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WEIGHTED SHARING AND UNIQUENESS OF ENTIRE OR MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we study the uniqueness problems of entire or meromorphic functions concerning differential polynomials that share one value with multiplicity using weighted sharing method. We prove two main theorems which generalize and improve the results of Fang and Fang [2], Dyavanal [1] and others and also solve the open problem posed by Dyavanal. This method yields some new results.

1. Introduction and main results

Let $f(z)$ be a non-constant meromorphic function in the whole complex plane $\mathbb{C}$. We shall use the following standard notations of value distribution theory: $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f), \ldots$ (see [8], [9]). We denote by $S(r,f)$ any function satisfying $S(r,f) = o\{T(r,f)\}$, as $r \to +\infty$, possibly outside of a set with finite measure.

Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_{k)}(r, \frac{1}{f-a})$ the counting function for the zeros of $f(z) - a$ in $|z| \leq r$ with multiplicity $\leq k$ and by $\overline{N}_{k)}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{f-a})$ be the counting function for the zeros of $f(z) - a$ in $|z| \leq r$ with multiplicity $\geq k$ and by $\overline{N}_{(k}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

We say that f and g share a CM (counting multiplicity) if $f - a$ and $g - a$ have same zeros with the same multiplicities. Similarly, we say that f and g share a IM (ignoring multiplicity) if $f - a$ and $g - a$ have same zeros with ignoring multiplicities.

In 2002, Fang and Fang [2] proved the following results.

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Theorem A. *Let f and g be two non-constant meromorphic functions, $n \geq 28$ be a positive integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share 1 IM, then $f \equiv g$.*

Theorem B. *Let f and g be two non-constant entire functions, $n \geq 17$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 IM, then $f \equiv g$.*

Recently, Dyavanal [1] obtained the following theorems by introducing the notion of multiplicity.

Theorem C. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-2)s \geq 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$ or*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function.

Theorem D. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-3)s \geq 10$. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share 1 CM, then $f \equiv g$.*

Theorem E. *Let f and g be two transcendental entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-2)s \geq 5$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In the same paper, Dyavanal posed the following open problem.

Open Problem. Can CM shared value be replaced by an IM shared value in Theorems C-E?

In this paper, we present a unified approach of investigating uniqueness problems of entire or meromorphic functions concerning differential polynomials that share one value with multiplicity using weighted sharing method and also give a positive answer to the above open problem.

Main results

Theorem 1.1. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n, m and l be positive integers. Suppose that $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$ share $(1, l)$,*

(i) *if $l = \infty$ and $(n-m-1)s \geq 10$ or*
(ii) *if $l \geq 2$ and $(n-m-2)s \geq 11$ or*
(iii) *if $l = 1$ and $(n-m-5)s \geq 16$ or*
(iv) *if $l = 0$ and $(n-m-7)s \geq 19$, then*
1. *for $m = 1$ either $f \equiv g$ or*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function.

2. *for $m=2$, $f\equiv g$;*

3. *for $m\geq 3$, either $f\equiv tg$ for a constant such that $t^d=1$, where $d=(n{+}m{+}1, n{+}m, \dots, n{+}1)$, or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where*

$$\begin{aligned}
&R(\omega_1,\omega_2)\\
&=\omega_1^{n+1}\left(\frac{\omega_1^m}{m+p+1}-m\frac{\omega_1^{m-1}}{m+n}+\frac{m(m-1)}{2}\frac{\omega_1^{m-2}}{m+n-1}+\cdots+\frac{(-1)^m}{n+1}\right)\\
&\quad-\omega_2^{n+1}\left(\frac{\omega_2^m}{m+p+1}-m\frac{\omega_2^{m-1}}{m+n}+\frac{m(m-1)}{2}\frac{\omega_2^{m-2}}{m+n-1}+\cdots+\frac{(-1)^m}{n+1}\right).
\end{aligned}$$

Remarks.

1.1. If $m=1$ in Theorem 1.1(i), then Theorem 1.1(i) reduces to Theorem C.

1.2. If $m=2$ in Theorem 1.1(i), then Theorem 1.1(i) reduces to Theorem D.

1.3. If $m=2$ and $s=1$ in Theorem 1.1(iv), then Theorem 1.1(iv) reduces to Theorem A.

1.4. The case $l=0$ in Theorem 1.1 solves the open problem posed by Dyavanal and also generalize Theorems C and D.

Theorem 1.2. *Let f and g be two non-constant entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let n, m and l be positive integers. Suppose that $f^n(f-1)^m f'$ and $g^n(g-1)^m g'$) share $(1,l)$,*

(i) *if $l=\infty$ and $(n-m-1)s\geq 5$ or*

(ii) *if $l\geq 2$ and $(n-m-2)s\geq 5$ or*

(iii) *if $l=1$ and $(n-m-5)s\geq 7$ or*

(iv) *if $l=0$ and $(n-m-7)s\geq 8$, then*

1. *for $m=1$ and $m=2$, $f\equiv g$;*

2. *for $m\geq 3$, either $f\equiv tg$ for a constant such that $t^d=1$, where $d=(n{+}m{+}1, n{+}m, \dots, n{+}1)$, or f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where*

$$\begin{aligned}
&R(\omega_1,\omega_2)\\
&=\omega_1^{n+1}\left(\frac{\omega_1^m}{m+p+1}-m\frac{\omega_1^{m-1}}{m+n}+\frac{m(m-1)}{2}\frac{\omega_1^{m-2}}{m+n-1}+\cdots+\frac{(-1)^m}{n+1}\right)\\
&\quad-\omega_2^{n+1}\left(\frac{\omega_2^m}{m+p+1}-m\frac{\omega_2^{m-1}}{m+n}+\frac{m(m-1)}{2}\frac{\omega_2^{m-2}}{m+n-1}+\cdots+\frac{(-1)^m}{n+1}\right).
\end{aligned}$$

Remarks.

1.1. If $m=1$ in Theorem 1.2(i), then Theorem 1.2(i) reduces to Theorem E.

1.2. If $m=1$ and $s=1$ in Theorem 1.2(iv), then Theorem 1.2(iv) improves Theorem B.

1.3. The case $l=0$ in Theorem 1.2 solves the open problem posed by Dyavanal and also generalize Theorem E.

2. Definitions and some lemmas

Definition 2.1 ([5]). Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. We note that f, g share the value a IM or CM if and only if they share $(a, 0)$ or (a, ∞), respectively.

Lemma 2.1 ([9]). *Let f be a non-constant meromorphic function and n be a positive integer. Let $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f$ where a_i is a meromorphic function satisfying $T(r, a_i) = S(r, f)$, $i = 1, 2, \ldots, n$. Then $T(r, P(f)) = nT(r, f) + S(r, f)$.*

Lemma 2.2 ([9]). *Let f be a meromorphic function and let k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.3 ([4]). *If f and g be two non-constant meromorphic functions. If f and g share 1 CM one of the cases holds*:

(i) $T(r, f) \leq N_2(r, f) + N_2(r, \frac{1}{f}) + N_2(r, g) + N_2(r, \frac{1}{g}) + S(r, f) + S(r, g)$, *the same inequality holding for* $T(r, g)$;

(ii) $f \equiv g$;

(iii) $fg \equiv 1$,

where $N_2(r, \frac{1}{f})$ is the counting function of zeros of f, where a simple zero is counted once and a multiple zero is counted two times. Similarly, we define $N_2(r, f)$.

Lemma 2.4 ([2]). *Let f and g be two non-constant meromorphic functions and let l be a positive integer. If $E_l(1, f) = E_l(1, g)$, then one of the following cases must occur*;

(i) $T(r, f)+T(r, g) \leq N_2(r, f)+N_2(r, \frac{1}{f})+N_2(r, g)+N_2(r, \frac{1}{g})+\overline{N}(r, \frac{1}{f-1})+\overline{N}(r, \frac{1}{g-1}) - N_{11}(r, \frac{1}{f-1}) + \overline{N}_{(l+1}(r, \frac{1}{f-1}) + \overline{N}_{(l+1}(r, \frac{1}{g-1}) + S(r, f) + S(r, g)$;

(ii) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where $a(\neq 0)$, b are two constants.*

where $N_{11}(r, \frac{1}{f-1})$ is the counting function of common simple 1-points of both f and g.

Lemma 2.5 ([2], [7]). *Let f and g be two non-constant meromorphic functions. If f and g share 1 IM, then one of the following cases must occur*;

(i) $T(r, f) + T(r, g) \leq 2(N_2(r, f) + N_2(r, g) + N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g})) + 3(\overline{N}(r, f) + \overline{N}(r, g) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g})) + S(r, f) + S(r, g)$;

(ii) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where* $a(\neq 0)$, b *are two constants.*

Lemma 2.6 ([3])**.** *Let* f *and* g *be two meromorphic functions, and let* $n \geq 6$ *be a positive integer. If*

$$\begin{aligned}&\frac{(n-1)(n-2)}{2}f^n - n(n-2)f^{n-1} + \frac{n(n-1)}{2}f^{n-2}\\ \equiv\ &\frac{(n-1)(n-2)}{2}g^n - n(n-2)g^{n-1} + \frac{n(n-1)}{2}g^{n-2},\end{aligned}$$

then $f \equiv g$.

Using the method in [2] and introducing the order of multiplicity, we prove the following lemmas.

Lemma 2.7. *Let* $f(z)$ *and* $g(z)$ *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s*, where* s *is a positive integer. Let* n *be a positive integer satisfying* $(n-3m)s \geq 2$*. Let* $F = f^n(f-1)^m f'$, $G = g^n(g-1)^m g'$*. If*

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \tag{1}$$

where $a(\neq 0)$, b *are two constants, then*

(i) *for* $m = 2$, $f \equiv g$;

(ii) *for* $m \geq 3$*, either* $f \equiv tg$ *for a constant such that* $t^d = 1$*, where* $d = (n+m+1, n+m, \ldots, n+1)$*, or* f *and* g *satisfy the algebraic equation* $R(f, g) \equiv 0$*, where*

$$\begin{aligned}&R(\omega_1, \omega_2)\\ &= \omega_1^{n+1}\left(\frac{\omega_1^m}{m+p+1} - m\frac{\omega_1^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{\omega_1^{m-2}}{m+n-1} + \cdots + \frac{(-1)^m}{n+1}\right)\\ &\quad - \omega_2^{n+1}\left(\frac{\omega_2^m}{m+p+1} - m\frac{\omega_2^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{\omega_2^{m-2}}{m+n-1} + \cdots + \frac{(-1)^m}{n+1}\right).\end{aligned}$$

Proof. By Lemma 2.1 and in view of $T(r, f') \leq T(r, f) + \overline{N}(r, f) + S(r, f)$, we have

$$\begin{aligned}T(r, F) &= T(r, f^n(f-1)^m f')\\ &\leq T(r, f^n(f-1)^m) + T(r, f') + O(1).\end{aligned}\tag{2}$$

By assumption, zeros and poles of f and g are of multiplicities at least s, i.e.,

$$\begin{aligned}&\overline{N}(r, f) \leq \frac{1}{s}N(r, f) \leq \frac{1}{s}T(r, f), \quad \overline{N}(r, g) \leq \frac{1}{s}T(r, g),\\ &\overline{N}\left(r, \frac{1}{f}\right) \leq \frac{1}{s}N\left(r, \frac{1}{f}\right) \leq \frac{1}{s}T\left(r, \frac{1}{f}\right), \quad \overline{N}\left(r, \frac{1}{g}\right) \leq \frac{1}{s}T(r, g)\end{aligned}$$

and

$$T(r, f') \leq \left(1 + \frac{1}{s}\right)T(r, f) + S(r, f). \tag{3}$$

Using (3) and Lemma 2.1, (2) becomes

$$T(r,F) \le \left(n+m+1+\frac{1}{s}\right)T(r,f)+S(r,f) \tag{4}$$

and

$$\begin{aligned}(n+m)T(r,f) &= T(r,f^n(f-1)^m)+S(r,f)\\ &= m(r,f^n(f-1)^m)+N(r,f^n(f-1)^m)+S(r,f)\\ &= (n+m)T(r,f)\\ &\le N(r,f^n(f-1)^m f')-N(r,f')+m(r,f^n(f-1)^m f')\\ &\quad + m\left(r,\frac{1}{f'}\right)+S(r,f)\end{aligned} \tag{5}$$

since $N(r,f^n(f-1)^m)=N(r,f^n(f-1)^m f')-N(r,f')$.

By (5) and the first fundamental theorem, we have

$$\begin{aligned}(n+m)T(r,f) &\le T(r,f^n(f-1)^m f')-N(r,f')+T(r,f')\\ &\quad - N\left(r,\frac{1}{f'}\right)+S(r,f)\\ &\le T(r,F)+m(r,f')-N\left(r,\frac{1}{f'}\right)+S(r,f)\\ &\le T(r,F)+T(r,f)-N(r,f)-N\left(r,\frac{1}{f'}\right)+S(r,f),\end{aligned}$$

$$T(r,F) \ge (n+m-1)T(r,f)+N(r,f)+N\left(r,\frac{1}{f'}\right)+S(r,f). \tag{6}$$

From (4) and (6), we get $S(r,F)=S(r,f)$. Similarly, we get

$$\begin{aligned}&T(r,G) \ge (n+m-1)T(r,g)+N(r,g)+N\left(r,\frac{1}{g'}\right)+S(r,g) \quad \text{and}\\ &S(r,G)=S(r,g).\end{aligned} \tag{7}$$

Without loss of generality, we suppose that $T(r,f) \le T(r,g)$, $r \in I$, where I is a set with infinite measure. Next, we consider 3 cases.

Case 1. $b \ne 0,-1$. If $a-b-1 \ne 0$, then by (1) we have

$$\overline{N}\left(r,\frac{1}{G+\frac{a-b-1}{b+1}}\right)=\overline{N}\left(r,\frac{1}{F}\right).$$

By the Nevanlinna's second fundamental theorem and Lemma 2.2, we have

$$\begin{aligned}T(r,G) &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G+\frac{a-b-1}{b+1}}\right)+S(r,G)\\ &\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,g)\end{aligned}$$

$$\leq \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+\overline{N}\left(r,\frac{1}{f}\right)$$
$$+\overline{N}\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)+S(r,g).$$

By using multiplicity and Lemma 2.2, we have

$$N\left(r,\frac{1}{f'}\right)\leq\left(1+\frac{1}{s}\right)T(r,f)+S(r,f).$$

So,

$$\begin{aligned}T(r,G)&\leq \overline{N}(r,g)+\frac{1}{s}T(r,g)+\overline{N}\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)\\&\quad+\frac{1}{s}T(r,f)+\overline{N}\left(r,\frac{1}{f-1}\right)+\left(1+\frac{1}{s}\right)T(r,f)+S(r,g)\\&\leq\left(1+\frac{1}{s}\right)T(r,g)+\overline{N}(r,g)+N\left(r,\frac{1}{g'}\right)+\left(2+\frac{2}{s}\right)T(r,f)+S(r,g).\end{aligned}$$

Therefore

$$T(r,G)\leq\left(3+\frac{3}{s}\right)T(r,g)+\overline{N}(r,g)+N\left(r,\frac{1}{g'}\right)+S(r,g).$$

By (7), we get

$$(n+m-1)T(r,g)\leq\left(3+\frac{3}{s}\right)T(r,g)+S(r,g),$$

we obtain $(n+m-4)s\leq 3$ which contradicts $(n-3m)s\geq 2$.

If $a-b-1=0$, then (1) becomes

$$F=\frac{(b+1)G}{bG+1}.$$

Clearly

$$\overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right)=\overline{N}(r,F).$$

By the Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}T(r,G)&\leq \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right)+S(r,g)\\&=\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,g)\\&\leq\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+\overline{N}(r,f)+S(r,g)\\&\leq\overline{N}(r,g)+\frac{1}{s}T(r,g)+\overline{N}\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+\frac{1}{s}T(r,f)+S(r,g)\\&\leq\left(1+\frac{2}{s}\right)T(r,g)+\overline{N}(r,g)+N\left(r,\frac{1}{g'}\right)+S(r,g).\end{aligned}$$

By (7), we get

$$(n+m-1)T(r,g) \le \left(1+\frac{2}{s}\right)T(r,g)+S(r,g),$$

we obtain that $(n+m-2)s \le 2$ which contradicts $(n-3m)s \ge 2$.

Case 2. $b=-1$. Then (1) takes the form as

$$F=\frac{a}{(a+1)-G}.$$

If $a+1\neq 0$, then

$$\overline{N}\left(r,\frac{1}{G-(a+1)}\right)=\overline{N}(r,F).$$

Similarly, we can deduce a contradiction as in Case 1.

If $a+1=0$, then (1) becomes $FG\equiv 1$, i.e.,

$$f^n(f-1)^m f' g^n (g-1)^m g' \equiv 1. \tag{8}$$

Let z_0 be a zero of $f(z)$ with multiplicity p. Then by (8), z_0 is a pole of $g(z)$ with multiplicity q. Thus by (8),

$$\begin{aligned} np+p-1 &= nq+mq+q+1\\ \Longrightarrow (p-q)(n+1) &= mq+2\\ \Longrightarrow p\ge q+1 \quad &\text{and} \quad mq+2\ge n+1. \end{aligned}$$

Hence, by simple computing we obtain

$$p \ge \frac{n+m-1}{m}. \tag{9}$$

Let z_1 be a zero of $(f(z)-1)$ with multiplicity p_1, then z_1 is a pole of G with multiplicity q_1. Then by (8), we get

$$\begin{aligned} mp_1+p_1-1 &= nq_1+mq_1+q_1+1,\\ (m+1)p_1 &= (n+m+1)q_1+2, \end{aligned}$$

i.e.,

$$p_1 \ge \frac{(n+m+1)s+2}{m+1} = \frac{(n-3m)s+4ms+s+2}{m+1} \ge \frac{(4m+1)s+4}{m+1}. \tag{10}$$

By (8), we know that

$$\overline{N}(r,f^n(f-1)^m f') = \overline{N}\left(r,\frac{1}{g^n(g-1)^m g'}\right),$$

i.e.,

$$\overline{N}(r,f) = \overline{N}\left(r,\frac{1}{g^n(g-1)^m g'}\right).$$

By the Nevanlinna's second fundamental theorem, we have

$$T(r,f) \le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)-N_0\left(r,\frac{1}{f'}\right)+S(r,f)$$

$$\leq \overline{N}\left(r, \frac{1}{g^n(g-1)^m g'}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right)$$
$$- N_0\left(r, \frac{1}{f'}\right) + S(r,f)$$
$$\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + N_0\left(r, \frac{1}{g'}\right) + \overline{N}\left(r, \frac{1}{f}\right)$$
$$+ \overline{N}\left(r, \frac{1}{f-1}\right) - N_0\left(r, \frac{1}{f'}\right) + S(r,f).$$

Similarly,

$$T(r,g) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + N_0\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{g}\right)$$
$$+ \overline{N}\left(r, \frac{1}{g-1}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r,g).$$

Combining above two equations, we obtain

$$T(r,f) + T(r,g) \leq 2\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{f-1}\right) + 2\overline{N}\left(r, \frac{1}{g}\right) + 2\overline{N}\left(r, \frac{1}{g-1}\right)$$
$$+ S(r,f) + S(r,g).$$

Using (9) and (10), we get

$$T(r,f) + T(r,g)$$
$$\leq \frac{2m}{n+m-1}T(r,f) + \frac{2(m+1)}{(4m+1)s+4}T(r,f) + \frac{2m}{n+m-1}T(r,g)$$
$$+ \frac{2(m+1)}{(4m+1)s+4}T(r,g) + S(r,f) + S(r,g)$$
$$= \left\{\frac{2m}{n+m-1} + \frac{2(m+1)}{(4m+1)s+4}\right\}(T(r,f) + T(r,g)) + S(r,f) + S(r,g).$$

For $m \geq 2$, giving specific values for n and s which satisfies $(n-3m)s \geq 2$, we deduce that

$$T(r,f) + T(r,g) \leq (0.8)(T(r,f) + T(r,g)) + S(r,f) + S(r,g),$$

which is a contradiction.

Case 3. $b = 0$. Then (1) gives

$$F = G + \frac{a-1}{a}.$$

If $a - 1 \neq 0$. Then

$$\overline{N}\left(r, \frac{1}{G + (a-1)}\right) = \overline{N}\left(r, \frac{1}{f}\right).$$

Similarly, we can again deduce a contradiction as in Case 1.

If $a-1=0$, then $F\equiv G$, i.e.,

$$f^n(f-1)^m f' \equiv g^n(g-1)^m g',$$

where $(f-1)^m = f^m - mf^{m-1} + \frac{m(m-1)}{2}f^{m-2} - \ldots + (-1)^m$.

On integrating, we obtain

$$\begin{aligned} F^* &= \frac{f^{m+n+1}}{m+n+1} - m\frac{f^{m+n}}{m+n} + \frac{m(m-1)}{2}\frac{f^{m+n-1}}{m+n-1} - \ldots + (-1)^m\frac{f^{n+1}}{n+1} \\ &= \frac{g^{m+n+1}}{m+n+1} - m\frac{g^{m+n}}{m+n} + \frac{m(m-1)}{2}\frac{g^{m+n-1}}{m+n-1} - \ldots + (-1)^m\frac{g^{n+1}}{n+1} \\ &= G^* + c, \end{aligned}$$

where c is a constant.

If $c \neq 0$, then by the Nevanlinna's second fundamental theorem and Lemma 2.1, we have

$$\begin{aligned} &(n+m+1)T(r,g) \\ &= T(r,G^*) + S(r,g) \\ &\leq \overline{N}(r,G^*) + \overline{N}\left(r,\frac{1}{G^*}\right) + \overline{N}\left(r,\frac{1}{G^*+c}\right) + S(r,g) \\ &\leq \overline{N}(r,G^*) + \overline{N}\left(r,\frac{1}{G^*}\right) + \overline{N}\left(r,\frac{1}{F^*}\right) + S(r,g) \\ &\leq \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{f}\right) \\ &\quad + \overline{N}\left(r,\frac{1}{\frac{g^m}{n+m+1} - m\frac{g^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{g^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m}{n+1}}\right) \\ &\quad + \overline{N}\left(r,\frac{1}{\frac{f^m}{n+m+1} - m\frac{f^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{f^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m}{n+1}}\right) \\ &\quad + S(r,g) \\ &\leq \frac{1}{s}T(r,g) + \frac{1}{s}T(r,g) + mT(r,g) + \frac{1}{s}T(r,f) + mT(r,f) + S(r,g) \\ &\leq \left(\frac{3}{s} + 2m\right)T(r,g) + S(r,g), \end{aligned}$$

we obtain that $(n-m+1)s \leq 3$ which contradicts $(n-3m)s \geq 2$.

Hence $c=0$. Therefore $F^* \equiv G^*$, i.e.,

$$\begin{aligned} &f^{n+1}\left\{\frac{f^m}{n+m+1} - m\frac{f^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{f^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m}{n+1}\right\} \\ &= g^{n+1}\left\{\frac{g^m}{n+m+1} - m\frac{g^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{g^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m}{n+1}\right\}. \end{aligned} \tag{11}$$

(i) For $m = 2$, we have

$$\frac{f^{n+3}}{n+3} - 2\frac{f^{n+2}}{n+2} + \frac{f^{n+1}}{n+1} = \frac{g^{n+3}}{n+3} - 2\frac{g^{n+2}}{n+2} + \frac{g^{n+1}}{n+1}$$
$$\Longrightarrow \quad \frac{(n+2)(n+1)}{2} f^{n+3} - (n+3)(n+1) f^{n+2} + \frac{(n+3)(n+2)}{2} f^{n+1}$$
$$= \frac{(n+2)(n+1)}{2} g^{n+3} - (n+3)(n+1) g^{n+2} + \frac{(n+3)(n+2)}{2} g^{n+1}.$$

Using Lemma 2.6, we obtain $f \equiv g$.

(ii) For $m \geq 3$. Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (11), we deduce

(12)
$$(gh)^{n+1}\left\{\frac{(gh)^m}{m+n+1} - m\frac{(gh)^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{(gh)^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m}{n+1}\right\}$$
$$- g^{n+1}\left\{\frac{g^m}{n+m+1} - m\frac{g^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{g^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m}{n+1}\right\} = 0,$$
$$\frac{g^{n+m+1}(h^{n+m+1}-1)}{m+n+1} - m\frac{g^{n+m}(h^{m+n}-1)}{m+n} + \frac{m(m-1)}{2}\frac{g^{m+n-1}(h^{m+n-1}-1)}{m+n-1}$$
$$- \cdots + (-1)^m \frac{g^{n+1}(h^{n+1}-1)}{n+1} = 0,$$

which implies that $h^d = 1$, where $d = (n+m+1, m+n, \ldots, n+1)$. Thus $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = (n+m+1, m+n, \ldots, n+1)$.

If h is not a constant, then we know by (12) that f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(\omega_1, \omega_2)$$
$$= \omega_1^{n+1}\left\{\frac{\omega_1^m}{n+m+1} - m\frac{\omega_1^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{\omega_1^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m}{n+1}\right\}$$
$$- \omega_2^{n+1}\left\{\frac{\omega_2^m}{n+m+1} - m\frac{\omega_2^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{\omega_2^{m-2}}{m+n-1} - \cdots + \frac{(-1)^m}{n+1}\right\}.$$

□

Lemma 2.8. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n be a positive integer satisfying $(n-3)s \geq 4$. Let $F = f^n(f-1)f'$, $G = g^n(g-1)g'$. If*

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)},$$

where $a(\neq 0)$, b are two constants, then either $f \equiv g$ or

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function.

Lemma 2.9. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let n be a positive integer satisfying $(n-m)s \geq 5$. Let $F = f^n(f-1)^m f'$, $G = g^n(g-1)^m g'$.*

If

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)},$$

where $a(\neq 0)$, b are two constants, then

(i) *for $m = 1$ and $m = 2$, $f \equiv g$;*

(ii) *for $m \geq 3$, either $f \equiv tg$ for a constant such that $t^d = 1$, where $d = (n+m+1, n+m, \ldots, n+1)$, or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where*

$$\begin{aligned} &R(\omega_1, \omega_2) \\ &= \omega_1^{n+1}\left(\frac{\omega_1^m}{m+p+1} - m\frac{\omega_1^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{\omega_1^{m-2}}{m+n-1} + \ldots\ldots + \frac{(-1)^m}{n+1}\right) \\ &\quad - \omega_2^{n+1}\left(\frac{\omega_2^m}{m+p+1} - m\frac{\omega_2^{m-1}}{m+n} + \frac{m(m-1)}{2}\frac{\omega_2^{m-2}}{m+n-1} + \ldots\ldots + \frac{(-1)^m}{n+1}\right). \end{aligned}$$

Using the same argument as in the proof of Lemma 2.7, we easily prove Lemmas 2.8 and 2.9.

3. Proof of theorems

Proof of Theorem 1.1. Let

$$F = f^n(f-1)^m f', \quad G = g^n(g-1)^m g', \tag{13}$$

where $(f-1)^m = f^m - mf^{m-1} + \frac{m(m-1)}{2}f^{m-2} - \cdots + (-1)^m$. Then

(14)
$$F^* = \frac{f^{m+n+1}}{m+n+1} - m\frac{f^{m+n}}{m+n} + \frac{m(m-1)}{2}\frac{f^{m+n-1}}{m+n-1} \ldots\ldots + (-1)^m\frac{f^{n+1}}{n+1},$$

(15)
$$G^* = \frac{g^{m+n+1}}{m+n+1} - m\frac{g^{m+n}}{m+n} + \frac{m(m-1)}{2}\frac{g^{m+n-1}}{m+n-1} \ldots\ldots + (-1)^m\frac{g^{n+1}}{n+1}.$$

By Lemma 2.1, we have

$$T(r, F^*) = (m+n+1)T(r,f) + s(r,f), \tag{16}$$

$$T(r, G^*) = (m+n+1)T(r,g) + s(r,g). \tag{17}$$

Since $(F^*)' = F$, we have

$$m\left(r, \frac{1}{F^*}\right) \leq m\left(r, \frac{(F^*)'}{F^*}\right) + m\left(r, \frac{1}{(F^*)'}\right)$$

$$\leq m\left(r,\frac{1}{F}\right)+S(r,f).$$

By the first fundamental theorem, we have

$$T(r,F^*)\leq T(r,F)+N\left(r,\frac{1}{F^*}\right)-N\left(r,\frac{1}{F}\right)+S(r,f). \tag{18}$$

Clearly,

(19)

$$N\left(r,\frac{1}{F^*}\right)=(n+1)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a_1}\right)++\cdots+N\left(r,\frac{1}{f-a_m}\right),$$

where $a_1,a_2,\ldots,a_m$ are distinct roots of algebraic equation

$$\frac{z^m}{m+n+1}-m\frac{z^{m-1}}{m+n}+\frac{m(m-1)}{2}\frac{z^{m-2}}{m+n-1}-\cdots\frac{(-1)^m}{n+1}=0$$

and

$$N\left(r,\frac{1}{F}\right)=nN\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right). \tag{20}$$

Substituting (19) and (20) in (18), we obtain

$$\begin{aligned}T(r,F^*)\leq{}&T(r,F)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a_1}\right)++\cdots+N\left(r,\frac{1}{f-a_m}\right)\\&-mN\left(r,\frac{1}{f-1}\right)-N\left(r,\frac{1}{f'}\right)+S(r,f).\end{aligned}$$

Similarly,

$$\begin{aligned}T(r,G^*)\leq{}&T(r,G)+N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-a_1}\right)++\cdots+N\left(r,\frac{1}{g-a_m}\right)\\&-mN\left(r,\frac{1}{g-1}\right)-N\left(r,\frac{1}{g'}\right)+S(r,g).\end{aligned}$$

Adding above two equations, we get

$$\begin{aligned}&T(r,F^*)+T(r,G^*)\\&\leq(T(r,F)+T(r,G))+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a_1}\right)\\&\quad+\cdots+N\left(r,\frac{1}{f-a_m}\right)-mN\left(r,\frac{1}{f-1}\right)-N\left(r,\frac{1}{f'}\right)\\&\quad+N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-a_1}\right)+\cdots+N\left(r,\frac{1}{g-a_m}\right)\\&\quad-mN\left(r,\frac{1}{g-1}\right)-N\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g).\end{aligned} \tag{21}$$

By (13), we have

$$N_2(r,F)+N_2\left(r,\frac{1}{F}\right)\le 2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right), \tag{22}$$

$$N_2(r,G)+N_2\left(r,\frac{1}{G}\right)\le 2\overline{N}(r,g)+2\overline{N}\left(r,\frac{1}{g}\right)+mN\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right) \tag{23}$$

and

$$\overline{N}\left(r,\frac{1}{F}\right)\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{f'}\right), \tag{24}$$

$$\overline{N}\left(r,\frac{1}{G}\right)\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right). \tag{25}$$

(i) If $l=\infty$, then by Definition 2.1, $F=f^n(f-1)^mf'$ and $G=g^n(g-1)^mg'$ share the value 1 CM.

Suppose that (i) in Lemma 2.3 holds, then we have

$$T(r,F)+T(r,G)\le 2\left[N_2(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)\right]+S(r,f)+S(r,g).$$

Using (22) and (23), we get

$$\begin{aligned}T(r,F)+T(r,G)\le{}& 4\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{f}\right)+2mN\left(r,\frac{1}{f-1}\right)+2N\left(r,\frac{1}{f'}\right)\\&+4\overline{N}(r,g)+4\overline{N}\left(r,\frac{1}{g}\right)+2mN\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)\\&+S(r,f)+S(r,g).\end{aligned} \tag{26}$$

From (21) and (26), we deduce that

$$\begin{aligned}T(r,F^*)+T(r,G^*)\le{}& 4\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f-1}\right)\\&+N\left(r,\frac{1}{f'}\right)+N\left(r,\frac{1}{f-a_1}\right)+\cdots+N\left(r,\frac{1}{f-a_m}\right)\\&+4\overline{N}(r,g)+4\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g}\right)+mN\left(r,\frac{1}{g-1}\right)\\&+N\left(r,\frac{1}{g'}\right)+N\left(r,\frac{1}{g-a_1}\right)+\cdots+N\left(r,\frac{1}{g-a_m}\right)\\&+S(r,f)+S(r,g).\end{aligned} \tag{27}$$

By hypothesis, zeros and poles of f and g are of multiplicities at least s, i.e.,

$$\overline{N}(r,f) \le \frac{1}{s}N(r,f) \le \frac{1}{s}T(r,f), \quad \overline{N}\left(r,\frac{1}{f}\right) \le \frac{1}{s}\left(r,\frac{1}{f}\right) \le \frac{1}{s}T(r,f)$$

and

$$N\left(r,\frac{1}{f'}\right) \le N\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f) \le \left(1+\frac{1}{s}\right)T(r,f) + S(r,f).$$

Therefore by (16) and (17), we have

$$\begin{aligned}
&(n+m+1)(T(r,f)+T(r,g))\\
\le\ &\left(1+\frac{8}{s}\right)T(r,f) + mN\left(r,\frac{1}{f-1}\right) + \left(1+\frac{1}{s}\right)T(r,f)\\
&+ N\left(r,\frac{1}{f-a_1}\right) + \cdots + N\left(r,\frac{1}{f-a_m}\right) + \left(1+\frac{8}{s}\right)T(r,g)\\
&+ mN\left(r,\frac{1}{g-1}\right) + \left(1+\frac{1}{s}\right)T(r,g) + N\left(r,\frac{1}{g-a_1}\right)\\
&+ \cdots + N\left(r,\frac{1}{g-a_m}\right) + S(r,f) + S(r,g),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&(n+m+1)(T(r,f)+T(r,g))\\
\le\ &\left(\frac{9}{s}+2m+2\right)(T(r,f)+T(r,g)) + S(r,f) + S(r,g).
\end{aligned}$$

We obtain $(n-m-1)s \le 9$ which contradicts our hypothesis $(n-m-1)s \ge 10$. Thus, by Lemma 2.3 we get either $FG \equiv 1$ or $F \equiv G$.

Next we consider two cases.

Case 1. $FG \equiv 1$, i.e., $f^n(f-1)^m f' g^n (g-1)^m g' \equiv 1$.

Proceeding as in the proof of Lemma 2.7, we get a contradiction.

Case 2. $F \equiv G$, i.e., $f^n(f-1)^m f' \equiv g^n(g-1)^m g'$.

Proceeding as in the proof of Lemma 2.7, we get the conclusion of Theorem 1.1.

(ii) Let $l \ge 2$. By using the definitions of counting functions $\overline{N}$, N_{11} and $\overline{N}_{(l+1}$ we get the following

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) - N_{11}\left(r,\frac{1}{F-1}\right) + \frac{1}{2}\overline{N}_{(l+1}\left(r,\frac{1}{F-1}\right)\\
&+ \frac{1}{2}\overline{N}_{(l+1}\left(r,\frac{1}{G-1}\right)\\
\le\ &\frac{1}{2}N\left(r,\frac{1}{F-1}\right) + \frac{1}{2}N\left(r,\frac{1}{G-1}\right)\\
\le\ &\frac{1}{2}T(r,F) + \frac{1}{2}T(r,G) + S(r,f) + S(r,g).
\end{aligned}$$

Suppose that (i) in Lemma 2.4 holds, then using above inequality we have

$$\begin{aligned} T(r,F)+T(r,G) &\le 2\left[N_2(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)\right] \\ &\quad +\overline{N}_{(l+1}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(l+1}\left(r,\frac{1}{G-1}\right) \\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{28}$$

If z_0 is a zero of $(F-1)$ with multiplicity $p \ge l+1$, then z_0 is a pole of $\frac{F}{F'}$, so we have

$$\begin{aligned} \overline{N}_{(l+1}\left(r,\frac{1}{F-1}\right) &\le \frac{1}{2}N\left(r,\frac{F}{F'}\right) \\ &= \frac{1}{2}N\left(r,\frac{F'}{F}\right)+S(r,f) \\ &\le \frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f) \\ &\le \frac{1}{2}\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)\right] \\ &\quad +S(r,f) \end{aligned}$$

using multiplicity, we can write this as

$$\overline{N}_{(l+1}\left(r,\frac{1}{F-1}\right) \le \frac{1}{2}\left(\frac{3}{s}+2\right)T(r,f)+S(r,f). \tag{29}$$

Similarly,

$$\overline{N}_{(l+1}\left(r,\frac{1}{G-1}\right) \le \frac{1}{2}\left(\frac{3}{s}+2\right)T(r,g)+S(r,g). \tag{30}$$

Substituting (22), (23), (29), (30) in (28), we obtain

$$\begin{aligned} T(r,F)+T(r,G) &\le 4\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{f}\right)+2m\overline{N}\left(r,\frac{1}{f-1}\right)+2\overline{N}\left(r,\frac{1}{f'}\right) \\ &\quad +4\overline{N}(r,g)+4\overline{N}\left(r,\frac{1}{g}\right)+2m\overline{N}\left(r,\frac{1}{g-1}\right)+2\overline{N}\left(r,\frac{1}{g'}\right) \\ &\quad +\frac{1}{2}\left(\frac{3}{s}+2\right)T(r,f)+\frac{1}{2}\left(\frac{3}{s}+2\right)T(r,g)+S(r,g)+S(r,f). \end{aligned}$$

Therefore (21) becomes

$$\begin{aligned} &T(r,F^*)+T(r,G^*) \\ &\le 4\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right) \\ &\quad +N\left(r,\frac{1}{f-a_1}\right)+\cdots+N\left(r,\frac{1}{f-a_m}\right)+4\overline{N}(r,g)+4\overline{N}\left(r,\frac{1}{g}\right) \end{aligned} \tag{31}$$

$$+N\left(r,\frac{1}{g}\right)+mN\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+N\left(r,\frac{1}{g-a_1}\right)$$
$$+\cdots+N\left(r,\frac{1}{g-a_m}\right)+\frac{1}{2}\left(\frac{3}{s}+2\right)T(r,f)+\frac{1}{2}\left(\frac{3}{s}+2\right)T(r,g)$$
$$+S(r,f)+S(r,g),$$

$$\begin{aligned}&(n+m+1)(T(r,f)+T(r,g))\\ \le\ &\left(\frac{8}{s}+1\right)T(r,f)+N\left(r,\frac{1}{f-a_1}\right)+\cdots+N\left(r,\frac{1}{f-a_m}\right)\\ &+mN\left(r,\frac{1}{f-1}\right)+\left(\frac{1}{s}+1\right)T(r,f)+\left(\frac{8}{s}+1\right)T(r,g)\\ &+N\left(r,\frac{1}{g-a_1}\right)+\cdots+N\left(r,\frac{1}{g-a_m}\right)+mN\left(r,\frac{1}{g-1}\right)\\ &+\left(\frac{1}{s}+1\right)T(r,g)+\frac{1}{2}\left(\frac{3}{s}+2\right)T(r,f)+\frac{1}{2}\left(\frac{3}{s}+2\right)T(r,g),\\ &+S(r,f)+S(r,g),\end{aligned}$$

$$\begin{aligned}&(n+m+1)(T(r,f)+T(r,g))\\ \le\ &\left(\frac{21}{2s}+2m+3\right)(T(r,f)+T(r,g))+S(r,f)+S(r,g).\end{aligned}$$

We obtain $(n-m-2)s\le\frac{21}{2}$ which contradicts our hypothesis $(n-m-2)s\ge 11$. Hence F and G satisfy (ii) in Lemma 2.4, i.e.,

$$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)},$$

where $a(\neq 0)$, b are two constants.

Then by Lemmas 2.7 and 2.8, we get the conclusion of Theorem 1.1.

(iii) Let $l=1$. Again by using the definitions of $\overline{N}$ and N_{11}, we have

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)\\ \le\ &\frac{1}{2}N\left(r,\frac{1}{F-1}\right)+\frac{1}{2}N\left(r,\frac{1}{G-1}\right)\\ \le\ &\frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g).\end{aligned}$$

Suppose that (i) in Lemma 2.4 holds, then we have

$$\begin{aligned}T(r,F)+T(r,G)\le\ &2\left[N_2(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)\right]\\ &+2\left[\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{G-1}\right)\right]\end{aligned}$$

$$+ S(r,f) + S(r,g). \tag{32}$$

Since

$$\begin{aligned}\overline{N}_{(2}\left(r,\frac{1}{F-1}\right) &\le N\left(r,\frac{F}{F'}\right) \le T\left(r,\frac{F}{F'}\right)\\ &= T\left(r,\frac{F'}{F}\right) + O(1)\\ &= N\left(r,\frac{F'}{F}\right) + S(r,f)\\ &\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f)\\ &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + N\left(r,\frac{1}{f'}\right) + S(r,f)\end{aligned}$$

using multiplicity, we have

$$\overline{N}_{(2}\left(r,\frac{1}{F-1}\right) \le \left(\frac{3}{s}+2\right)T(r,f) + S(r,f). \tag{33}$$

Similarly,

$$\overline{N}_{(2}\left(r,\frac{1}{G-1}\right) \le \left(\frac{3}{s}+2\right)T(r,g) + S(r,g). \tag{34}$$

Substituting (22), (23), (33), (34) in (32), we obtain

$$\begin{aligned}T(r,F) + T(r,G) \le\ & 4\overline{N}(r,f) + 4\overline{N}\left(r,\frac{1}{f}\right) + 2m\overline{N}\left(r,\frac{1}{f-1}\right) + 2N\left(r,\frac{1}{f'}\right)\\ &+ 4\overline{N}(r,g) + 4\overline{N}\left(r,\frac{1}{g}\right) + 2m\overline{N}\left(r,\frac{1}{g-1}\right) + 2N\left(r,\frac{1}{g'}\right)\\ &+ 2\left(\frac{3}{s}+2\right)T(r,f) + 2\left(\frac{3}{s}+2\right)T(r,g) + S(r,g) + S(r,f).\end{aligned}$$

Therefore (21) becomes

$$\begin{aligned}&T(r,F^*) + T(r,G^*) \\ &\le 4\overline{N}(r,f) + 4\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f-1}\right) + N\left(r,\frac{1}{f'}\right)\\ &\quad + N\left(r,\frac{1}{f-a_1}\right) + \cdots + N\left(r,\frac{1}{f-a_m}\right) + 4\overline{N}(r,g) + 4\overline{N}\left(r,\frac{1}{g}\right)\\ &\quad + N\left(r,\frac{1}{g}\right) + mN\left(r,\frac{1}{g-1}\right) + N\left(r,\frac{1}{g'}\right) + N\left(r,\frac{1}{g-a_1}\right)\\ &\quad + \cdots + N\left(r,\frac{1}{g-a_m}\right) + 2\left(\frac{3}{s}+2\right)T(r,f) + 2\left(\frac{3}{s}+2\right)T(r,g)\\ &\quad + S(r,f) + S(r,g),\end{aligned} \tag{35}$$

$$
\begin{aligned}
&(n+m+1)(T(r,f)+T(r,g)) \\
\leq\ & \left(\frac{8}{s}+1\right)T(r,f)+N\left(r,\frac{1}{f-a_1}\right)+\cdots+N\left(r,\frac{1}{f-a_m}\right) \\
&+mN\left(r,\frac{1}{f-1}\right)+\left(\frac{1}{s}+1\right)T(r,f)+\left(\frac{8}{s}+1\right)T(r,g) \\
&+N\left(r,\frac{1}{g-a_1}\right)+\cdots+N\left(r,\frac{1}{g-a_m}\right)+mN\left(r,\frac{1}{g-1}\right) \\
&+\left(\frac{1}{s}+1\right)T(r,g)+2\left(\frac{3}{s}+2\right)T(r,f)+2\left(\frac{3}{s}+2\right)T(r,g) \\
&+S(r,f)+S(r,g), \\
&(n+m+1)(T(r,f)+T(r,g)) \\
\leq\ & \left(\frac{15}{s}+2m+6\right)(T(r,f)+T(r,g))+S(r,f)+S(r,g).
\end{aligned}
$$

We obtain $(n-m-5)s \leq 15$ which contradicts our hypothesis $(n-m-5)s \geq 16$. Hence F and G satisfy (ii) in Lemma 2.4, i.e.,

$$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)},$$

where $a(\neq 0)$, b are two constants.

Then by Lemmas 2.7 and 2.8, we get the conclusion of Theorem 1.1.

(iv) If $l=0$, then by Definition 2.1, $F=f^n(f-1)^m f'$ and $G=g^n(g-1)^m g'$ share the value 1 IM.

Suppose that (i) in Lemma 2.5 holds, then we have

$$
\begin{aligned}
T(r,F)+T(r,G) \leq\ & 2\left[N_2(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)\right] \\
&+3\left[\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)\right] \\
&+S(r,f)+S(r,g).
\end{aligned}
$$

Substituting (22), (23), (24) and (25), we obtain

$$
\begin{aligned}
&T(r,F)+T(r,G) \\
\leq\ & 4\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{f}\right)+2mN\left(r,\frac{1}{f-1}\right)+2N\left(r,\frac{1}{f'}\right) \\
&+4\overline{N}(r,g)+4\overline{N}\left(r,\frac{1}{g}\right)+2mN\left(r,\frac{1}{g-1}\right)+2N\left(r,\frac{1}{g'}\right) \\
&+3\overline{N}(r,f)+3\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}\left(r,\frac{1}{f-1}\right)+3N\left(r,\frac{1}{f'}\right) \\
&+3\overline{N}(r,g)+3\overline{N}\left(r,\frac{1}{g}\right)+3\overline{N}\left(r,\frac{1}{g-1}\right)+3N\left(r,\frac{1}{g'}\right)
\end{aligned}
$$

$$+ S(r,f) + S(r,g).$$

Therefore (21) becomes

$$(36)\qquad T(r,F^*) + T(r,G^*)$$
$$\le 7\overline{N}(r,f) + 7\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f-1}\right)$$
$$+ 4N\left(r,\frac{1}{f'}\right) + 3\overline{N}\left(r,\frac{1}{f-1}\right) + N\left(r,\frac{1}{f-a_1}\right) + \cdots$$
$$+ N\left(r,\frac{1}{f-a_m}\right) + 7\overline{N}(r,g) + 7\overline{N}\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g}\right)$$
$$+ mN\left(r,\frac{1}{g-1}\right) + 4N\left(r,\frac{1}{g'}\right) + 3\overline{N}\left(r,\frac{1}{g-1}\right) + N\left(r,\frac{1}{g-a_1}\right),$$
$$+ \cdots + N\left(r,\frac{1}{g-a_m}\right) + S(r,f) + S(r,g).$$

Using (16) and (17), we get

$$(n+m+1)(T(r,f)+T(r,g))$$
$$\le \left(1+\frac{14}{s}\right)T(r,f) + mN\left(r,\frac{1}{f-1}\right) + 4\left(1+\frac{1}{s}\right)T(r,f)$$
$$+ 3\overline{N}\left(r,\frac{1}{f-1}\right) + N\left(r,\frac{1}{f-a_1}\right) + \cdots + N\left(r,\frac{1}{f-a_m}\right)$$
$$+ \left(1+\frac{14}{s}\right)T(r,g) + mN\left(r,\frac{1}{g-1}\right) + 4\left(1+\frac{1}{s}\right)T(r,g)$$
$$+ 3\overline{N}\left(r,\frac{1}{g-1}\right) + N\left(r,\frac{1}{g-a_1}\right) + \cdots + N\left(r,\frac{1}{g-a_m}\right)$$
$$+ S(r,f) + S(r,g),$$
$$(n+m+1)(T(r,f)+T(r,g))$$
$$\le \left(\frac{18}{s} + 2m + 8\right)(T(r,f)+T(r,g)) + S(r,f) + S(r,g).$$

We obtain $(n-m-7)s \le 18$ which contradicts our hypothesis $(n-m-7)s \ge 19$. Hence F and G satisfy (ii) in Lemma 2.5, i.e.,

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)},$$

where $a(\ne 0)$, b are two constants.

Then by Lemmas 2.7 and 2.8, we get the conclusion of Theorem 1.1. □

Proof of Theorem 1.2. Using the same argument as in the proof of Theorem 1.1 and putting $N(r,f)=0$ and $N(r,g)=0$ and using Lemma 2.9, we easily obtain the conclusion of Theorem 1.2. □

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