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BALANCE FOR RELATIVE HOMOLOGY WITH RESPECT TO SEMIDUALIZING MODULES

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Abstract. We derive in the paper the tensor product functor $-\otimes_R-$ by using proper $\mathcal{GP}_C$-resolutions, where C is a semidualizing module. After giving several cases in which different relative homologies agree, we use the Pontryagin duals of $\mathcal{G}_C$-projective modules to establish a balance result for such relative homology over a Cohen-Macaulay ring with a dualizing module D.

1. Introduction

Unless otherwise stated, throughout this paper R is a commutative ring with identity, and all modules are unitary. We denote by $\mathcal{P}$ the class of projective modules. For a module M, the Pontryagin dual or character module $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ is denoted by M^*.

In 2009, Sather-Wagstaff et al. [16] introduced and studied several relative cohomology functors with respect to a fixed semidualizing module C (see Section 2 for definition), such as $\mathrm{Ext}^i_{\mathcal{GP}_C\mathcal{M}}(-,-)$ derived from $\mathrm{Hom}_R(-,-)$ using proper $\mathcal{GP}_C$-resolutions of the first variable and $\mathrm{Ext}^i_{\mathcal{MGI}_C}(-,-)$ derived from $\mathrm{Hom}_R(-,-)$ using proper $\mathcal{GI}_C$-coresolutions of the second variable, where $\mathcal{GP}_C$ (resp., $\mathcal{GI}_C$) denotes the class of $\mathcal{G}_C$-projective (resp., $\mathcal{G}_C$-injective) modules. In particular, they demonstrated in [16, Example 5.3] that the natural version of balance for above relative cohomologies fails even in a very special case. More specifically, they showed that $\mathrm{Ext}^i_{\mathcal{GP}_C\mathcal{M}}(M,N)\ncong\mathrm{Ext}^i_{\mathcal{MGI}_C}(M,N)$ even if the module M has finite $\mathcal{P}_C$-projective dimension and the module N has finite $\mathcal{I}_C$-injective dimension, where $\mathcal{P}_C$ (resp., $\mathcal{I}_C$) denotes the class of C-projective (resp., C-injective) modules. However, if R is a Cohen-Macaulay ring with a dualizing module D, then they can obtain the correct balance result by using coresolutions with respect to the semidualizing module $C^\dagger=\mathrm{Hom}_R(C,D)$.

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They proved that if M has finite $\mathcal{GP}_C$-projective dimension and N has finite $\mathcal{GI}_{C^\dagger}$-injective dimension, then $\mathrm{Ext}^i_{\mathcal{GP}_C\mathcal{M}}(M,N) \cong \mathrm{Ext}^i_{\mathcal{MGI}_{C^\dagger}}(M,N)$ for each $i \in \mathbb{Z}$ (see [16, Theorem A]).

Inspired by their work, we introduce in this paper the relative homology functor $\mathrm{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(-,-)$, which is derived from $-\otimes_R -$ by using proper $\mathcal{GP}_C$-resolutions, and prove the following balance result by using a combination of the techniques of Holm [9], Sather-Wagstaff et al. [16] and Emmanouil [4].

Main Theorem. *Let R be a Cohen-Macaulay ring with a dualizing module D, and let M, N be modules such that $\mathcal{GP}_C$-$\mathrm{pd}(M) < \infty$ and $\mathcal{GP}_{C^\dagger}$-$\mathrm{pd}(N) < \infty$. Then there exists an isomorphism*

$$\mathrm{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M,N) \cong \mathrm{Tor}_i^{\mathcal{MGP}_{C^\dagger}}(M,N)$$

for each $i \in \mathbb{Z}$.

The contents of this paper are summarized as follows. In Section 2, we review some basic notation and notions. Also, some necessary facts appeared in [11, 20] are listed. We focus on in Section 3 comparison of different relative homologies. Finally, Section 4 is devoted to give the proof of our main theorem.

2. Preliminaries

A complex

$$\cdots \to X_1 \xrightarrow{\delta_1^{\mathbf{X}}} X_0 \xrightarrow{\delta_0^{\mathbf{X}}} X_{-1} \to \cdots$$

of modules will be simply denoted by $\mathbf{X}$. The complex $\mathbf{X}$ is *bounded* if $X_n = 0$ for $|n| \gg 0$. We identify a module M with the complex $\cdots \to 0 \to M \to 0 \to \cdots$, where M is in degree zero and 0 elsewhere. The nth *homology module* of $\mathbf{X}$ is defined as module $\mathrm{Ker}(\delta_n^{\mathbf{X}})/\mathrm{Im}(\delta_{n+1}^{\mathbf{X}})$, and it is denoted by $\mathrm{H}_n(\mathbf{X})$.

A *morphism* $f : \mathbf{X} \to \mathbf{Y}$ of complexes is a family of morphisms $f = (f_n : X_n \to Y_n)_{n\in\mathbb{Z}}$ of modules satisfying $\delta_{n+1}^{\mathbf{Y}} f_{n+1} = f_n \delta_{n+1}^{\mathbf{X}}$ for all $n \in \mathbb{Z}$. A *quasi-isomorphism* is a morphism $f : \mathbf{X} \to \mathbf{Y}$ of complexes with $\mathrm{H}_n(f) : \mathrm{H}_n(\mathbf{X}) \to \mathrm{H}_n(\mathbf{Y})$ bijective for all $n \in \mathbb{Z}$.

Let $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{Y}$ be classes of modules such that $\mathcal{W} \subseteq \mathcal{X}$. We write $\mathcal{X} \perp \mathcal{Y}$ (resp., $\mathcal{X} \top \mathcal{Y}$) in case $\mathrm{Ext}_R^{\geqslant 1}(X,Y) = 0$ (resp., $\mathrm{Tor}^R_{\geqslant 1}(X,Y) = 0$) for each $X \in \mathcal{X}$ and each $Y \in \mathcal{Y}$. When $\mathcal{X} = \{M\}$, we use the notation $M \perp \mathcal{Y}$ instead of $\{M\} \perp \mathcal{Y}$. There are some analogues, such as $M \top \mathcal{Y}$, $\mathcal{X} \perp M$ and $\mathcal{X} \top M$. According to [17], $\mathcal{W}$ is said to be a *cogenerator* for $\mathcal{X}$ if for each $X \in \mathcal{X}$, there exists a short exact sequence $0 \to X \to W \to X' \to 0$ of modules such that $W \in \mathcal{W}$ and $X' \in \mathcal{X}$.

A complex $\mathbf{X}$ is called an $\mathcal{X}$-*resolution* of M if $X_i \in \mathcal{X}$ for all $i \geqslant 0$, $X_i = 0$ for all $i < 0$, $\mathrm{H}_i(\mathbf{X}) = 0$ for all $i > 0$ and $\mathrm{H}_0(\mathbf{X}) \cong M$. In this case, the associated exact sequence

$$\cdots \to X_1 \to X_0 \to M \to 0$$

is denoted by $\mathbf{X}^+$. Sometimes we call the quasi-isomorphism $\mathbf{X} \xrightarrow{\simeq} M$ an $\mathcal{X}$-resolution of M. If $\mathbf{X}^+$ is $\mathrm{Hom}_R(\mathcal{X}, -)$-exact, we say that $\mathbf{X}$ is a *proper* $\mathcal{X}$-resolution of M. Let $\mathrm{res}\,\widetilde{\mathcal{X}}$ denote the class of modules M admitting a proper $\mathcal{X}$-resolution.

If M admits a proper $\mathcal{X}$-resolution $\mathbf{X}$, then such proper resolution is unique up to homotopy equivalence, and yields the relative cohomology functor $\mathrm{Ext}^i_{\mathcal{X}\mathcal{M}}(M, -) = \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathbf{X}, -))$ and the relative homology functor $\mathrm{Tor}_i^{\mathcal{X}\mathcal{M}}(M, -) = \mathrm{H}_i(\mathbf{X} \otimes_R -)$.

The $\mathcal{X}$-*projective dimension* of M is the least non-negative n such that there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$, where each $X_i \in \mathcal{X}$. In this case, we write $\mathcal{X}\text{-pd}(M) = n$. If no such n exists, we write $\mathcal{X}\text{-pd}(M) = \infty$. The class of modules M with finite $\mathcal{X}$-projective dimension is denoted by $\mathrm{res}\,\widehat{\mathcal{X}}$.

The study of semidualizing modules was initiated independently (with different names) by Foxby [6], Vasconcelos [19], and Golod [8]. Recall from [20] that a module C is called *semidualizing* if C admits a degree-wise finite projective resolution, $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$ and the natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism.

More examples of semidualizing modules can be found in [1, 11, 12, 19]. In the remainder of the paper, let C be a fixed semidualizing module.

Recall from [20] that a module is called C-*projective* if it is isomorphic to a module of the form $C \otimes_R P$ for some projective module P, and let $\mathcal{P}_C$ denote the class of C-projective modules. A *complete* $\mathcal{P}\mathcal{P}_C$-*resolution* is an exact and $\mathrm{Hom}_R(-, \mathcal{P}_C)$-exact complex $\mathbf{X}$ with X_i projective for $i \geqslant 0$ and X_i C-projective for $i < 0$. A module M is $\mathcal{G}_C$-*projective* if there exists a complete $\mathcal{P}\mathcal{P}_C$-resolution $\mathbf{X}$ such that $M \cong \mathrm{Ker}(\delta_{-1}^{\mathbf{X}})$. Let $\mathcal{G}\mathcal{P}_C$ denote the class of $\mathcal{G}_C$-projective modules.

Over a local Cohen-Macaulay ring admitting a dualizing module (Note that dualizing modules are just those semidualizing modules with finite injective dimensions), Foxby [7] introduced the next two important classes of modules via the associated dualizing module.

Following Enochs and Yassemi [5], the *Auslander class* $\mathcal{A}_C$ with respect to C consists of all modules M satisfying

(1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0 = \mathrm{Ext}_R^{\geqslant 1}(C, C \otimes_R M)$ and

(2) the canonical map $\mu_{CCM}\colon M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

Dually, the *Bass class* $\mathcal{B}_C$ with respect to C consists of all modules N satisfying

(1) $\mathrm{Ext}_R^{\geqslant 1}(C, N) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, N))$ and

(2) the canonical map $\nu_{CCN} : C \otimes_R \mathrm{Hom}_R(C, N) \to N$ is an isomorphism.

For more details about semidualizing modules and their related categories, we refer the reader to [3, 11, 12, 13, 18, 20, 21, 22].

The following facts will be used in the sequel. Note that each semidualizing module is faithfully in the sense of Holm and White [11] when R is commutative (see [11, Proposition 3.1]).

Fact 2.1. (1) *The class $\mathcal{GP}_C$ is closed under arbitrary direct sums, direct summands, extensions, and kernels of epimorphisms* (*see* [20, Proposition 2.8]).

(2) *If* $\mathbf{X}$ *is a complete* $\mathcal{PP}_C$*-resolution, then* $\operatorname{Ker}(\delta_n^{\mathbf{X}}) \in \mathcal{GP}_C$ *for each* $n \in \mathbb{Z}$ (*see* [20, Proposition 2.9]).

(3) *One has* $\mathcal{P}_C \subseteq \mathcal{GP}_C$, $\mathcal{P} \subseteq \mathcal{GP}_C$ *and* $\mathcal{GP}_C \perp \mathcal{P}_C$ (*see* [20, Propositions 2.6 and 2.7]).

(4) *The classes* $\mathcal{A}_C$ *and* $\mathcal{B}_C$ *are closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms* (*see* [11, Corollary 6.3]).

(5) *One has* $\operatorname{res}\widehat{\mathcal{P}_C} \subseteq \mathcal{B}_C$ *and* $\operatorname{res}\widehat{\mathcal{P}} \subseteq \mathcal{A}_C$ (*see* [11, Corollary 6.1]).

(6) *Let* M *be a module. Then* $\mathcal{A}_C = \{M \mid C \otimes_R M \in \mathcal{B}_C\}$ *and* $\mathcal{B}_C = \{M \mid \operatorname{Hom}_R(C, M) \in \mathcal{A}_C\}$ (*see* [11, Theorem 1]).

The next result is a direct corollary of [11, Theorem 6.4] and Fact 2.1(5).

Lemma 2.2. *One has* $\mathcal{P}_C \perp \mathcal{B}_C$ *and* $\mathcal{P}_C \top \mathcal{A}_C$. *In particular,* $\mathcal{P}_C \perp \mathcal{P}_C$.

Remark 2.3. Let M be a module.

(1) Assume that $M \in \mathcal{B}_C$. Then M admits a proper $\mathcal{P}_C$-resolution. We can verify this assertion as follows. Suppose that $\mathbf{X}$ is a projective resolution of $\operatorname{Hom}_R(C, M)$. Since $M \in \mathcal{B}_C$, $\operatorname{Hom}_R(C, M) \in \mathcal{A}_C$ by Fact 2.1(6). Thus, every kernel of $\mathbf{X}^+$ belongs to $\mathcal{A}_C$ by Fact 2.1(4) and (5). It follows that $C \otimes_R \mathbf{X}^+$ is exact. Then we can easily check that $C \otimes_R \mathbf{X}$ is a proper $\mathcal{P}_C$-resolution of M.

(2) Assume that $M \in \operatorname{res}\widehat{\mathcal{GP}_C}$. In view of Fact 2.1(1-3), we know that $\mathcal{GP}_C$ is closed under extensions and $\mathcal{P}_C$ is a cogenerator for $\mathcal{GP}_C$. Hence, it follows from [15, Lemma 3.3(a)] that M admits a proper bounded $\mathcal{GP}_C$-resolution $\mathbf{X} \xrightarrow{\simeq} M$ such that $X_0 \in \mathcal{GP}_C$, $X_i \in \mathcal{P}_C$ for $1 \leqslant i \leqslant \mathcal{GP}_C\text{-pd}(M)$, and $X_i = 0$ for $i > \mathcal{GP}_C\text{-pd}(M)$.

3. Comparison of relative homologies

In what follows, let M and N be modules. We display in this section several situations in which different relative homologies agree. All results here compare to those in [16, Section 4] on relative cohomologies.

Proposition 3.1. *If* $M \in \operatorname{res}\widehat{\mathcal{P}_C}$, *then there exists an isomorphism*

$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \cong \operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M, N)$$

for each $i \in \mathbb{Z}$, *and so* $\operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M, N) = 0$ *for* $i > \mathcal{P}_C\text{-pd}(M)$.

Proof. Assume that $\mathbf{X} \xrightarrow{\simeq} M$ is a $\mathcal{P}_C$-resolution of M such that $X_i = 0$ for each $i > \mathcal{P}_C\text{-pd}(M)$. In view of Fact 2.1(3) and Lemma 2.2, $\mathbf{X}$ is indeed $\mathcal{GP}_C$-proper and $\mathcal{P}_C$-proper. Thus, $\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) = \mathrm{H}_i(\mathbf{X} \otimes_R N) = \operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M, N)$ for each $i \in \mathbb{Z}$. The vanishing conclusion follows readily since $X_i = 0$ for each $i > \mathcal{P}_C\text{-pd}(M)$. □

Assume that R is a Cohen-Macaulay ring with a dualizing module D. Set $C^\dagger = \operatorname{Hom}_R(C, D)$. Since D has finite injective dimension, $C^{\dagger\dagger} \cong C$ and

$D \in \mathcal{B}_C$ by [11, Corollary 6.2]. Hence, according to [16, Lemma 3.14], $C^\dagger$ is also semidualizing.

One can find the first part of the following result in [16, Fact 3.13], and the other part is symmetrical.

Lemma 3.2. *Let R be a Cohen-Macaulay ring with a dualizing module D. Then $\operatorname{res}\widehat{\mathcal{P}_C} \subseteq \mathcal{B}_C \cap \mathcal{A}_{C^\dagger}$ and $\operatorname{res}\widehat{\mathcal{P}_{C^\dagger}} \subseteq \mathcal{B}_{C^\dagger} \cap \mathcal{A}_C$.*

The next lemma will be applied in the proofs of Proposition 3.4, Proposition 3.5 and Main Theorem in the introduction.

Lemma 3.3. *Let R be a Cohen-Macaulay ring with a dualizing module D. Then $\mathcal{GP}_C \top \operatorname{res}\widehat{\mathcal{P}_{C^\dagger}}$ and $\mathcal{GP}_{C^\dagger} \top \operatorname{res}\widehat{\mathcal{P}_C}$.*

Proof. We prove the first relation and the second one is symmetrical. Assume that $M \in \mathcal{GP}_C$ and $N \in \operatorname{res}\widehat{\mathcal{P}_{C^\dagger}}$. Set $j = \mathcal{P}_{C^\dagger}\text{-pd}(N)$. According to Fact 2.1(2), there exists an exact sequence

$$0 \to M \to W_0 \to \cdots \to W_{j-1} \to M' \to 0$$

with each $W_i \in \mathcal{P}_C$ and $M' \in \mathcal{GP}_C$. Since $N \in \operatorname{res}\widehat{\mathcal{P}_{C^\dagger}}$, $N \in \mathcal{B}_{C^\dagger} \cap \mathcal{A}_C$ by Lemma 3.2. Hence, by Lemma 2.2, $W_i \top N$ for $0 \leqslant i \leqslant j-1$. Then for $n > 0$, the standard dimension shift yields the first isomorphism in the following sequence

$$\operatorname{Tor}_n^R(N, M) \cong \operatorname{Tor}_{n+j}^R(N, M') \cong \operatorname{Tor}_{n+j}^{\mathcal{P}_{C^\dagger}\mathcal{M}}(N, M') = 0.$$

The second isomorphism follows from [14, Proposition 4.3] since $M' \in \mathcal{GP}_C \subseteq \mathcal{A}_{C^\dagger}$ by [10, Theorem 4.6], and the vanishing holds because $n + j > j = \mathcal{P}_{C^\dagger}$-pd$(N)$, as desired. □

Let $\mathcal{X}$ be a class of modules. Recall from [17, Definition 1.12] that an exact complex $\mathbf{X}$ with each $X_i \in \mathcal{X}$ is called *totally $\mathcal{X}$-acyclic* if it is $\operatorname{Hom}_R(\mathcal{X}, -)$-exact and $\operatorname{Hom}_R(-, \mathcal{X})$-exact. We denote by $\mathcal{G}(\mathcal{X})$ the class of modules of the form $M \cong \operatorname{Ker}(\delta_{-1}^{\mathbf{X}})$ for some totally $\mathcal{X}$-acyclic complex $\mathbf{X}$.

By virtue of [17, Lemma 2.9], $M \in \operatorname{res}\widehat{\mathcal{G}(\mathcal{P}_C)}$ if and only if $M \in \operatorname{res}\widehat{\mathcal{GP}_C} \cap \mathcal{B}_C$. Hence, if $M \in \operatorname{res}\widehat{\mathcal{G}(\mathcal{P}_C)}$, then $M \in \operatorname{res}\widetilde{\mathcal{GP}_C} \cap \operatorname{res}\widetilde{\mathcal{P}_C}$ by Remark 2.3(2) and (1).

Proposition 3.4. *Let R be a Cohen-Macaulay ring with a dualizing module D. If $M \in \operatorname{res}\widetilde{\mathcal{GP}_C} \cap \operatorname{res}\widetilde{\mathcal{P}_C}$ and $N \in \operatorname{res}\widehat{\mathcal{P}_{C^\dagger}}$, then there exists an isomorphism*

$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M, N) \cong \operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M, N)$$

for each $i \in \mathbb{Z}$.

Proof. Let $\mathbf{W} \xrightarrow{\gamma'} M$ and $\mathbf{X} \xrightarrow{\gamma} M$ be proper $\mathcal{P}_C$-resolution and proper $\mathcal{GP}_C$-resolution of M, respectively. In view of the fact $\mathcal{P}_C \subseteq \mathcal{GP}_C$, we can get a quasi-isomorphism $\alpha : \mathbf{W} \to \mathbf{X}$ such that $\gamma\alpha = \operatorname{Id}_M\gamma'$. Now it suffices to prove the mapping cone cone(α) is $-\otimes_R N$-exact. Indeed, Lemma 3.3 guarantees the fact, as desired. □

Note that $\mathcal{P} \subseteq \mathcal{GP}_C$. We omit the proof of the next result, which is similar to Proposition 3.4.

Proposition 3.5. *Let R be a Cohen-Macaulay ring with a dualizing module D. If $M \in \operatorname{res}\widetilde{\mathcal{GP}_C}$ and $N \in \operatorname{res}\widehat{\mathcal{P}_{C^\dagger}}$, then there exists an isomorphism*

$$\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M,N)$$

for each $i \in \mathbb{Z}$.

One can obtain the next three results by using similar proofs for Lemma 3.3, Proposition 3.4 and Proposition 3.5, respectively.

Lemma 3.6. *One has $\mathcal{GP}_C \top \operatorname{res}\widehat{\mathcal{P}}$.*

Proposition 3.7. *If $M \in \operatorname{res}\widetilde{\mathcal{GP}_C} \cap \operatorname{res}\widetilde{\mathcal{P}_C}$ and $N \in \operatorname{res}\widehat{\mathcal{P}}$, then there exists an isomorphism*

$$\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M,N) \cong \operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M,N)$$

for each $i \in \mathbb{Z}$.

Proposition 3.8. *If $M \in \operatorname{res}\widetilde{\mathcal{GP}_C}$ and $N \in \operatorname{res}\widehat{\mathcal{P}}$, then there exists an isomorphism*

$$\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M,N)$$

for each $i \in \mathbb{Z}$.

The following result is a direct corollary of [16, Lemma 3.16]. It will be used in Proposition 3.10.

Lemma 3.9. *Assume that R is a local ring. Then $C \cong R$ if and only if $C \in \operatorname{res}\widehat{\mathcal{P}}$.*

Inspired by [16, Proposition 6.1] on relative cohomologies, we conclude this section with two cases in which the relative homologies do not agree. In [19, Proposition 4.9], Vasconcelos gave us an example to show that there exists a local ring with a non-trivial semidualizing module. Moreover, by [16, Lemma 6.2], the hypotheses in (2) of the following proposition can be satisfied.

Proposition 3.10. *Assume that (R,m,k) is a local ring such that $C \ncong R$. Then*

(1) $\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(C,k) = \operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(C,k) = 0$ *and* $\operatorname{Tor}_i^R(C,k) \neq 0$ *for $i \geqslant 1$.*

(2) *Suppose that M admits a proper $\mathcal{P}_C$-resolution, $\mathcal{GP}_C$-pd$(M) < \infty$ and $\mathcal{P}_C$-pd$_R(M) = \infty$. Then $\operatorname{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M,N) = 0$ for $i > \mathcal{GP}_C$-pd(M) and $\operatorname{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M,N) \neq 0$.*

Proof. (1) Since $C \in \mathcal{P}_C$, Proposition 3.1 implies that

$$\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(C,k) = \mathrm{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(C,k) = 0$$

for $i \geqslant 1$. On the other hand, note that C is a finitely generated module with infinite projective dimension by Lemma 3.9. It follows that $\mathrm{Tor}_i^R(C,k) \neq 0$ for $i \geqslant 1$.

(2) According to Remark 2.3(2), M admits a proper bounded $\mathcal{GP}_C$-resolution $\mathbf{X} \xrightarrow{\simeq} M$ such that $X_i = 0$ for $i > \mathcal{GP}_C\text{-pd}(M)$. Thus, $\mathrm{Tor}_i^{\mathcal{GP}_C\mathcal{M}}(M,N) = 0$ for $i > \mathcal{GP}_C\text{-pd}(M)$. On the other hand, $\mathrm{Tor}_i^{\mathcal{P}_C\mathcal{M}}(M,N) \neq 0$ follows directly from [14, Theorem 3.10 and Theorem 5.4]. □

4. Proof of Main Theorem

In this section, we always assume that R is a Cohen-Macaulay ring with a dualizing module D and $C^\dagger = \mathrm{Hom}_R(C,D)$. We begin with the following assertion, which will be used in the proofs of Lemma 4.1 and Lemma 4.2.

Assume that M is a module such that $M \in \mathcal{GP}_{C^\dagger}$. Then for any integer $n \geqslant 1$, there exists an exact sequence

$$0 \to N^* \to P_{C^\dagger}^{n\,*} \to \cdots \to P_{C^\dagger}^{1\,*} \to M^* \to 0$$

with $N \in \mathcal{GP}_{C^\dagger}$ and each $P_{C^\dagger}^i \in \mathcal{P}_{C^\dagger}$.

Indeed, by a complete $\mathcal{PP}_{C^\dagger}$-resolution of M, we can get an exact sequence

$$0 \to M \to P_{C^\dagger}^1 \to \cdots \to P_{C^\dagger}^n \to N \to 0$$

with each $P_{C^\dagger}^i \in \mathcal{P}_{C^\dagger}$. According to Fact 2.1(2), N is also in $\mathcal{GP}_{C^\dagger}$. Applying now the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$, we obtain an exact sequence with required properties.

Lemma 4.1. *Let K and M be modules such that $K \in \mathrm{res}\,\widehat{\mathcal{P}_C}$ and $M \in \mathcal{GP}_{C^\dagger}$. Then $\mathrm{Ext}^1_{\mathcal{P}_C\mathcal{M}}(K,M^*) = 0$.*

Proof. Assume that $\mathcal{P}_C\text{-pd}(K) = n$ for some non-negative integer n. As we showed above, there exists an exact sequence

$$(\sharp) \qquad 0 \to N^* \to P_{C^\dagger}^{1\,*} \to \cdots \to P_{C^\dagger}^{n\,*} \to M^* \to 0$$

with each $P_{C^\dagger}^i \in \mathcal{P}_{C^\dagger}$ and $N \in \mathcal{GP}_{C^\dagger}$.

For any module $P_{C^\dagger} \in \mathcal{P}_{C^\dagger}$, we have

$$\mathrm{Ext}^{i\geqslant 1}_{\mathcal{P}_C\mathcal{M}}(K,P_{C^\dagger}^*) \cong \mathrm{Tor}_{i\geqslant 1}^{\mathcal{P}_C\mathcal{M}}(K,P_{C^\dagger})^* \cong \mathrm{Tor}_{i\geqslant 1}^{\mathcal{M}\mathcal{P}_{C^\dagger}}(K,P_{C^\dagger})^* = 0.$$

The first isomorphism follows from [14, Corollary 3.13], and the second one holds by [14, Corollary 4.4], Lemma 3.2 and [14, Theorem 3.10].

On the other hand, Lemma 3.3 implies that $\mathcal{P}_C \perp N^*$ and $\mathcal{P}_C \perp P_{C^\dagger}^{i\,*}$ for each i. It follows that $(\sharp)$ is $\mathrm{Hom}_R(\mathcal{P}_C,-)$-exact. Hence, $\mathrm{Ext}^1_{\mathcal{P}_C\mathcal{M}}(K,M^*) \cong \mathrm{Ext}^{n+1}_{\mathcal{P}_C\mathcal{M}}(K,N^*) = 0$ by [15, Lemma 4.4(a)] and [15, Lemma 4.5(b)(1)]. This completes the proof. □

The next result is given in service of Lemma 4.3.

Lemma 4.2. *Let* $\mathbf{L} = \ 0 \longrightarrow K \xrightarrow{\zeta} G \longrightarrow H \longrightarrow 0$ *be a short exact sequence of modules with* $K \in \operatorname{res}\widehat{\mathcal{P}_C}$ *and* $G \in \mathcal{GP}_C$*. Then for each module* $M \in \mathcal{GP}_{C^\dagger}$*,* $\mathbf{L}$ *remains exact after applying* $\operatorname{Hom}_R(-, M^*)$*, i.e., the sequence*

$$0 \longrightarrow \operatorname{Hom}_R(H, M^*) \longrightarrow \operatorname{Hom}_R(G, M^*) \xrightarrow{\zeta_{M^*}} \operatorname{Hom}_R(K, M^*) \longrightarrow 0$$

is exact.

Proof. Using the assertion above again, we get a short exact sequence

$$(\natural) \qquad 0 \to N^* \to P^*_{C^\dagger} \to M^* \to 0$$

with $P_{C^\dagger} \in \mathcal{P}_{C^\dagger}$ and $N \in \mathcal{GP}_{C^\dagger}$. Note that $\mathcal{P}_C \perp N^*$ by Lemma 3.3. We have that ($\natural$) is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact. Then we can obtain the following commutative diagram with exact rows by [15, Lemma 4.3(a)]:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \operatorname{Hom}_R(G, N^*) & \longrightarrow & \operatorname{Hom}_R(G, P^*_{C^\dagger}) & \longrightarrow & \operatorname{Hom}_R(G, M^*) & & \\
 & \downarrow & & \downarrow \zeta' & & \downarrow \zeta_{M^*} & & \\
0 \longrightarrow & \operatorname{Hom}_R(K, N^*) & \longrightarrow & \operatorname{Hom}_R(K, P^*_{C^\dagger}) & \xrightarrow{\rho^*} & \operatorname{Hom}_R(K, M^*) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{P}_C\mathcal{M}}(K, N^*)
\end{array}$$

In view of Lemma 4.1, we know $\operatorname{Ext}^1_{\mathcal{P}_C\mathcal{M}}(K, N^*) = 0$, and so ρ^* is surjective.

On the other hand, we claim that ζ' is surjective. It suffices to show that $H \perp P^*_{C^\dagger}$. Since $G \in \mathcal{GP}_C$, Fact 2.1(2) implies that there exists a short exact sequence $0 \longrightarrow G \longrightarrow P_C \longrightarrow G' \longrightarrow 0$ with $P_C \in \mathcal{P}_C$ and $G' \in \mathcal{GP}_C$. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P_C & \longrightarrow & T & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the third column, we get a short exact sequence

$$0 \longrightarrow H \longrightarrow T \longrightarrow G' \longrightarrow 0$$

such that $T \in \operatorname{res}\widehat{\mathcal{P}_C} \subseteq \mathcal{A}_{C^\dagger}$ (see Lemma 3.2). Lemma 3.3 yields $G' \perp P^*_{C^\dagger}$. Moreover, $T \perp P^*_{C^\dagger}$ follows from Lemma 2.2. Thus, $H \perp P^*_{C^\dagger}$.

Finally, in view of the commutativity of the diagram, it follows that ζ_{M^*} is surjective, as desired. □

The following lemma will play a key role in the proof of Main Theorem in the introduction.

Lemma 4.3. *Let M be a module. Assume that $\mathbf{X} \xrightarrow{\simeq} M$ is a bounded $\mathcal{GP}_C$-resolution of M such that $X_0 \in \mathcal{GP}_C$ and $X_i \in \mathcal{P}_C$ for all $i \neq 0$. Then the complex* $\mathrm{Hom}_R(\mathbf{X}^+, N^*)$ *is exact for any module $N \in \mathcal{GP}_{C^\dagger}$.*

Proof. Fix a module $N \in \mathcal{GP}_{C^\dagger}$. We set $M_n = \mathrm{Im}(X_{n+1} \to X_n)$ for $n \geqslant 0$ and $M_{-1} = M$. Then $M_n \in \mathrm{res}\,\widehat{\mathcal{P}_C}$ for all $n \geqslant 0$. In order to verify the exactness of $\mathrm{Hom}_R(\mathbf{X}^+, N^*)$, it suffices to show that each of the following short exact sequence

$$0 \longrightarrow M_n \xrightarrow{\zeta^n} X_n \longrightarrow M_{n-1} \longrightarrow 0$$

is $\mathrm{Hom}_R(-, N^*)$-exact. Indeed, Lemma 4.2 guarantees the fact. This completes the proof. □

Now, we can give the proof of Main Theorem.

Proof of Main Theorem. Assume that $M \in \mathrm{res}\,\widehat{\mathcal{GP}_C}$ and $N \in \mathrm{res}\,\widehat{\mathcal{GP}_{C^\dagger}}$. According to Remark 2.3(2), there exist proper bounded $\mathcal{GP}_C$-resolution $\mathbf{X} \xrightarrow{\simeq} M$ and proper bounded $\mathcal{GP}_{C^\dagger}$-resolution $\mathbf{X}' \xrightarrow{\simeq} N$, respectively, such that $X_0 \in \mathcal{GP}_C$, $X_i \in \mathcal{P}_C$ for all $i \neq 0$, $X'_0 \in \mathcal{GP}_{C^\dagger}$, and $X'_i \in \mathcal{P}_{C^\dagger}$ for all $i \neq 0$.

The proof will be completed once we verify the following quasi-isomorphisms

$$M \otimes_R \mathbf{X}' \simeq \mathbf{X} \otimes_R \mathbf{X}' \simeq \mathbf{X} \otimes_R N.$$

By symmetry, it suffices to show the first one. To see this, by [2, Lemma 2.13], we only need to prove that the complex $\mathbf{X}^+ \otimes_R G'$ is exact for any module $G' \in \mathcal{GP}_{C^\dagger}$. Equivalently, the complex $\mathrm{Hom}_R(\mathbf{X}^+, G'^*)$ is exact. Indeed, Lemma 4.3 guarantees the fact. This completes the proof. □

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