REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WHOSE SHAPE OPERATOR IS OF CODAZZI TYPE IN GENERALIZED TANAKA-WEBSTER CONNECTION

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ABSTRACT. In this paper, we give a non-existence theorem of Hopf hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose shape operator is of Codazzi type in generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$.

Introduction

Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$, which is said to be a *complex two-plane Grassmannian*. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J, for details we refer to [2], [3] and [4]. In particular, when $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

Moreover, naturally we could consider two geometric conditions for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M (see Berndt and Suh [3]). Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields ξ_ν for the 3-dimensional distribution

$\mathfrak{D}^{\perp}$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_{\nu} = -J_{\nu}N$ ($\nu = 1, 2, 3$), where J_{ν} denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$, $x \in M$.

By using such two geometric conditions and the results in Alekseevskii [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^{\perp}$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, we say that the Reeb vector field ξ on M is Killing. This means that the metric tensor g is invariant under the Reeb flow of ξ on M. In [4], Berndt and Suh showed that this notion is equivalent that *the shape operator A commutes with the structure tensor ϕ*. From this, they also gave a characterization of real hypersurfaces of type (A) in Theorem A in terms of the Reeb flow on M as follows (see [4]):

Theorem B. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

On the other hand, Lee and Suh [11] gave a new characterization of real hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$.

Theorem C. *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

As a generalization of the well-known connection defined by Tanaka in [14] and, independently, by Webster in [16], Tanno [15] introduced the notion of *generalized Tanaka Webster connection* (in short, *g-Tanaka-Webster connection*). This connection coincides with Tanaka-Webster connection if the associated CR-structure is integrable. Here Tanaka-Webster connection was defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. Moreover, on real hypersurfaces in Kähler manifolds with almost contact metric structure (ϕ, ξ, η, g), the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for a non-zero real number k was given by Cho (see [5]). In particular, if a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

Recently, by using the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ Jeong, Suh and the second author have studied some parallelism of the shape operator on real hypersurfaces in complex two-plane Grassmannians ([8] and [9]). For example,

they proved that the shape operator on a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}^\perp$-parallel with respect to the g-Tanaka-Webster connection, that is, the shape operator A of M satisfies the condition $(\hat{\nabla}_X^{(k)}A)Y = 0$ for any tangent vector fields $X \in \mathfrak{D}^\perp$ and $Y \in TM$ if and only if a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$ (see [9]).

In this paper, let us consider a new notion which becomes another extension for the parallelism of the shape operator on real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with respect to the g-Tanaka-Webster connection. For given a tensor T of type $(1,1)$ on M we will say that T *is of Codazzi type with respect to the g-Tanaka-Webster connection* if it satisfies $(\hat{\nabla}_X^{(k)}T)Y = (\hat{\nabla}_Y^{(k)}T)X$ for any tangent vector fields X and Y on M. By virtue of this notion we will consider a real hypersurface in $G_2(\mathbb{C}^{m+2})$ whose *shape operator A is of Codazzi type with respect to $\hat{\nabla}^{(k)}$*, that is, the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ satisfies the property

$$(*) \qquad (\hat{\nabla}_X^{(k)}A)Y = (\hat{\nabla}_Y^{(k)}A)X$$

for any tangent vector fields X and Y on M. By using such the notion for the shape operator, we give a classification theorem for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows:

Main Theorem. *There does not exist any Hopf hypersurface, $\alpha \neq 2k$, in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose shape operator is of Codazzi type with respect to the generalized Tanaka-Webster connection if the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of the Reeb vector field are invariant by the shape operator.*

Remark. In [7], the authors have remarked the case $\alpha = 2k$ (k is a nonzero real number) on Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}^\perp$ (see Proposition 3.7). That is, under assumptions, $\xi \in \mathfrak{D}^\perp$ and $\alpha = 2k$, the shape operator A of M becomes naturally g-Tanaka-Webster Reeb parallel. Thus in this paper, we only consider for a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\alpha \neq 2k$, when the Reeb vector field ξ belongs to $\mathfrak{D}^\perp$.

1. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ and the generalized Tanaka-Webster connection

In this section, we first review some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ introduced in [3], [4], [9], [12], [13], etc.

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. Now let us put

$$(1.1) \qquad JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$
\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0, \\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned}
$$

(1.2)

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = J J_\nu$, $\nu = 1, 2, 3$ and (1.1), the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$
\begin{aligned}
&\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\
&\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi.
\end{aligned}
$$

(1.3)

On the other hand, as J is a Kähler structure (i.e., $\tilde{\nabla}J = 0$) and $\mathfrak{J}$ a quaternionic Kähler structure (i.e., $\tilde{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$ for any tangent vector fields X on $G_2(\mathbb{C}^{m+2})$), together with Gauss and Weingarten formulas it follows that

$$(1.4) \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

$$(1.5) \qquad \nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,$$

$$
(1.6) \qquad
\begin{aligned}
(\nabla_X \phi_\nu)Y = & -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y \\
& + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu.
\end{aligned}
$$

Using the expression for the curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$ in [9], the equation of Codazzi is given by

(1.7)
$$
\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
& + \sum_{\nu=1}^{3} \left\{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \right\} \\
& + \sum_{\nu=1}^{3} \left\{ \eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X \right\} \\
& + \sum_{\nu=1}^{3} \left\{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \right\}\xi_\nu.
\end{aligned}
$$

As mentioned in Theorem A, the complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with two kinds of A-invariancy for the distributions $[\xi] = \mathrm{Span}\{\xi\}$ and $\mathfrak{D}^{\perp} = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ was obtained by Berndt and Suh [3]. Accordingly we introduce the following two propositions related to the principal curvatures of the model spaces (A) and (B), respectively.

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has the following three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures:*

principal curvature	*multiplicity*	*eigenspace*
$\alpha = \sqrt{8}\cot(\sqrt{8}r)$	1	$T_{\alpha} = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\}$
$\beta = \sqrt{2}\cot(\sqrt{2}r)$	2	$T_{\beta} = \mathbb{C}^{\perp}\xi = \mathrm{Span}\{\xi_2, \xi_3\}$
$\lambda = -\sqrt{2}\tan(\sqrt{2}r)$	$2(m-1)$	$T_{\lambda} = \{X \mid X \perp \mathbb{H}\xi,\ JX = J_1 X\}$
$\mu = 0$	$2(m-1)$	$T_{\mu} = \{X \mid X \perp \mathbb{H}\xi,\ JX = -J_1 X\}$

with some $r \in (0, \pi/\sqrt{8})$. Here $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector field ξ and $\mathbb{C}^{\perp}\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \in \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures and the corresponding multiplicities with respect to the eigenspaces:*

principal curvature	*multiplicity*	*eigenspace*
$\alpha = -2\tan(2r)$	1	$T_{\alpha} = \mathrm{Span}\{\xi\}$
$\beta = 2\cot(2r)$	3	$T_{\beta} = \mathrm{Span}\{\xi_{\nu} \mid \nu = 1, 2, 3\}$
$\gamma = 0$	3	$T_{\gamma} = \mathrm{Span}\{\phi_{\nu}\xi \mid \nu = 1, 2, 3\}$
$\lambda = \cot(r)$	$4n-4$	T_{λ}
$\mu = -\tan(r)$	$4n-4$	T_{μ}

where $r \in (0, \pi/\sqrt{8})$ and $T_{\lambda} \oplus T_{\mu} = (\mathbb{H}\mathbb{C}\xi)^{\perp}$, $\mathfrak{J}T_{\lambda} = T_{\lambda}$, $\mathfrak{J}T_{\mu} = T_{\mu}$, $JT_{\lambda} = T_{\mu}$.

From now on, we introduce the generalized Tanaka-Webster (shortly, g-Tanaka-Webster) connection for a real hypersurface in Kähler manifolds (see [5], [6], [8], [9] and [10]).

As stated in the introduction, the Tanaka-Webster connection is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold (see [14] and [16]). Tanno [15] defined the g-Tanaka-Webster connection for contact metric manifolds by the canonical connection as follows:

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y$$

for any vector fields X and Y. It coincides with the Tanaka-Webster connection if the associated CR-structure is integrable.

On the other hand, a real hypersurface M in Kähler manifolds has the property $A\phi + \phi A = \pm 2\phi$ (the sign depends on the orientation) for the shape operator A and the structure tensor ϕ of M if and only if the almost contact metric structure of M is contact metric (see [5]). From such a point of view, Cho defined the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ on M by the naturally extended one of Tanno's g-Tanaka-Webster connection $\hat{\nabla}$ for contact metric manifolds:

$$(1.8) \qquad \hat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$$

for a non-zero real number k. Moreover, if a real hypersurface in Kähler manifolds satisfies $A\phi + \phi A = 2k\phi$, then the associated CR-structure is pseudo-hermitian, strongly pseudo-convex, integrable. Thus under the assumption $A\phi + \phi A = 2k\phi$, we obtain the following facts (see [5], [8] and [10]) :

(1) if $k = 1$, then we see that $\hat{\nabla}^{(k)} = \hat{\nabla}$,
(2) the connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold.

Thus we see that the connection $\hat{\nabla}^{(k)}$ is a natural generalization of Tanaka-Webster connection and call it generalized Tanaka-Webster connection (or shortly, g-Tanaka-Webster connection) on real hypersurfaces of Kähler manifolds.

2. Proof of Main Theorem

First of all, we will prove that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ for a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ whose shape operator is of Codazzi type with respect to the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$.

Lemma 2.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose shape operator is of Codazzi type with respect to the generalized Tanaka-Webster connection. If $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of ξ are invariant under the shape operator of M, then the principal curvature $\alpha = g(A\xi, \xi)$ is constant. Furthermore, the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. To show this fact, we assume that the Reeb vector field ξ satisfies

$$(**) \qquad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \ \ \text{such that} \ \ \eta(X_0)\eta(\xi_1) \neq 0$$

for some unit vectors $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$.

Let us consider the case that the smooth function $\alpha = g(A\xi, \xi)$ vanishes. In [3] Berndt and Suh gave

$$(2.1) \qquad Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any tangent vector field Y on M under the assumption that M is Hopf. Now, as $\alpha = 0$, it implies that

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu\xi = 0.$$

Moreover, using the equation $(\ast\ast)$, we have $\eta(X_0)\phi X_0 = 0$. Since $\eta(X_0) \neq 0$, the tangent vector $\phi X_0 \in T_x M$ becomes a zero one. This gives a contradiction and we see that ξ belongs to either $\mathfrak{D}$ or $\mathfrak{D}^\perp$ when α is vanishing. In addition, from the equation (2.1) we see that if α is constant, then ξ belongs to either $\mathfrak{D}$ or $\mathfrak{D}^\perp$.

From now on, let us consider the case $\alpha \neq 0$. Since the shape operator A of M satisfies the Codazzi type equation for $\hat{\nabla}^{(k)}$, that is, $(\hat{\nabla}_X^{(k)} A)Y = (\hat{\nabla}_Y^{(k)} A)X$ for any tangent vector fields X, Y on M, we have

$$(\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY$$
$$- g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y$$
$$= (\nabla_Y A)X + g(\phi AY, AX)\xi - \eta(AX)\phi AY - k\eta(Y)\phi AX$$
$$- g(\phi AY, X)A\xi + \eta(X)A\phi AY + k\eta(Y)A\phi X,$$

together with the equation (1.8). Moreover, it can be written as

$$(2.2) \quad
\begin{aligned}
&\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
&+ 2g(A\phi AX, Y)\xi - \alpha\eta(Y)\phi AX - k\eta(X)\phi AY - \alpha g(\phi AX, Y)\xi \\
&+ \eta(Y)A\phi AX + k\eta(X)A\phi Y + \alpha\eta(X)\phi AY + k\eta(Y)\phi AX \\
&- \alpha g(A\phi X, Y)\xi - \eta(X)A\phi AY - k\eta(Y)A\phi X \\
&+ \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu + \eta_\nu(\phi X)\phi_\nu\phi Y \\
&\qquad - \eta_\nu(\phi Y)\phi_\nu\phi X + \eta(X)\eta_\nu(\phi Y)\xi_\nu - \eta(Y)\eta_\nu(\phi X)\xi_\nu\Big\} = 0,
\end{aligned}$$

using the assumption $A\xi = \alpha\xi$ and the Codazzi equation (1.7). Putting Y by ξ in (2.2), we get

$$(2.3) \quad
\begin{aligned}
&- \phi X + \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3g(\phi_\nu X, \xi)\xi_\nu\Big\} \\
&- \alpha\phi AX + A\phi AX + k\phi AX - kA\phi X = 0
\end{aligned}$$

for any vector field X tangent on M.

Substituting $X = X_0$ in (2.3) and using $(**)$, we have

$$(2.4) \qquad \phi X_0 + \eta(\xi_1)\phi_1 X_0 + \alpha\phi A X_0 - A\phi A X_0 - k\phi A X_0 + kA\phi X_0 = 0.$$

Since $\phi\xi = 0$ and $(**)$, we see that $\phi X_0 = -\eta(\xi_1)\phi_1 X_0$. From this, the equation (2.4) is written as

$$(2.5) \qquad \alpha\phi A X_0 - A\phi A X_0 - k\phi A X_0 + kA\phi X_0 = 0.$$

On the other hand, from our assumptions that the $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of ξ are invariant under the shape operator of M and M is Hopf, we see that the unit vector field $X_0 \in \mathfrak{D}$ becomes a principal one, $AX_0 = \alpha X_0$. Thus (2.5) changes to

$$\alpha^2 \phi X_0 - \alpha A\phi X_0 - k\alpha\phi X_0 + kA\phi X_0 = 0.$$

In addition, if the $\mathfrak{D}$-component of ξ is a principal vector field with the corresponding principal curvature $\alpha(\neq 0)$, then ϕX_0 is also principal vector field, that is, $A\phi X_0 = ((\alpha^2 + 4\eta^2(X_0))/\alpha)\phi X_0$ (see Proposition 2 in [4]). Thus we have

$$\alpha^2 \phi X_0 - \big(\alpha^2 + 4\eta^2(X_0)\big)\phi X_0 - k\alpha\phi X_0 + \frac{k\big(\alpha^2 + 4\eta^2(X_0)\big)}{\alpha}\phi X_0 = 0,$$

it follows that $4\eta^2(X_0)(k - \alpha)\phi X_0 = 0$. Taking the inner product with ϕX_0, we obtain $-4\eta^2(X_0)\eta^2(\xi_1)(k - \alpha) = 0$. Since $\eta(X_0)\eta(\xi_1) \neq 0$, we have $\alpha = k$. It implies that the smooth function α is constant. By virtue of the proof for the case $\alpha = 0$, the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$. $\qquad\square$

From Lemma 2.1, we can consider the following two cases:

- The Reeb vector field ξ belongs to distribution $\mathfrak{D}^\perp$, that is, $\xi \in \mathfrak{D}^\perp$,
- ξ belongs to $\mathfrak{D}$, that is, $\xi \in \mathfrak{D}$.

Now, we assume that $\xi \in \mathfrak{D}^\perp$. For the sake of convenience we put $\xi = \xi_1$.

Lemma 2.2. *Let M be a Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose shape operator is of Codazzi type in generalized Tanaka-Webster connection. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then the shape operator A commutes with the structure tensor ϕ.*

Proof. Using $\xi = \xi_1$ and (2.3), we have

$$-\phi X + \big\{\eta_2(X)\phi_2\xi + \eta_3(X)\phi_3\xi - \phi_1 X - 3g(\phi_2 X, \xi)\xi_2 - 3g(\phi_3 X, \xi)\xi_3\big\}$$
$$- \alpha\phi A X + A\phi A X + k\phi A X - kA\phi X = 0,$$

that is,

$$(2.6) \quad -\phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 - \alpha\phi A X + A\phi A X + k\phi A X - kA\phi X = 0$$

for any tangent vector field X on M.

Now we introduce the formula derived from $A\xi = \alpha\xi$ (see [4]) as follows:

$$\alpha A\phi X + \alpha\phi AX - 2A\phi AX + 2\phi X$$

$$(2.7) \qquad = -2\sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu\xi + \eta_\nu(\phi X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X$$

$$- 2\eta(X)\eta_\nu(\xi)\phi_\nu\xi - 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\Big\}.$$

Combining above equations, we have

$$\phi X + \phi_1 X - 2\eta_2(X)\xi_3 + 2\eta_3(X)\xi_2 + \alpha\phi AX - k\phi AX + kA\phi X$$

$$= A\phi AX$$

$$= \frac{\alpha}{2}A\phi X + \frac{\alpha}{2}\phi AX + \phi X$$

$$+ \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu\xi + \eta_\nu(\phi X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X$$

$$- 2\eta(X)\eta_\nu(\xi)\phi_\nu\xi - 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\Big\}$$

$$= \frac{\alpha}{2}A\phi X + \frac{\alpha}{2}\phi AX + \phi X$$

$$+ \Big\{\eta_2(X)\phi_2\xi + \eta_3(X)\phi_3\xi + \eta_2(\phi X)\xi_2 + \eta_3(\phi X)\xi_3 + \phi_1 X\Big\}$$

$$= \frac{\alpha}{2}A\phi X + \frac{\alpha}{2}\phi AX + \phi X$$

$$- \eta_2(X)\xi_3 + \eta_3(X)\xi_2 + \eta_3(X)\xi_2 - \eta_2(X)\xi_3 + \phi_1 X.$$

Therefore, we get

$$\Big(\frac{\alpha}{2} - k\Big)\phi AX - \Big(\frac{\alpha}{2} - k\Big)A\phi X = 0,$$

that is,

$$\Big(\frac{\alpha}{2} - k\Big)\big(\phi A - A\phi\big)X = 0.$$

Since $\alpha \neq 2k$, we obtain $(\phi A - A\phi)X = 0$ for any tangent vector field X on M. It means that the shape operator A commutes with the structure tensor ϕ. $\qquad\square$

Due to Berdnt and Suh [4], the Reeb flow on M is isometric if and only if the structure tensor field ϕ commutes with the shape operator A of M, that is, $A\phi = \phi A$. Thus, from Lemma 2.2 and Theorem B, we conclude that M is of type (A) under our assumptions.

Conversely, let M be a real hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$ and check if the shape operator A of M satisfies the Codazzi type equation $(*)$ for the g-Tanaka-Webster connection. In order to do this, we suppose that the shape operator A of M satisfies $(*)$.

Putting $X \in T_\lambda$ and $Y \in T_\mu$ in (2.2), we get

$$0 = -2\sum_{\nu=1}^{3} g(\phi_\nu X, Y)\xi_\nu$$
$$= -2g(\phi_1 X, Y)\xi_1 - 2g(\phi_2 X, Y)\xi_2 - 2g(\phi_3 X, Y)\xi_3.$$

Taking the inner product with ξ_2, we have $g(\phi_2 X, Y) = 0$ for any tangent vector fields $X \in T_\lambda$ and $Y \in T_\mu$. Since $\phi_2 X \in T_\mu$ for any tangent vector field $X \in T_\lambda$, it follows that $g(\phi_2 X, \phi_2 X) = 0$, which means that $\phi_2 X$ becomes a zero vector field. This gives a contradiction. So we assert that the shape operator of a real hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$ does not satisfy the Codazzi type equation $(*)$ with respect to the generalized Tanaka-Webster connection.

Next, we consider the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$. In fact, under this assumption we conclude that a Hopf hypersurface M satisfying $(*)$ is locally congruent to a model space of (B) by virtue of Theorem C. Hence, to prove our main theorem, we may check the condition $(*)$ for the real hypersurface of type (B) in $G_2(\mathbb{C}^{m+2})$.

By putting $Y = \xi$ in (2.2), we already obtain the equation (2.3). As setting $X = \xi_1$ in (2.3), we have

$$- \phi\xi_1 + \sum_{\nu=1}^{3} \left\{ \eta_\nu(\xi_1)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu\xi_1 - 3g(\phi_\nu\xi_1, \xi)\xi_\nu \right\}$$
$$- \alpha\phi A\xi_1 + A\phi A\xi_1 + k\phi A\xi_1 - kA\phi\xi_1 = 0.$$

Since $A\xi_1 = \beta\xi_1$ (see Proposition B), we get

$$-\phi\xi_1 + \phi_1\xi - \alpha\beta\phi\xi_1 + \beta A\phi\xi_1 + k\beta\phi\xi_1 - kA\phi\xi_1 = 0,$$

that is,

$$\beta(\alpha - k)\phi\xi_1 = 0.$$

For some $r \in (0, \pi/4)$, the principal curvature β can not equal to 0. So, we know that $\alpha = k$ where k is non-zero constant. Substituting $\alpha = k$ into (2.3), we have

$$(2.8) \quad -\phi X + \sum_{\nu=1}^{3} \left\{ \eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3g(\phi_\nu X, \xi)\xi_\nu \right\} + A\phi AX - kA\phi X = 0$$

for any tangent vector field X on M.

By putting $X \in T_\lambda$ in (2.8), we get $(-\lambda\mu + \alpha\mu + 1)\phi X = 0$, because if $X \in T_\lambda$, then X is orthogonal to ξ_ν and $\phi\xi_\nu$ for $\nu = 1, 2, 3$. Moreover, we see that $\phi T_\lambda \subset T_\mu$ for $X \in T_\lambda$. From this, we see that $\lambda\mu - \alpha\mu - 1 = 0$. By using the properties in Proposition B, we have $1 + \tan^2 r = 0$ for some $r \in (0, \pi/4)$. This gives a contradiction.

Hence summing up these discussions, we give a complete proof of our Main Theorem in the introduction.

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