

A PROPER TOTAL COLORING DISTINGUISHING ADJACENT VERTICES BY SUMS OF SOME PRODUCT GRAPHS

HANA CHOI, DONGSEOK KIM, SUNGJIN LEE, AND YEONHEE LEE

ABSTRACT. In this article, we consider a proper total coloring distinguishes adjacent vertices by sums, if every two adjacent vertices have different total sum of colors of the edges incident to the vertex and the color of the vertex. Pilsniak and Wozniak [15] first introduced this coloring and made a conjecture that the minimal number of colors need to have a proper total coloring distinguishes adjacent vertices by sums is less than or equal to the maximum degree plus 3. We study proper total colorings distinguishing adjacent vertices by sums of some graphs and their products. We prove that these graphs satisfy the conjecture.

1. Introduction

Let Γ be a finite simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. An n -coloring ϕ of Γ is a function from $V(\Gamma)$ to $\{1, 2, \dots, n\}$. A coloring ϕ is *proper* if $\phi(u) \neq \phi(v)$ for all edges $\{u, v\} \in E(\Gamma)$. The *chromatic number* $\chi(\Gamma)$ of a graph Γ is the smallest number of colors needed to color properly the vertices of Γ . Since the exploratory paper by Dirac [6], the chromatic number has been in the center of graph theory research. Its rich history can be found in several articles [9, 16].

This concept naturally expand to an *edge coloring* which is a function from $E(\Gamma)$ to $\{1, 2, \dots, n\}$. An edge coloring ϕ is *proper* if $\phi(e_1) \neq \phi(e_2)$ if e_1 and e_2 have a common vertex. The minimum required number of colors for a proper edge coloring of a given graph is called *the chromatic index* of the graph, denoted by $\chi'(\Gamma)$. A very famous theorem by Vizing [17] slanted that for simple graph, the chromatic index of the graph is either its maximum degree Δ or $\Delta + 1$. For some graphs, such as bipartite graphs and high-degree planar graphs, the chromatic index is always Δ .

A *total coloring* is a coloring on the vertices and edges of a graph Γ such that no adjacent vertices have the same color, no adjacent edges have the same

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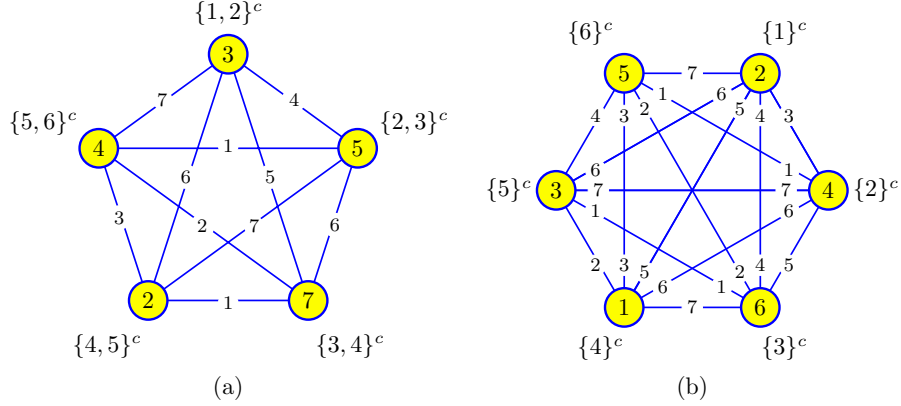


FIGURE 1. Proper total colorings distinguishing adjacent vertices by sums of the complete graph (a) K_5 and (b) K_6 .

color and no edge and its end-vertices are assigned the same color. The *total chromatic number* $\chi''(\Gamma)$ of a graph Γ is the least number of colors needed in a total coloring of Γ . Some properties of total chromatic number $\chi''(\Gamma)$ are as follows; $\chi''(\Gamma) \geq \Delta + 1$ and for upper bound for $\chi''(\Gamma)$, Molloy and Reed [14] first found that $\chi''(\Gamma) \leq \Delta + 10^{26}$ and $\chi''(\Gamma) \leq ch'(\Gamma) + 2$, where $ch'(\Gamma)$ is *the edge choosability* which is the least number k such that every instance of the list edge-coloring that has Γ as its underlying graph and that provides at least k allowed colors for each edge of Γ has a proper coloring. A long standing open problem about the total coloring is $\chi''(\Gamma) \leq \Delta + 2$ arose by Behzad and Vizing [11]. We refer to [11] for more problems on vertex, edge and total colorings.

For total coloring ϕ on a simple graph Γ , the *color set of a vertex* v is defined by $C(v) = \{\phi(u), \phi(\{u, v\}) \mid \{u, v\} \in E(\Gamma)\}$. Z. Zhang et al. [21] introduced a new concept that a total coloring of a graph Γ is an *adjacent vertex distinguishing total coloring (AVD total coloring)* if $C(u) \neq C(v)$ for all $\{u, v\} \in E(\Gamma)$. The *adjacent-vertex-distinguishing-total-chromatic number*, denoted by $\chi_{at}(\Gamma)$ of a graph Γ is the least number of colors needed in an AVD-total-coloring of Γ . There have been several articles studying AVD-total-coloring of graphs [3, 4, 8, 10, 13, 18, 19]

M. Piłśniak and M. Woźniak [15] first introduced that a proper total coloring of Γ is a *proper total colorings distinguishing adjacent vertices by sums* if for a vertex $v \in V(\Gamma)$, the total sum of colors of the edges incident to v and the color of v , denoted by $f(v)$, are distinct for adjacent vertices. An example of a proper total 7 coloring distinguishing adjacent vertices by sums of the complete graph K_6 is given Figure 1. The smallest number of color k such that Γ admits a proper total k colorings distinguishing adjacent vertices by sums is called *adjacent vertex distinguishing index by sum*, denoted by $tndi_{\Sigma}(\Gamma)$. They

also find that the adjacent vertex distinguishing index by sums of some special graphs including paths, cycles, stars, complete and complete bipartite graphs. They made a well-believed conjecture that:

Conjecture 1.1 ([15]). Let G be a graph with the maximum degree Δ . Then

$$tndi_{\Sigma}(G) \leq \Delta + 3.$$

Consequently, they showed that the above graphs hold Conjecture 1.1 and they also proved that the conjecture holds for regular bipartite graphs, cubic graphs, graphs with $\Delta \leq 3$. Li, Liu and Wang [12] proved that Conjecture 1.1 holds for K_4 -minor free graphs.

It is obvious that $\Delta + 1 \leq tndi_{\Sigma}(G)$. Thus if the conjecture is true, then we can divide graphs into three groups; $tndi_{\Sigma}(G) = \Delta + 1$, $\Delta + 2$ and $\Delta + 3$, we called them $tndi_{\Sigma}$ class I, II and III, respectively.

Once some types of graphs were studied, it is natural to consider their products for the next step. Chen, Zhang and Sun [5] found the adjacent vertex distinguishing total chromatic numbers of $P_m \times P_n$, $P_m \times C_n$ and $C_m \times C_n$.

The aim of the present article is two-fold. First we find that the wheel graphs are $tndi_{\Sigma}$ class I except W_3 which is K_4 . Second, we find that the product graphs $P_m \times P_n$, $C_m \times P_n$, $S_m \times P_n$, $W_m \times P_n$ and $K_m \times P_n$, $S_m \times C_n$ and $C_m \times C_n$ are $tndi_{\Sigma}$ class II except $P_3 \times P_3$, $S_m \times P_3$ and $W_m \times P_3$ which are $tndi_{\Sigma}$ class I. These results not only find the adjacent vertex distinguishing indices by sums but also strongly support that Conjecture 1.1 holds for these product graphs.

The outline of this paper is as follows. We first provide some preliminary definitions and results in Section 2. In Section 3, we find the adjacent vertex distinguishing indices by sums of the star graphs and wheel graphs. In Section 4, we investigate the adjacent vertex distinguishing indices by sum of the products of two graphs. At last, we make a conclusive remark in Section 5.

2. Preliminaries

For a graph Γ , the maximal and minimal degree of Γ are denoted by Δ and δ respectively. The following theorems will be used in proofs of theorems in Sections 3 and 4. For general terminology in graph theory, we refer the reader to [1].

Theorem 2.1 ([15, Observation 3]). *If a graph Γ contains two adjacent vertices x, y such that $\deg(x) = \deg(y) = \Delta$, then $tndi_{\Sigma}(\Gamma) \geq \Delta + 2$.*

Theorem 2.2 ([15, Propositions 4, 5, 6, 7, 8, 9 & 10]).

- (1) *Let P_n be a path of the size $n \geq 2$. Then*

$$tndi_{\Sigma}(P_n) = \begin{cases} \Delta + 1 & \text{if } n = 3, \\ \Delta + 2 & \text{if } n = 2 \text{ or } n \geq 4. \end{cases}$$

(2) Let C_n be a cycle of the size $n \geq 3$. Then

$$tndi_{\Sigma}(C_n) = \begin{cases} \Delta + 3 & \text{if } n = 3, \\ \Delta + 2 & \text{if } n \geq 4. \end{cases}$$

(3) Let S_n be a star graph of size $n \geq 2$. Then

$$tndi_{\Sigma}(S_n) = \Delta + 1.$$

(4) Let K_n be a complete graph of the size $n \geq 2$. Then

$$tndi_{\Sigma}(K_n) = \begin{cases} \Delta + 2 & \text{if } n \text{ is even,} \\ \Delta + 3 & \text{if } n \text{ is odd.} \end{cases}$$

(5) Let $K_{p,q}$ be a complete bipartite graph. Then

$$tndi_{\Sigma}(K_{m,n}) = \begin{cases} \Delta + 1 & \text{if } n \neq m, \\ \Delta + 2 & \text{if } n = m. \end{cases}$$

(6) Let Γ be a regular bipartite graph. Then $tndi_{\Sigma}(\Gamma) = \Delta + 2$.

(7) Let T be a tree of order n . Then $\Delta + 1 \leq tndi_{\Sigma}(T) \leq \Delta + 2$. Furthermore, if there exist two adjacent vertices x, y such that $\deg(x) = \deg(y) = \Delta$, then $tndi_{\Sigma}(T) = \Delta + 2$, and otherwise, we have $tndi_{\Sigma}(T) = \Delta + 1$.

Throughout the article, we will often use $\equiv \pmod{n}$. Since this modulo will be used for colorings, we use the standard complete residue system which is $\{1, 2, \dots, n-1, n\}$ for modulo n , unless stated differently.

3. Some other special graphs

In this section, we will discuss the adjacent vertex distinguishing indices by sums of the star graph and the wheel graphs. Although the adjacent vertex distinguishing indices by sums of the star graphs are already known in Theorem 2.2(3), we will find two proper total colorings distinguishing adjacent vertices by sums for these two graphs because they will be used in Section 4.

Theorem 3.1. *Let S_n be a star graph of size $n \geq 2$. Then*

$$tndi_{\Sigma}(S_n) = \Delta + 1.$$

Proof. A proper total coloring c distinguishing adjacent vertices by sums provided in Theorem 2.2 is as follows; the edge of S_n is colored by $1, 2, \dots, \Delta$, the vertex in the middle is colored by $\Delta + 1$, the vertex incident to the edge colored by 1 is colored by 2 and all remaining vertices are colored by 1.

For the second proper total coloring c' distinguishing adjacent vertices by sums of S_n , we consider $c' \equiv c + 1 \pmod{\Delta + 1}$. One can easily check this is also a proper total coloring distinguishing adjacent vertices by sums of S_n . \square

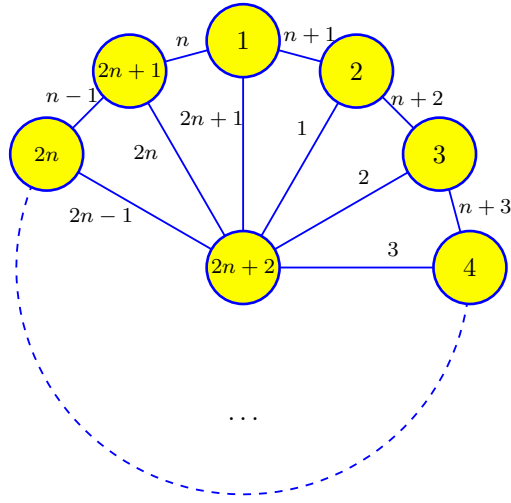


FIGURE 2. A proper total coloring distinguishing adjacent vertices by sums of the wheel graph W_{2n+1} , $n \geq 2$.

The wheel graphs are highly well structured, for example, they are planar and any maximal planar graph, other than K_4 , contains a subgraph isomorphic to either W_5 or W_6 . For $n \geq 3$, the wheel graph W_n contains $n^2 - 3n + 3$ cycles and at least one of them is a Hamiltonian cycle. Their (vertex) chromatic numbers are already known as

$$\chi(W_m) = \begin{cases} 3 & \text{if } m \text{ is odd,} \\ 4 & \text{if } m \text{ is even.} \end{cases}$$

The following theorem finds the proper total coloring distinguishing adjacent vertices by sums of the wheel graph.

Theorem 3.2. *Let W_m be a wheel graph of size $m \geq 3$. Then*

$$tndi_{\Sigma}(W_m) = \begin{cases} \Delta + 1 & \text{for } m \geq 4, \\ \Delta + 2 & \text{for } m = 3. \end{cases}$$

Proof. The smallest m for which the wheel graph makes a natural geometric shape is 3. But, because $W_3 \equiv K_4$ and $tndi_{\Sigma}(K_4)$ was given in Theorem 2.2(4). Now we assume $m \geq 4$. We prove the theorem dividing cases depending on the parity of m .

First we assume m is odd, so write $m = 2n + 1$ for some $n \geq 2$. We color the center vertex of the wheel graph by color $2n + 2$ and the adjacent edges by $1, 2, \dots, 2n + 1 (= \Delta)$ clockwise. Next we use color 2 for a vertex connected to the center vertex by the edge colored by 1, and for all remaining vertices, we similarly color by $3, \dots, 2n + 1, 1$ clockwise. Last we will color the edges in

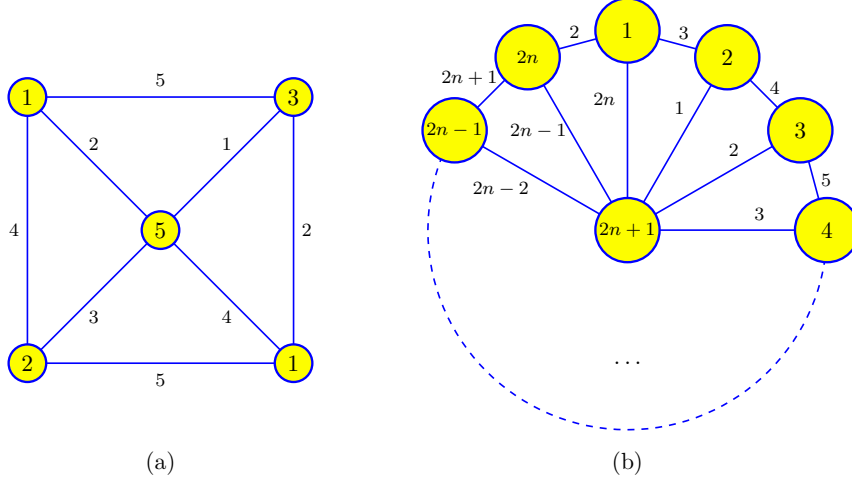


FIGURE 3. A proper total coloring distinguishing adjacent vertices by sums of the wheel graph (a) W_4 and (b) W_{2n} , $n \geq 3$.

the circumference of the wheel graph. The edge between vertices colored by 1 and 2 will be colored by $n+1$, and we increase the color by 1 up to $\text{mod}(2n+1)$ clockwise. Therefore, the wheel graph of an odd size has $\text{tndi}_\Sigma(W_m) \leq m+1$ as illustrated in Figure 2. By combining the fact $m+1 = \Delta+1 \leq \text{tndi}_\Sigma(W_m)$, we find $\text{tndi}_\Sigma(W_m) = m+1$.

Next, we assume m is even. One may find $\text{tndi}_\Sigma(W_4) = 5$ as shown in Figure 3(a). Now we assume that $m = 2n$ is bigger than or equal to 6. Similar to the $m = 2n+1$ we color the vertices and edges as illustrated in Figure 3(b). By the same reason, we find $\text{tndi}_\Sigma(W_n) = \Delta+1$. It completes the proof of the theorem.

Let c be the coloring of W_m discussed above. For the second proper total coloring c' distinguishing adjacent vertices by sums of W_m , we consider $c' \equiv c+1$ modulo $\Delta+1$ ($\Delta+2$, resp) for $m \geq 4$ ($m = 3$, respectively). \square

Although we are not discussing the hypercubes, one may find it is regular bipartite. So by using Theorem 2.2(6), it is tndi_Σ class II. The ladder graph can be considered as a product of two graphs and will be handled in the following section.

4. Products of two graphs

Throughout this section, the product means the graph product unless stated differently. A formal definition of the product is as follows. Let $G_1 = (V(G_1), E(G_1))$, $G_2 = (V(G_2), E(G_2))$ be two graphs. The *product of G_1 and G_2* which

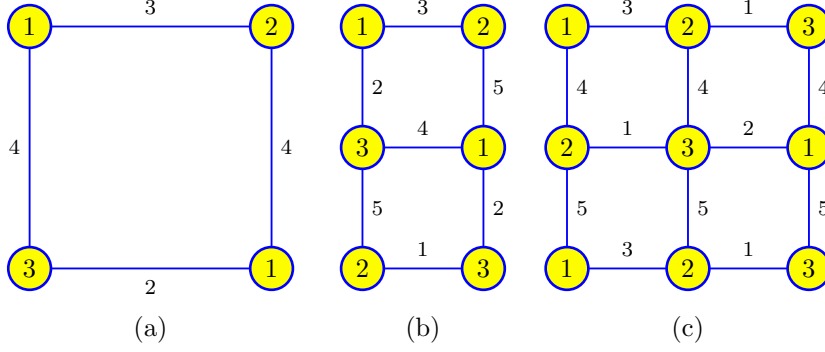


FIGURE 4. A proper total coloring distinguishing adjacent vertices by sums of the product graph (a) $P_2 \times P_2$, (b) $P_2 \times P_3$ and (c) $P_3 \times P_3$.

is denoted by $G_1 \times G_2$ is consists of the vertex set

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

and the edge set

$$E(G_1 \times G_2) = \{((u_1, u_2), (v_1, v_2)) \mid [u_1 = v_1 \text{ and } (u_2, v_2) \in E(G_2)] \text{ or } [(u_1, v_1) \in E(G_1) \text{ and } u_2 = v_2]\}.$$

Throughout the section, we use the following notations; let $V(G_2) = \{v_1, v_2, \dots, v_l\}$. $G_1 \times \{v_i\}$ is the i -th copy of G_1 denoted by $(G_1 \times G_2)_i$, and the set of the edges between $(G_1 \times G_2)_i$ and $(G_1 \times G_2)_{i+1}$ is denoted by $h((G_1 \times G_2)_i)$ where $i = 1, 2, \dots, l$.

Theorem 4.1. *Let P_m, P_n be a path of order $m, n \geq 2$. Then*

$$tndi_{\Sigma}(P_m \times P_n) = \begin{cases} \Delta + 1 & \text{for } m = n = 3, \\ \Delta + 2 & \text{otherwise.} \end{cases}$$

Proof. First, if $m = n = 2$ or 3 , then maximum degree Δ is 2 or 4 , respectively. Proper total colorings distinguishing adjacent vertices by sums of $P_2 \times P_2$ and $P_3 \times P_3$ are depicted in Figure 4. For the case $P_2 \times P_3$, then maximum degree Δ is 3 and two adjacent vertices have the maximum degree 3 , by Theorem 2.1 and the coloring in Figure 4(b), we find that $tndi_{\Sigma}(P_2 \times P_3) = 5$. Let us remark that $P_3 \times P_3$ is the unique case that there is only one vertex of the maximum degree and $tndi_{\Sigma}(P_3 \times P_3) = 5 = \Delta + 1$ which is the only exception that $P_m \times P_n$ is not $tndi_{\Sigma}$ class II.

Second, if just one of m, n is 2 , then without loss of generality, we may assume $n = 2$ and $m \geq 3$. The maximum degree Δ of $P_m \times P_2$ is 3 . Color the vertices of $(P_m \times P_2)_1$ alternatively with colors $1, 3, 1, 3, \dots$, and the edges

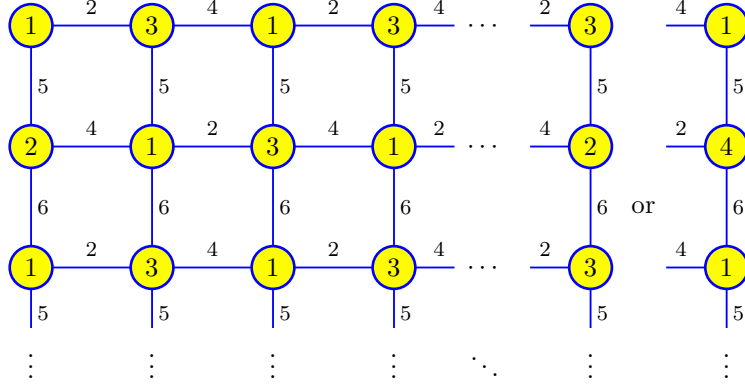


FIGURE 5. A proper total coloring distinguishing adjacent vertices by sums of the product graph $P_m \times P_n$ where $m \geq 3$.

with $2, 4, 2, 4, \dots$. For $(P_m \times P_2)_2$, we first color the edges with $4, 2, 4, 2, \dots$, and the vertices $2, 1, 3, 1, \dots, 2(4)$ if the last color of edges was 4 (2 , *reps.*), as two different rightmost figures as depicted in Figure 5. Color the edges in $h((P_m \times P_2)_1)$ by 5 then this is a proper total 5 coloring distinguishing adjacent vertices by sums. We get $tndi_{\Sigma}(P_m \times P_2) \leq 5$ but Theorem 2.1 implies that $5 = \Delta + 2 \leq tndi_{\Sigma}(P_m \times P_2)$. Thus, $tndi_{\Sigma}(P_m \times P_2) = \Delta + 2$. This coloring will be used for the remaining products $P_m \times P_n$, let us denote it by ϕ .

Last, if $m, n \geq 3$, then the maximum degree Δ of $P_m \times P_n$ is 4 . If there exists only one vertex of the maximum degree, then it must be $P_3 \times P_3$ which was handled previously. Now, we assume there exist two adjacent vertices of the maximum degree. The method we are going to use is solely depends on the parity of i . We define a coloring ψ on $P_m \times P_n$ by

$$\psi((P_m \times P_n)_i) = \begin{cases} \phi((P_m \times P_2)_1) & \text{if } i \text{ is odd,} \\ \phi((P_m \times P_2)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((P_m \times P_n)_i)) = \begin{cases} 5 & \text{if } i \text{ is odd,} \\ 6 & \text{if } i \text{ is even,} \end{cases}$$

as illustrated in Figure 5. One can easily check that ψ is a proper total 6 coloring distinguishing adjacent vertices by sums of the graph $P_m \times P_n$, which means $tndi_{\Sigma}(P_m \times P_n) \leq \Delta + 2$. By Theorem 2.1, the opposite inequality $\Delta + 2 \leq tndi_{\Sigma}(P_m \times P_n)$ holds. Therefore, it completes the proof of theorem. \square

Theorem 4.2. *Let C_m be a cycle of order $m \geq 3$ and P_n be a path of order $n \geq 2$. Then $tndi_{\Sigma}(C_m \times P_n) = \Delta + 2$.*

Proof. These $C_m \times P_n$ graphs always have two adjacent vertices x, y such that $\deg(x) = \deg(y) = \Delta$. By Theorem 2.1, we obtain the inequality $tndi_{\Sigma}(C_m \times P_n) \geq \Delta + 2$.

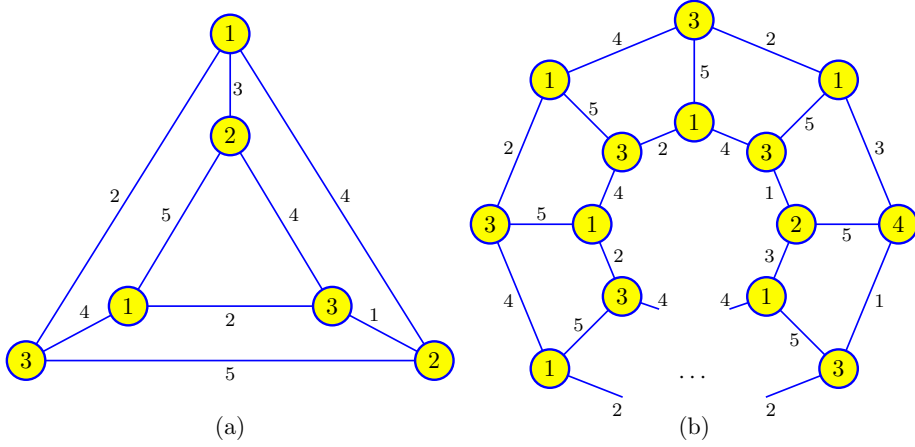


FIGURE 6. A proper total coloring distinguishing adjacent vertices by sums of the product graph (a) $C_3 \times P_2$ and (b) $C_{2k+1} \times P_2$, $k \geq 2$.

First look at the case $n = 2$. Since $\Delta = 3$, to show $C_m \times P_2$ is $tndi_\Sigma$ class II, we need to find a proper total 5 coloring distinguishing adjacent vertices by sums. If m is even, then one can see that $C_m \times P_2$ is regular bipartite. By Theorem 2.2(6), it is $tndi_\Sigma$ class II. To handle the remaining general cases, let us find a coloring of $C_m \times P_2$ as follows.

If m is odd, let us denote it by $m = 2k + 1$. For $m = 3$, we color it as depicted in Figure 6(a). If $2k + 1 \geq 5$, we color the vertices of $(C_{2k+1} \times P_2)_1$ with colors 4, 1, 3, 1, 3, \dots , 1, 3 and the edges with 1, 3, 2, 4, 2, 4, \dots , 4, 2. Similarly, color the consecutive vertices of $(C_{2k+1} \times P_2)_2$ with 2, 3, 1, 3, 1, \dots , 3, 1 and consecutive edges with 3, 1, 4, 2, 4, 2, \dots , 4. Color the reminding edges with the color 5 as illustrated in Figure 6(b).

If $m = 2k \geq 4$, we color the vertices of $(C_{2k} \times P_2)_1$ with colors 1, 3, 1, 3, \dots , 1, 3 and the edges with 2, 4, 2, 4, \dots , 2, 4. Similarly, color the consecutive vertices of $(C_{2k} \times P_2)_2$ with 3, 1, 3, 1, \dots , 3, 1 and consecutive edges with 4, 2, 4, 2, \dots , 4, 2. Color the reminder edges with the color 5 then this is a proper total 5 coloring distinguishing adjacent vertices by sums of $C_{2k} \times P_2$.

Now we are going to deal with the case $n \geq 3$. Since the maximum degree Δ of $C_m \times P_n$ is 4, to show $C_m \times P_n$ is $tndi_\Sigma$ class II, we need to find a proper total 6 coloring distinguishing adjacent vertices by sums. The above coloring for $C_m \times P_2$ is denoted by ϕ .

If $m = 3$, then color the consecutive vertices of $(C_3 \times P_n)_1$ with 1, 3, 2 and corresponding consecutive edges with 2, 4, 5. Similarly, we color the consecutive vertices of $(C_3 \times P_n)_2$ with 3, 2, 1 and corresponding consecutive edges with 5, 4, 2 as we have used for $C_3 \times P_2$ as illustrated in Figure 6(a). We expand

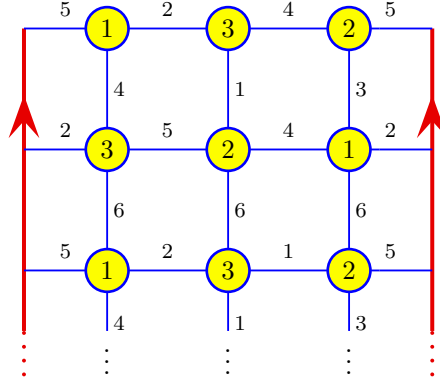


FIGURE 7. A proper total coloring distinguishing adjacent vertices by sums of the product graph $C_3 \times P_n$.

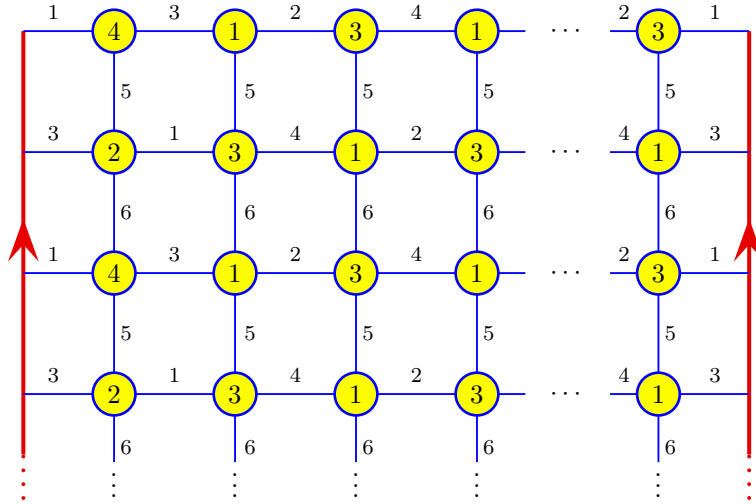


FIGURE 8. A proper total coloring distinguishing adjacent vertices by sums of the product graph $C_m \times P_n$ where m is bigger than 3 and odd.

these colorings on $(C_3 \times P_n)_1$ and $(C_3 \times P_n)_2$ to a coloring ψ of $(C_3 \times P_n)$ by

$$\psi((C_3 \times P_n)_i) = \begin{cases} \phi((C_3 \times P_2)_1) & \text{if } i \text{ is odd,} \\ \phi((C_3 \times P_2)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((C_3 \times P_n)_i)) = \begin{cases} 4, 1, 3 & \text{if } i \text{ is odd,} \\ 6, 6, 6 & \text{if } i \text{ is even,} \end{cases}$$

as illustrated in Figure 7.

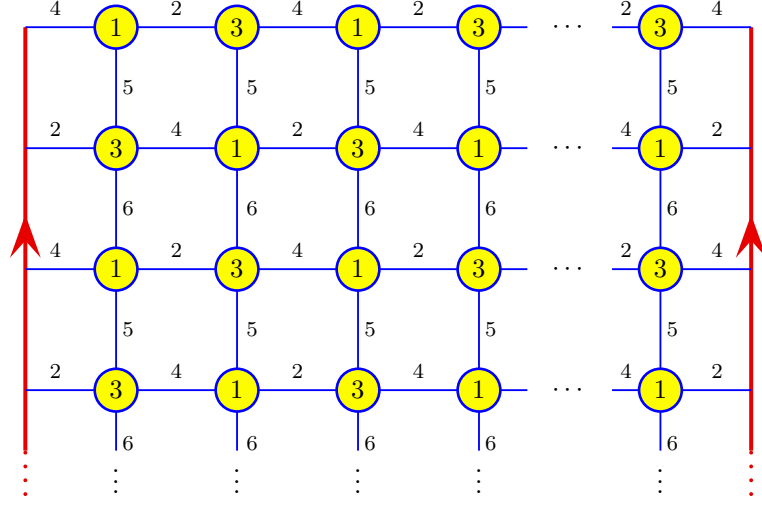


FIGURE 9. A proper total coloring distinguishing adjacent vertices by sums of the product graph $C_m \times P_n$ where m is bigger than 3 and even.

If $m = 2k + 1 \geq 5$, we define a coloring ψ by

$$\psi((C_{2k+1} \times P_n)_i) = \begin{cases} \phi((C_{2k+1} \times P_2)_1) & \text{if } i \text{ is odd,} \\ \phi((C_{2k+1} \times P_2)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((C_{2k+1} \times P_n)_i)) = \begin{cases} 5 & \text{if } i \text{ is odd,} \\ 6 & \text{if } i \text{ is even,} \end{cases}$$

as depicted in Figure 8.

If $m = 2k \geq 4$, we define a coloring ψ by

$$\psi((C_{2k} \times P_n)_i) = \begin{cases} \phi((C_{2k} \times P_2)_1) & \text{if } i \text{ is odd,} \\ \phi((C_{2k} \times P_2)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((C_{2k} \times P_n)_i)) = \begin{cases} 5 & \text{if } i \text{ is odd,} \\ 6 & \text{if } i \text{ is even,} \end{cases}$$

as illustrated in Figure 9.

It is not difficult check that these three colorings ψ 's are proper total 6 colorings distinguishing adjacent vertices by sums. Therefore, it completes the proof of theorem. \square

Theorem 4.3. Let S_m be the star graph of order $m \geq 2$ and P_n be the path of order $n \geq 2$. Then

$$tndi_{\Sigma}(S_m \times P_n) = \begin{cases} \Delta + 1 & \text{if } n = 3, \\ \Delta + 2 & \text{if } n = 2 \text{ or } n \geq 4. \end{cases}$$

Proof. First let us deal with the case $n = 2$. Then it has two adjacent vertices of the maximum degree $\Delta = m + 1$. To prove $S_m \times P_n$ is $tndi_{\Sigma}$ class II, we need to find a proper total $m + 3 (= \Delta + 2)$ coloring distinguishing adjacent vertices by sums on $S_m \times P_2$ by Theorem 2.1.

On the other hand, there exists a proper total $m + 1$ coloring distinguishing adjacent vertices by sums for S_m by Theorem 2.2(3) denoted by c . Define a new coloring c' by $c' = c + 1$. Color $(S_m \times P_2)_1$ by c and $(S_m \times P_2)_2$ by c' and $h((S_m \times P_2)_1) = m + 3$. Then one can easily check that this is a proper total $(m + 3)$ coloring distinguishing adjacent vertices by sums of $S_m \times P_2$. This coloring for $S_m \times P_2$ is denoted by ϕ .

Second, we assume $n \geq 4$, then $\Delta + 2 = m + 4$ and $S_m \times P_n$ has two adjacent vertices of the maximum degree. By Theorem 2.1, we need to find a proper total $(m + 4)$ coloring distinguishing adjacent vertices by sums. We define a coloring ψ on $(S_m \times P_n)$ as follows,

$$\psi((S_m \times P_n)_i) = \begin{cases} \phi((S_m \times P_2)_1) & \text{if } i \text{ is odd,} \\ \phi((S_m \times P_2)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((S_m \times P_n)_i)) = \begin{cases} m + 3 & \text{if } i \text{ is odd,} \\ m + 4 & \text{if } i \text{ is even.} \end{cases}$$

One may easily check that ψ is a proper total $m + 4$ coloring distinguishing adjacent vertices by sums. Therefore, $(S_m \times P_n)$ is $tndi_{\Sigma}$ class II.

We are discussing the last case $n = 3$. For this case, we get just one vertex of the maximum degree. As we have seen for $P_3 \times P_3$ in Figure 4(b), $S_2 \times P_3 = P_3 \times P_3$ is $tndi_{\Sigma}$ class I. For $m \geq 3$, we can show that $S_m \times P_3$ is $tndi_{\Sigma}$ class I as follows. Let c be a proper total $m + 1$ coloring distinguishing adjacent vertices by sums of S_m , and $c' \equiv c + 1 \pmod{m + 1}$ be the second coloring provided in Theorem 3.1. We define a coloring ψ on $S_m \times P_3$ as follows,

$$\psi((S_m \times P_3)_i) = \begin{cases} c'((S_m \times P_2)_1) & \text{if } i = 1, 3, \\ c((S_m \times P_2)_2) & \text{if } i = 2, \end{cases}$$

$$\psi(h((S_m \times P_3)_i)) = \begin{cases} m + 2 & \text{if } i = 1, \\ m + 3 & \text{if } i = 2. \end{cases}$$

Then, one can check that ψ is a proper total $m + 3$ coloring distinguishing adjacent vertices by sums for $S_m \times P_3$. It completes the proof of theorem. \square

Theorem 4.4. *Let K_m be a complete graph of order $m \geq 2$ and P_n be a path of order $n \geq 2$. Then $tndi_{\Sigma}(K_m \times P_n) = \Delta + 2$.*

Proof. Since $K_m \times P_n$ has two adjacent vertices of the maximum degree, by Theorem 2.1, to prove $K_m \times P_n$ is $tndi_{\Sigma}$ class II, we need find a proper total $\Delta + 2$ coloring distinguishing adjacent vertices by sums of $K_m \times P_n$.

First, let us assume that $n = 2$ and m is odd. Then there exists a proper total coloring c_0 distinguishing adjacent vertices by sums of K_m by Theorem 2.2(4).

This coloring c_0 originally found in [15] has a very nice formula that for each vertex $v_i \in K_m$,

$$f(v_i) = \frac{m^2 + 5m + 6}{2} - (2i + 1)$$

by the fact that $C(v_i) = \{1, 2, \dots, m+2\} \setminus \{i, i+1\}$ as illustrated in Figure 1(a) where the vertex v_i is colored $2i+1 \pmod{7}$. We define a coloring ϕ on $K_m \times P_2$ by coloring the $(K_m \times P_2)_2$ with the coloring c_0 , coloring the $(K_m \times P_2)_1$ with $c'_0 \equiv c_0 + 1 \pmod{m+2}$ and $\phi(\{(v_i, w_1), (v_i, w_2)\}) = g(i) + 1$ for $i = 1, 2, \dots, m$.

If $n = 2$ and m is even, the coloring found in [15] has a similar property that

$$f(v_i) = \frac{m^2 + 3m + 2}{2} - i \pmod{m+1}$$

by the fact that $C(v_i) = \{1, 2, \dots, m+1\} \setminus \{i\}$ as illustrated in Figure 1(b) where the vertex v_i is colored $2i \pmod{7}$. We define a coloring ϕ on $K_m \times P_2$ by coloring $(K_m \times P_2)_2$ with the coloring c_0 , and by coloring $(K_m \times P_2)_1$ with $c'_0 \equiv c_0 + 1 \pmod{m+1}$ and $\phi(\{(v_i, w_1), (v_i, w_2)\}) = m+2$ for $i = 1, 2, \dots, m$. Then one may easily check that this is a proper total coloring distinguishing adjacent vertices by sums. The above coloring for $K_m \times P_2$ is denoted by ϕ .

Let $m \geq 3$, then $\Delta + 2 = m + 3$. If m is odd, we define a coloring ψ on $K_m \times P_n$ by

$$\psi((K_m \times P_n)_i) = \begin{cases} \phi((K_m \times P_2)_1) & \text{if } i \text{ is odd,} \\ \phi((K_m \times P_2)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((K_m \times P_n)_i)) = \begin{cases} m+3 & \text{if } i \text{ is odd,} \\ c_0(h(K_m \times P_n)_i) & \text{if } i \text{ is even.} \end{cases}$$

Clearly, ψ is a proper total $m+3$ coloring distinguishing adjacent vertices by sums of $K_m \times P_n$.

If m is even, we define a coloring ψ on $K_m \times P_n$ by

$$\psi((K_m \times P_n)_i) = \begin{cases} \phi((K_m \times P_2)_1) & \text{if } i \text{ is odd,} \\ \phi((K_m \times P_2)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((K_m \times P_n)_i)) = \begin{cases} m+2 & \text{if } i \text{ is odd,} \\ m+3 & \text{if } i \text{ is even.} \end{cases}$$

It is easy to see that ψ is a proper total $m+3$ coloring distinguishing adjacent vertices by sums of $K_m \times P_n$. \square

Theorem 4.5. *Let W_m be a wheel graph of order $m \geq 3$ and P_n be a path of order $n \geq 2$. Then*

$$tndi_{\Sigma}(W_m \times P_n) = \begin{cases} \Delta + 1 & \text{if } n = 3 \text{ and } m \geq 4, \\ \Delta + 2 & \text{otherwise.} \end{cases}$$

Proof. Since $W_3 = K_4$, $W_3 \times P_n$ is $tndi_\Sigma$ class II as proved in Theorem 4.4. If $n = 2$ and $m \geq 4$, then the maximum degree is $m + 1$. There exists a proper total $m + 1$ coloring c distinguishing adjacent vertices by sums of W_m by Theorem 3.2. Let $c' = c + 1$. Let us define a coloring ϕ of $W_m \times P_2$ by coloring $(W_m \times P_2)_1$ with the coloring c , $(W_m \times P_2)_2$ with the coloring c' and $\phi(h(W_m \times P_2)_1) = m + 3$. Then $tndi_\Sigma(W_m \times P_2) = m + 3$.

Next if $n \geq 4$ and $m \geq 4$, then maximum degree is $m + 2$ and there exist two adjacent vertices of the maximum degree. We define a coloring ψ on $W_m \times P_n$ as follows

$$\psi((W_m \times P_n)_i) = \begin{cases} \phi((W_m \times P_2)_1) & \text{if } i \text{ is odd,} \\ \phi((W_m \times P_2)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((W_m \times P_n)_i)) = \begin{cases} m + 3 & \text{if } i \text{ is odd,} \\ m + 4 & \text{if } i \text{ is even.} \end{cases}$$

One may easily check that ψ is a proper total coloring distinguishing adjacent vertices by sums. Therefore, by Theorem 2.1 we find that $W_m \times P_n$ is $tndi_\Sigma$ class II.

The remaining case is $n = 3$ and $m \geq 4$. By the similar proof in Theorem 4.3 and the second coloring provided in Theorem 3.2, one can prove that $W_m \times P_3$ is $tndi_\Sigma$ class I. \square

Theorem 4.6. *Let W_m be a wheel of order $m \geq 3$ and C_n be a cycle of order $n \geq 3$. Then $tndi_\Sigma(W_m \times C_n) = \Delta + 2$.*

Proof. If n is even, by Theorem 4.5 we know that $tndi_\Sigma(W_m \times P_n) = \Delta + 2$. So we have $tndi_\Sigma(W_m \times C_n) = \Delta + 2$ by using the same coloring ψ and the edges between $(W_m \times C_n)_n$ and $(W_m \times C_n)_1$ are colored by $m + 4$.

Next, let $n = 2k + 1$ be odd for some $k \geq 1$. There exists a proper total coloring distinguishing adjacent vertices by sums by Theorem 4.5. For $n = 1, 2, \dots, 2k$, we use the same coloring ψ . Suppose the coloring of $(W_m \times P_n)_1$ is c and $c' = c + 2$. Color $(W_m \times C_n)_{2k+1}$ by the coloring c' . We define a coloring ϕ on $W_m \times C_n$ as follows

$$\phi((W_m \times P_n)_i) = \begin{cases} \psi(W_m \times C_n) & \text{if } n = 1, 2, \dots, 2k, \\ c'((W_m \times C_n)_{2k+1}) & \text{if } n = 2k + 1, \end{cases}$$

$$\phi(h((W_m \times C_n)_i)) = \begin{cases} \psi(h((W_m \times P_n)_i)) & \text{if } i = 1, 2, \dots, 2k - 1, \\ 1 & \text{if } i = 2k, \\ m + 4 & \text{if } i = 2k + 1. \end{cases}$$

As a result, the adjacent vertex distinguishing index by sum of $W_m \times C_n$ is $\Delta + 2$. \square

Corollary 4.7. *Let S_m be a star of order $m \geq 2$ and C_n be a cycle of order $n \geq 3$. Then $tndi_\Sigma(S_m \times C_n) = \Delta + 2$.*

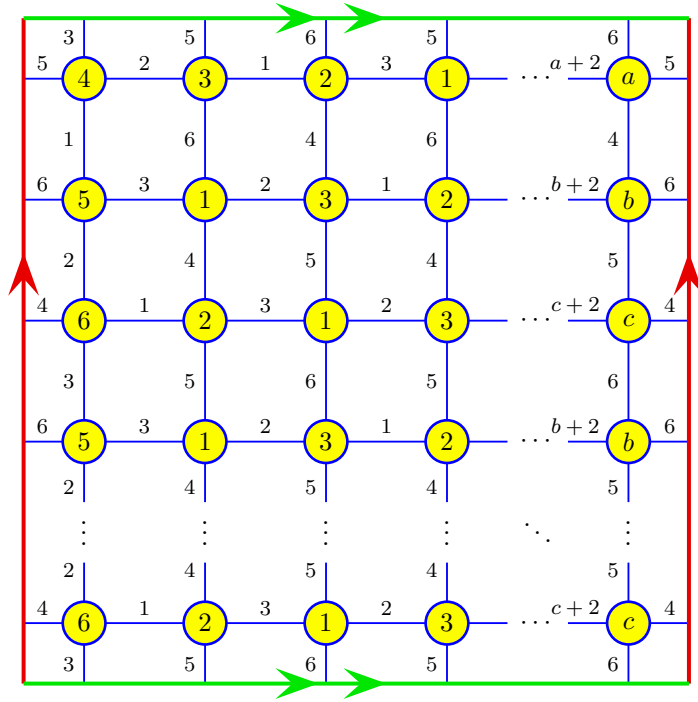


FIGURE 10. A proper total coloring distinguishing adjacent vertices by sums of the product graph $C_n \times C_m$ where m, n are odd.

Proof. $S_m \times C_n$ has two adjacent vertices of the maximum degree $\Delta = m + 2$ and the coloring ϕ used in Theorem 4.6 is a proper total $m + 4$ coloring distinguishing adjacent vertices by sums of $S_m \times C_n$. By Theorem 2.1, we find $tndi_{\Sigma}(S_m \times C_n) = \Delta + 2$. \square

Theorem 4.8. *Let C_n, C_m be a cycle of order $n, m \geq 3$. Then $tndi_{\Sigma}(C_n \times C_m) = \Delta + 2$.*

Proof. We proceed the proof depend on the parity of n, m . First suppose n, m are odd. To define a coloring ψ of $C_n \times C_m$, we make a coloring ϕ on the first three copies by coloring the vertices of $(C_n \times C_m)_1$ with colors 4, 3, 2, 1, 3, 2, 1, ..., and the edges of $(C_n \times C_m)_1$ with 5, 2, 1, 3, 2, 1, 3, 2, 1, 3, Color the vertices of $(C_n \times C_m)_2$ with colors 5, 1, 3, 2, 1, 3, 2, 1, 3, 2, ..., and the edges with 6, 3, 2, 1, 3, 2, 1, Color the vertices of $(C_n \times C_m)_3$ with colors 6, 2, 1, 3, 2, 1, 3, 2, 1, 3, ..., and the edges with 4, 1, 3, 2, 1, 3, 2, 1, 3, 2, Color the edges of $h((C_n \times C_m)_0)$ with colors 3, 5, 6, 5, 6, ..., 5, 6, and the edges of $h((C_n \times C_m)_1)$

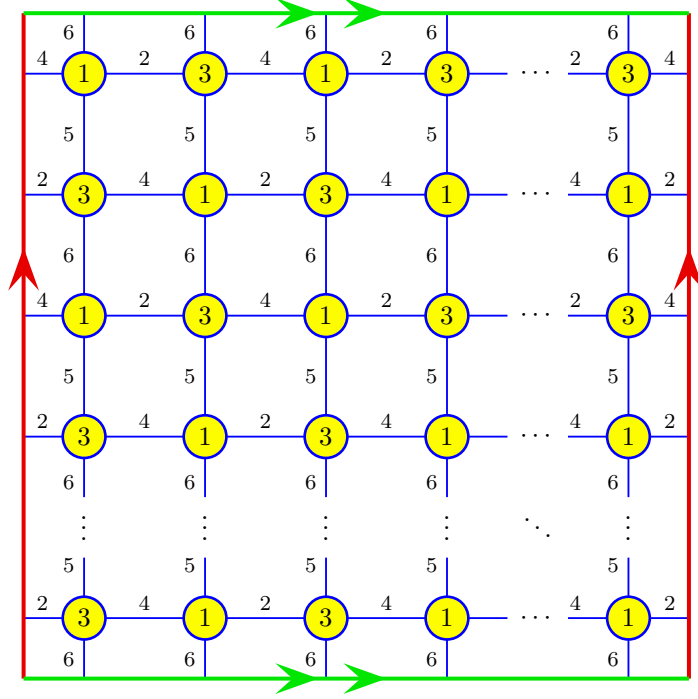


FIGURE 11. A proper total coloring distinguishing adjacent vertices by sums of the product graph $C_n \times C_m$ where m, n are even.

with colors $1, 6, 4, 6, 4, 6, 4, \dots, 6, 4$, and the edges of $h((C_n \times C_m)_2)$ with colors $2, 4, 5, 4, 5, \dots, 4, 5$. Then we extend these coloring ϕ to ψ of $C_n \times C_m$ by

$$\psi((C_n \times C_m)_i) = \begin{cases} \phi((C_n \times C_m)_1) & \text{if } i = 1, \\ \phi((C_n \times C_m)_2) & \text{if } i \text{ is even,} \\ \phi((C_n \times C_m)_3) & \text{if } i \text{ is odd and bigger than 3,} \end{cases}$$

$$\psi(h((C_n \times C_m)_i)) = \begin{cases} \phi(h((C_n \times C_m)_i)) & \text{if } i = 0 \text{ and } i = 1, \\ \phi(h((C_n \times C_m)_2)) & \text{if } i \text{ is even and nonzero,} \\ \phi(h((C_n \times C_m)_0)) & \text{if } i \text{ is odd and } \geq 3, \end{cases}$$

as illustrated in Figure 10 where $a \equiv 2 - n$, $b \equiv 3 - n$ and $c \equiv 4 - n \pmod{3}$ are taken in the standard complete residue system $\{1, 2, 3\}$.

For the rest of cases, we first make a coloring ϕ on some copies, then expand it to the coloring ψ of $C_n \times C_m$ as we did for the case n, m are odd.

Let n, m be even. We color the vertices of $(C_n \times C_m)_1$ with colors $1, 3, 1, 3, \dots, 1, 3$, and the edges with $4, 2, 4, 2, \dots, 4, 2$. Color the vertices of $(C_n \times C_m)_2$

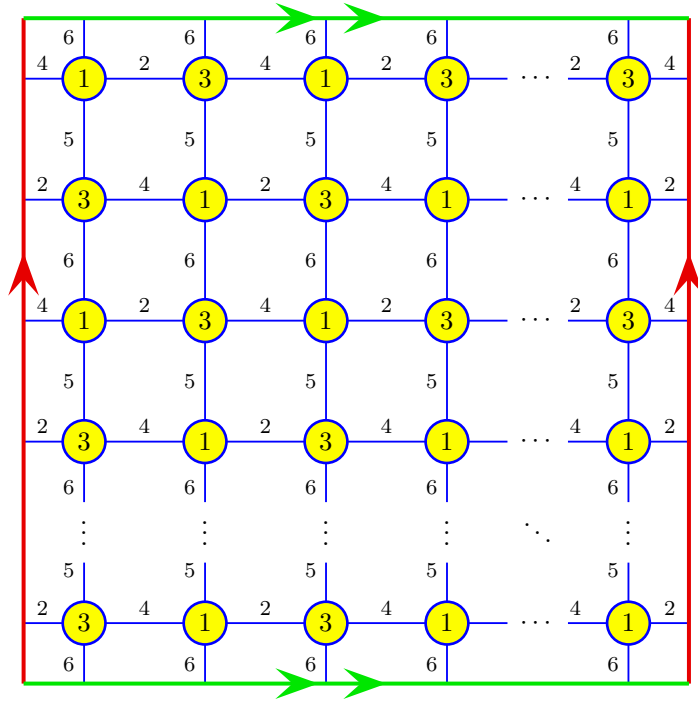


FIGURE 12. A proper total coloring distinguishing adjacent vertices by sums of the product graph $C_n \times C_m$ where one of n, m is even.

with colors $3, 1, 3, 1, \dots, 3, 1$, and the edges with $2, 4, 2, 4, \dots, 2, 4$.

$$\psi((C_n \times C_m)_i) = \begin{cases} \phi((C_n \times C_m)_1) & \text{if } i \text{ is odd,} \\ \phi((C_n \times C_m)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((C_n \times C_m)_i)) = \begin{cases} 5 & \text{if } i \text{ is odd,} \\ 6 & \text{if } i \text{ is even,} \end{cases}$$

as illustrated in Figure 11.

If one of n, m is even and the other is odd. By exchanging n, m , we may assume that m is even. If $n = 3$, we use the coloring of $C_3 \times P_m$ as depicted in Figure 7 since m is even, we can naturally extend this coloring to a coloring of $C_3 \times C_m$ and it is a proper total coloring distinguishing adjacent vertices by sums. Now we assume $n \geq 5$. We color the vertices of $(C_n \times C_m)_1$ with colors $4, 1, 3, 1, 3, \dots, 1, 3$, and the edges with $1, 3, 2, 4, 2, 4, \dots, 4, 2$. Color the vertices of $(C_n \times C_m)_2$ with colors $2, 3, 1, 3, 1, \dots, 3, 1$, and the edges with

3, 1, 4, 2, 4, \dots , 2, 4.

$$\psi((C_n \times C_m)_i) = \begin{cases} \phi((C_n \times C_m)_1) & \text{if } i \text{ is odd,} \\ \phi((C_n \times C_m)_2) & \text{if } i \text{ is even,} \end{cases}$$

$$\psi(h((C_n \times C_m)_i)) = \begin{cases} 5 & \text{if } i \text{ is odd,} \\ 6 & \text{if } i \text{ is even,} \end{cases}$$

as illustrated in Figure 12. These graphs always have two adjacent vertices x, y such that $\deg(x) = \deg(y) = \Delta$. By Theorem 2.1, we get $\text{tndi}_\Sigma(C_n \times C_m) = \Delta + 2$. \square

5. Conclusion and discussion

As we have seen the results in Section 4, the most of cases, the product graphs are tndi_Σ Class II. Although we are only able to find the adjacent vertex distinguishing index of a few product graphs for which one of component of the product is a path P_n , we expect that the adjacent vertex distinguishing index of $\Gamma \times P_n$ can be found where Γ is a regular graph.

It has been more than three years since a proper total colorings distinguishing adjacent vertices by sums was first invented, however, it is still unknown that the adjacent vertex distinguishing index by sum and adjacent vertex distinguishing index are different or not. If these two are the same, then this will benefit the computer program to find adjacent vertex distinguishing index because comparing the total sum is immensely faster than comparing two sets for all adjacent vertices. However, we know the following inequality between these two indices as follows.

Since $f(v) = \sum_{\alpha \in C(v)} \alpha$ and the implication, if $f(u) \neq f(v)$, then $C(u) \neq C(v)$, one can easily see that if a graph Γ has a proper total k -colorings distinguishing adjacent vertices by sums, then Γ has an AVD total k -coloring. Therefore,

$$\chi_{at}(\Gamma) \leq \text{tndi}_\Sigma(\Gamma).$$

At last, we state two conjectures regarding tndi_Σ classes.

Conjecture 5.1. A graph Γ is tndi_Σ class III if and only if $\Gamma = K_{2n+1}$ for some $n \geq 1$.

It is fairly easy to see that if a graph Γ is regular with valency k and it is tndi_Σ class II, then at each vertex the set of colors of the vertex and its incident edges have a cardinality $k + 1$ out of $k + 2$ colors as demonstrated in the complete graph K_{2n} . Thus if we assign the missing color to the vertex, we get a proper $k + 2$ coloring. We expect the converse also can be proven.

Conjecture 5.2. A k regular graph Γ is tndi_Σ class II if and only if Γ has a proper vertex $k + 2$ coloring.

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DEPARTMENT OF MATHEMATICS
SUNGKYUNKWAN UNIVERSITY
SUWON 440-746, KOREA
E-mail address: chn6025@naver.com

DEPARTMENT OF MATHEMATICS
KYONGGI UNIVERSITY
SUWON 443-760, KOREA
E-mail address: `dongseok@kgu.ac.kr`

DEPARTMENT OF MATHEMATICS
YONSEI UNIVERSITY
SEOUL 120-749, KOREA
E-mail address: `SungjinLee@yonsei.ac.kr`

DEPARTMENT OF MATHEMATICS
KYONGGI UNIVERSITY
SUWON 443-760, KOREA
E-mail address: `dusgmlrndi@hanmail.net`