Lambda M$ leaves a continuous invariant splitting $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ satisfying

$$\left\|DX_t|_{E^s(x)}\right\| \leq Ce^{-\lambda t} \quad \text{and} \quad \left\|DX_{-t}|_{E^u(x)}\right\| \leq Ce^{-\lambda t}$$

for any $x \in \Lambda$ and $t > 0$, where $\langle X \rangle$ denotes the subspace generated by the vector field X. For any hyperbolic periodic orbit γ of X, we define $\dim(E^s_\gamma) = \mathrm{index}(\gamma)$. Moreover, the sets

$$W^s(\gamma) = \{x \in M : X_t(x) \to \gamma \text{ as } t \to \infty\} \text{ and}$$

$$W^u(\gamma) = \{x \in M : X_t(x) \to \gamma \text{ as } t \to -\infty\}$$

are said to be the *stable manifold* and *unstable manifold*, respectively.

Let γ be a hyperbolic periodic orbit of X. Denote by $C_X(\gamma)$ the chain component of X containing γ. The *homoclinic class* of X associated to γ, denoted by $H_X(\gamma)$, is defined as the closure of the transversal intersection of the stable and unstable manifolds of γ, that is,

$$H_X(\gamma) = \overline{W^u(\gamma) \pitchfork W^s(\gamma)}\,.$$

By definition, we can easily see that both $C_X(\gamma)$ and $H_X(\gamma)$ are closed and X_t-invariant. Moreover we see that $H_X(\gamma) \subset C_X(\gamma)$. Bonatti *et al.* [2] proved the following remarkable results: C^1-generically, $H_X(\gamma) = C_X(\gamma)$.

We recall that a compact invariant set $\Lambda \subset M$ is *locally maximal* if there exists a neighborhood U of Λ such that $\bigcap_{t\in\mathbb{R}} X_t(U) = \Lambda$. In this paper, we use the terminology "for C^1-generic X" to express "there is a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R} \cdots$".

In this paper, we prove the following main theorem.

Main Theorem. *For C^1-generic $X \in \mathfrak{X}^1(M)$, the chain component $C_X(\gamma)$ is hyperbolic if and only if it is shadowable and locally maximal.*

2. Proof of Main Theorem

Let Λ be a closed, X_t-invariant subset of M. If Λ is locally maximal, then there exists $\epsilon > 0$ such that $\bigcap_{t\in\mathbb{R}} X_t(B_\epsilon(\Lambda)) = \Lambda$, where $B_\epsilon(\Lambda)$ is the ϵ-neighborhood of Λ. Suppose Λ is shadowable. Then there exists $\delta > 0$ such that if $\xi = \{(x_i, t_i) : t_i \geq 1,\ i \in \mathbb{Z}\}$ is a δ-pseudo orbit in Λ, then there are $y \in M$ and $h \in Rep(\mathbb{R})$ satisfying

$$d(X_{h(t)}(y), X_{t-T_i}(x_i)) < \epsilon,$$

where $T_i \le t < T_{i+1}$ for all $i \in \mathbb{Z}$. Then we have

$$\mathcal{O}(y) \subset \bigcap_{t\in\mathbb{R}} X_t(B_\epsilon(\Lambda)) = \Lambda.$$

This means that the shadowing point y of the δ-pseudo orbit $\xi = \{(x_i, t_i) : t_i \ge 1,\ i \in \mathbb{Z}\}$ in Λ can be taken from Λ. Let γ be a hyperbolic periodic orbit of X, and let $p \in \gamma$. Then there is $\epsilon = \epsilon(p) > 0$ such that

$$W^s_\epsilon(p) = \{y \in M : d(X_t(y), X_t(p)) < \epsilon,\ t \ge 0\}, \text{ and}$$

$$W^u_\epsilon(p) = \{y \in M : d(X_t(y), X_t(p)) < \epsilon,\ t \le 0\}$$

are C^1-embedded submanifolds of M. Clearly we have $W^s_\epsilon(p) \subset W^s(\gamma)$ and $W^u_\epsilon(p) \subset W^u(\gamma)$.

Lemma 2.1. *Let Λ be a transitive set of M. Suppose Λ is shadowable. Then for any hyperbolic critical elements η, τ in Λ, we have*

$$W^s(\eta) \cap W^u(\tau) \ne \emptyset \text{ and } W^u(\eta) \cap W^s(\tau) \ne \emptyset.$$

Proof. Let η, τ be hyperbolic critical elements in Λ, and take $p \in \eta$ and $q \in \tau$. (If η or τ is singularity, we can put $p = \eta$ or $q = \tau$, respectively.) Choose $\epsilon(p) > 0$ and $\epsilon(q) > 0$ such that $W^s_{\epsilon(p)}(p)$, $W^u_{\epsilon(p)}(p)$, $W^s_{\epsilon(q)}(q)$, and $W^u_{\epsilon(q)}(q)$ are well-defined. Let $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$, and take a constant $0 < \delta < \epsilon$ corresponding to ϵ by the shadowing property of Λ. Since Λ is transitive, we can take $x \in \Lambda$ and constants $l, m > 0$ such that

$$d(X_l(x), p) < \delta \text{ and } d(X_{l+m}(x), q) < \delta.$$

Construct a δ-pseudo orbit $\xi = \{(x_i, t_i) : t_i = 1\}_{i\in\mathbb{Z}}$ in Λ containing p and q as follows:

$$\begin{cases} x_{-i} = X_{-i}(p) & \text{for } \ i \ge 1 \\ x_i = X_{l+i}(x) & \text{for } \ 0 \le i \le m-1 \\ x_{m+i} = X_i(q) & \text{for } \ i \ge 0. \end{cases}$$

Then there is $z \in M$ and $h \in Rep(\mathbb{R})$ such that

$$d(X_{h(t)}(z), X_{t-T_i}(x_i)) < \epsilon$$

for all $i \in \mathbb{Z}$, $T_i \le t < T_{i+1}$. Since $t(1-\epsilon) < h(t) < t(1+\epsilon)$ and $T_i \le t < T_{i+1}$, we know that $h(t) \to \pm\infty$ as $t \to \pm\infty$ which implies that

$$d(X_{h(t)}(z), X_{t-T_i}(x_i)) = d(X_{h(t)}(z), X_{t-T_i}(q)) < \epsilon$$

for $T_i \le i < T_{i+1}$ for all $i \ge m$, and $d(X_t(z), X_t(p)) < \epsilon$ for $t \le 0$. This means that $X_t(z) \in W^s_\epsilon(q)$ for $t > T_m$, and so, $z \in W^s(\tau)$, and $z \in W^u_\epsilon(p) \subset W^u(\eta)$. Thus we can conclude that $W^u(\eta) \cap W^s(\tau) \ne \emptyset$. Similarly, we can show that $W^s(\eta) \cap W^u(\tau) \ne \emptyset$. $\square$

Note that, we can apply Lemma 2.1 to the homoclinic class $H_X(\gamma)$ since the homoclinic class $H_X(\gamma)$ is transitive. A vector field $X \in \mathfrak{X}^1(M)$ is said to be *Kupka-Smale* if any element of $Crit(X)$ is hyperbolic and its invariant manifolds intersect transversely. It is well known that the set of Kupka-Smale vector fields, denoted by $\mathcal{KS}(M)$, is residual in $\mathfrak{X}^1(M)$.

Lemma 2.2. *Let $X \in \mathcal{KS}(M)$, and let γ be a hyperbolic periodic orbit of X. If γ_1, γ_2 are hyperbolic critical elements in $H_X(\gamma)$ with $\dim W^s(\gamma_1) + \dim W^u(\gamma_2) \le \dim M$, then $W^s(\gamma_1) \cap W^u(\gamma_2) = \emptyset$.*

Proof. See [1, Lemma 3.4]. □

Lemma 2.3. *There is a residual subset $\mathcal{R}_1$ of $\mathfrak{X}^1(M)$ such that for any hyperbolic periodic orbit γ of $X \in \mathcal{R}_1$, if the chain component $C_X(\gamma)$ is shadowable, locally maximal and $W^s(\eta) \cap W^u(\tau) \ne \emptyset$ for any critical elements η, τ in $C_X(\gamma)$, then X has no singularities in $C_X(\gamma)$.*

Proof. Take a residual subset $\tilde{\mathcal{R}}$ of $\mathfrak{X}^1(M)$ such that for any $X \in \tilde{\mathcal{R}}$ and a hyperbolic periodic orbit γ of X, we have $C_X(\gamma) = H_X(\gamma)$. Let $\mathcal{R}_1 = \tilde{\mathcal{R}} \cap \mathcal{KS}(M)$, and let γ be a hyperbolic periodic orbit of $X \in \mathcal{R}_1$ with index j. Suppose that X has a hyperbolic singularity σ in $H_X(\gamma)$ with index i. By [8, Lemma 7], if $j > i$, then

$$\dim W^u(\sigma) + \dim W^s(\gamma) < \dim M.$$

This is a contradiction since X is Kupka-Smale. Similarly, we can get a same contradiction for the case of $j < i$. This completes the proof. □

From Lemma 2.3, for C^1-generic vector field $X \in \mathfrak{X}^1(M)$, if $C_X(\gamma)$ is shadowable and locally maximal, we can see that there is no singularity for X.

Proposition 2.4. *There is a residual subset $\mathcal{R}_2$ of $\mathfrak{X}^1(M)$ such that for any hyperbolic periodic orbit γ of $X \in \mathcal{R}_2$, if the chain component $C_X(\gamma)$ is shadowable and locally maximal, then* $\operatorname{index}(\eta) = \operatorname{index}(\tau)$ *for any hyperbolic periodic orbits η, τ in $C_X(\gamma)$.*

Proof. Take a residual subset $\mathcal{R}_2 = \mathcal{R}_1$ of $\mathfrak{X}^1(M)$, and let for any hyperbolic periodic orbit γ of $X \in \mathcal{R}_2$, $C_X(\gamma)$ be shadowable and two hyperbolic periodic orbits of X in $C_X(\gamma)$ be saddles. Suppose that $\operatorname{index}(\eta) \ne \operatorname{index}(\tau)$. Then we have

$$\mathrm{dim W^s}(\eta) + \mathrm{dim W^u}(\tau) \le \mathrm{dimM}.$$

Since $X \in \mathcal{KS}(M)$, by Lemma 2.2, $W^s(\eta) \cap W^u(\tau) = \emptyset$. This is a contradiction to Lemma 2.1. □

Let $X \in \mathfrak{X}^1(M)$, $p \in M$ a point in a periodic orbit of X_t with period $T > 0$ and $T_pM(s) = \{v \in T_pM : \|v\| \le s\}$. Define $\langle X(p) \rangle$ as the subspace generated by $X(p)$, and set

$$N_p = \langle X(p) \rangle^{\perp} \text{ and } N_{p,s} = N_p \cap T_pM(s) \quad \text{for } 0 < s < 1$$

such that the exponential map $\exp_p : T_pM(s) \to M$ is well defined for all $p \in M$. Finally define $\mathcal{N}_{p,s} = \exp_p(N_{p,s})$. Then for a given $p' = X_{t_0}(p)$ with $t_0 > 0$, there are $r_0 > 0$ and a C^1 map defined by

$$\tau : \mathcal{N}_{p,r_0} \to \mathbb{R} \text{ such that } X_{\tau(y)}(y) \in \mathcal{N}_{p',s}$$

for all $y \in \mathcal{N}_{p,r_0}$ with $\tau(p) = t_0$.

The flow X_t uniquely defines the *Pincaré map*

$$\begin{aligned} f : \mathcal{N}_{p,r_0} &\to \mathcal{N}_{p',s}, \\ y &\mapsto X_{\tau(y)}(y). \end{aligned}$$

This map is a C^1 embedding whose image set is contained in the interior of $\mathcal{N}_{p',s}$ if r_0 is small. If $X_t(p) \neq p$ for $0 < t \le t_0$ and r_0 is sufficiently small, then the map $(t, y) \mapsto X_t(y)$ is a C^1 embedding from the set $\{(t, y) \in \mathbb{R} \times \mathcal{N}_{p,r} : 0 \le t \le \tau(y)\}$ into M for $0 < r \le r_0$. The image will be denoted by

$$F_p(X_t, r, t_0) = \{X_t(y) : y \in \mathcal{N}_{p,r} \text{ and } 0 \le t \le \tau(y)\}.$$

For $\epsilon > 0$, let $U_\epsilon(\mathcal{N}_{p,r})$ be the set of diffeomorphisms $\varphi : \mathcal{N}_{p,r} \to \mathcal{N}_{p,r}$ such that $\mathrm{supp}(\varphi) \subset \mathcal{N}_{p,r/2}$ and $d_1(\varphi, id) < \epsilon$. Here d_1 is the usual C^1 metric, $id : \mathcal{N}_{p,r} \to \mathcal{N}_{p,r}$ is the identity map, and $\mathrm{supp}(\varphi)$ is the closure of the set where it differs from id. Note that p is a hyperbolic fixed point of f if and only if its orbit $O(p)$ is hyperbolic.

Lemma 2.5. *Let $X \in \mathfrak{X}^1(M)$, p a periodic point of X with period $T > 0$, let $f : \mathcal{N}_{p,r_0} \to \mathcal{N}_{p,s}$ be as above, and let $\mathcal{U} \subset \mathfrak{X}^1(M)$ be a C^1-neighborhood of X and $0 < r \le r_0$ be given. Then there are $\delta_0 > 0$ and $0 < \epsilon_0 < r/2$ such that for a linear isomorphism $L_\delta : N_p \to N_p$ with $\|L_\delta - D_pf\| < \delta < \delta_0$, there is $Y^\delta \in \mathcal{U}$ satisfying:*

(1) *$Y^\delta(x) = X(x)$, if $x \notin F_t(X_t, r, T)$,*
(2) *p belongs to a periodic orbit for Y_t^δ,*
(3)

$$g_{Y^\delta}(x) = \begin{cases} \exp_p \circ H_\delta \circ \exp_p^{-1}(x) & \text{if } x \in B_{\epsilon_0/4}(p) \cap \mathcal{N}_{p,r} \\ f(x) & \text{if } x \notin B_{\epsilon_0}(p) \cap \mathcal{N}_{p,r} \end{cases}$$

where $g_{Y^\delta}(x) : \mathcal{N}_{p,r} \to \mathcal{N}_{p,s}$ is the Poincaré map of Y_t^δ. Furthermore, let Y^0 be the vector field for $H_0 = D_pf$. Then we have

(4) *$d_1(Y^\delta, Y^0) \to 0$ as $\delta \to 0$.*

Proof. See [7, Lemma 1.3, p. 3395]. □

Lemma 2.6. *There is a residual subset $\mathcal{R}_3$ of $\mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}_3$, X satisfies the following: let $C_X(\gamma)$ be shadowable and locally maximal in U for a hyperbolic periodic orbit γ of $X \in \mathcal{R}_3$, and let $\mathcal{U}(X)$ be a C^1-neighborhood of X. If a periodic orbit of X in $C_X(\gamma)$ is not hyperbolic, then there is $Y \in \mathcal{U}(X)$ such that Y has two hyperbolic periodic orbits $\eta_Y, \tau_Y \in C_Y(\gamma_Y) \subset U$ with* $\mathrm{index}(\eta_Y) \neq \mathrm{index}(\tau_Y)$. *Here, γ_Y is the continuation of γ for Y.*

Proof. Since, by [3], for any $X \in \mathcal{R}_3$, $C_X(\gamma)$ is robustly isolated. So, there exist a neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of $C_X(\gamma)$ such that for any $Y \in \mathcal{U}(X)$,

$$C_Y(\gamma_Y) = \Lambda_Y(U)(= \bigcap_{t\in\mathbb{R}} Y_t(U)).$$

Therefore we can apply the result of [1, Lemma 4.3]. □

Note that since $C_X(\gamma)$ is locally maximal in an open set U we know that $\eta_Y, \tau_Y \in C_Y(\gamma_Y) \subset U$.

Lemma 2.7. *There is a residual subset $\mathcal{R}_4$ of $\mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}_4$, if for any C^1-neighborhood $\mathcal{U}(X)$ of X, there exists $Y \in \mathcal{U}(X)$ such that Y has two hyperbolic periodic orbits η_Y, τ_Y with* $\operatorname{index}(\eta_Y) \neq \operatorname{index}(\tau_Y)$*, then X has two hyperbolic periodic orbits η, τ with* $\operatorname{index}(\eta) \neq \operatorname{index}(\tau)$.

Proof. See, [1, Lemma 5.1]. □

We say that a vector field X is a *star vector field* if there exists a C^1-neighborhood $\mathcal{U}(X)$ of X in $\mathfrak{X}^1(M)$ such that for any $Y \in \mathcal{U}(X)$, every critical element of Y is hyperbolic, and M is said to satisfy the *star condition* for X if X is a star vector field. Moreover, we say that a compact invariant set $\Lambda \subset M$ satisfies the *star condition* for $X \in \mathfrak{X}^1(M)$ if there exist a C^1-neighborhood $\mathcal{U}(X)$ of X in $\mathfrak{X}^1(M)$ and a neighborhood U of Λ such that for any $Y \in \mathcal{U}(X)$, every critical element of Y in $\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(U)$ is hyperbolic.

Proposition 2.8. *There is a residual subset $\mathcal{R}'$ of $\mathfrak{X}^1(M)$ such that for any hyperbolic periodic orbit γ of $X \in \mathcal{R}'$, if $C_X(\gamma)$ is shadowable and locally maximal, then $C_X(\gamma)$ satisfies the star condition.*

To prove Proposition 2.8, we need the notion of a δ-weak eigenvalue (see [10]). Recall that if p is in a periodic orbit of X with period $\pi(p)$, then $DX_{\pi(p)}(p)$ has 1 as eigenvalue with eigenvector $X(p)$, and all the other eigenvalues are called the *characteristic multipliers* of p. We say that a point p in a hyperbolic periodic orbit of X has a *δ-weak eigenvalue* if there is a characteristic multiplier σ of the orbit of p such that

$$(1-\delta) < |\delta| < (1+\delta).$$

Lemma 2.9. *There is a residual subset $\mathcal{R}_5$ of $\mathfrak{X}^1(M)$ such that for any hyperbolic periodic orbit γ of $X \in \mathcal{R}_5$, if the chain component $C_X(\gamma)$ is shadowable and locally maximal, then there is $\delta > 0$ such that every periodic orbit in $C_X(\gamma)$ does not have a δ-weak eigenvalue.*

Proof. Take a residual subset $\mathcal{R}_5 = \mathcal{R}_2 \cap \mathcal{R}_3 \cap \mathcal{R}_4$, and let $C_X(\gamma)$ is shadowable and locally maximal for any hyperbolic periodic orbit γ of $X \in \mathcal{R}_5$. To derive a contradiction, we may assume that there is a point p which belongs to a periodic orbit η in $C_X(\gamma)$ such that for any $\delta > 0$, p has a δ-weak eigenvalue. Then by Lemmas 2.5 and 2.6, there is Y C^1-close to X such that Y has two hyperbolic periodic orbits η_Y, τ_Y in $\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(U)$ with index$(\eta_Y) \neq$ index(τ_Y). Since $C_X(\gamma)$ is locally maximal, by Lemma 2.7, X has two hyperbolic periodic orbits η, τ in $C_X(\gamma)$ with index$(\eta) \neq$ index(τ). This contradicts to Proposition 2.4. □

Proof of Proposition 2.8. Take a residual subset $\mathcal{R}' = \mathcal{R}_5$ and suppose that for any hyperbolic periodic orbit γ of $X \in \mathcal{R}'$, $C_X(\gamma)$ dose not satisfy the star condition. Then we may assume that there is a periodic point p in $C_X(\gamma)$ such that for any $\delta > 0$, p has a δ-weak eigenvalue. This is a contradiction to Lemma 2.9. □

Let $PO_h(X)$ be the set of hyperbolic periodic orbits of X.

Proposition 2.10. *Let $\gamma \in PO_h(X)$ and let $T \geq 1$, $\eta > 0$ and $\tilde{T} > 0$ be given. For any $\gamma' \sim \gamma$, if the period τ of γ' is bigger than $\tilde{T}$, we assume $H_X(\gamma)$ satisfies the following properties* (P1) *to* (P3):

(P1) *For any $x \in \gamma'$ and $t \geq T$,*

$$\frac{1}{t}(\log \|\Psi_t|_{\triangle^s(x)}\| - \log m(\Psi_t|_{\triangle^u(x)})) < -2\eta.$$

(P2) *Let $x \in \gamma'$ and $0 = T_0 < T_1 < \cdots < T_\iota = \tau$ be a partition with $T \leq T_i - T_{i-1} < 2T$ for any $i = 1, \ldots, \iota$. Then we have*

$$\frac{1}{\tau}\sum_{i=1}^{i=\iota} \log \|\Psi_{T_i - T_{i-1}}|_{\triangle^s(X_{T_{i-1}}x)}\| < -\eta,$$

$$\frac{1}{\tau}\sum_{i=1}^{i=\iota} \log m(\Psi_{T_i - T_{i-1}}|_{\triangle^u(X_{T_{i-1}}x)}) > \eta.$$

(P3) *X has the shadowing property on $H_X(\gamma)$.*

Then $H_X(\gamma)$ is hyperbolic.

Proof. See [5]. □

Proof of Main Theorem. Take a residual subset $\mathcal{R} = \mathcal{R}'$ of $\mathfrak{X}^1(M)$ and suppose that for any hyperbolic periodic orbit γ of $X \in \mathcal{R}$, $C_X(\gamma)$ is shadowable and locally maximal. Then by Proposition 2.8, $C_X(\gamma)$ satisfies the star condition. If we apply the result [4, Lemma VII.1], we can see that X satisfies (P1), (P2) of Proposition 2.10. This completes the proof by Proposition 2.10. □

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