SINGULARITY ORDER OF
THE RIESZ-NÁGY-TAKÁCS FUNCTION

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ABSTRACT. We give the characterization of Hölder differentiability points and non-differentiability points of the Riesz-Nágy-Takács (RNT) singular function $\Psi_{a,p}$ satisfying $\Psi_{a,p}(a) = p$. It generalizes recent multifractal and metric number theoretical results associated with the RNT function. Besides, we classify the singular functions using the singularity order deduced from the Hölder derivative giving the information that a strictly increasing smooth function having a positive derivative Lebesgue almost everywhere has the singularity order 1 and the RNT function $\Psi_{a,p}$ has the singularity order $g(a,p) = \frac{a \log p + (1-a) \log(1-p)}{a \log a + (1-a) \log(1-a)} \geq 1$.

1. Introduction

Recently many ([8, 9, 10, 16]) studied the Cantor function, a singular function which is not strictly increasing and the Minkowski's Question Mark function which is a strictly increasing singular function. Also J. Paradís *et al.* ([17]) studied some conditions of the null and infinite derivatives of the RNT strictly increasing singular function using metric number theory.

Recently we ([4]) also studied multifractal characterization of the null and infinite derivative sets and the non-differentiability set of the RNT singular function, which is the typical singular function related to mutifractal theory. In this paper, we employ the Hölder derivative, which is a generalized form of the usual derivative, of the RNT function on the unit interval. This definition extends the concept of the singularity to the general singularity for a strictly increasing continuous function. For every point in the unit interval, we ([4]) can give some code or dyadic expansion using digit 0 and 1, generating the distribution set determined by the frequency of the zero in its expansion. The distribution sets in the unit interval are the local dimension sets by a self-similar

measure on the unit interval ([2]). We note that the Hausdorff and packing dimensions of the local dimension sets in the unit interval were obtained in [2] by the cylinder density theorem ([7, 13]) instead of the usual density theorem ([11]).

Using the information of the distribution sets and the local dimension sets, we give the multifractal characterization of the Hölder derivative sets of Hölder order $0 < q < \infty$ and the Hölder non-differentiability set of Hölder order $0 < q < \infty$ of the RNT singular function. As a result, the RNT singular function is Hölder differentiable only on a meager subset ([14]) of the unit interval. Further the Hausdorff dimension of the Hölder non-differentiability set of the RNT singular function is greater than 0 whereas its packing dimension is 1. Further, for some Hölder order, the packing dimension of the infinite Hölder derivative set of the RNT singular function is less than 1, giving full Lebesgue measure for the null Hölder derivative set, and *vice versa*. That is, the packing dimension of the null Hölder derivative set of the RNT singular function is less than 1, giving full Lebesgue measure for the infinite Hölder derivative set for some different Hölder order.

The RNT function $\Psi_{a,p}$ ([17]) is generated by two positive numbers $a, p \in (0, 1)$ respectively which give the slope equation

$$(\frac{p}{a^q})^r (\frac{1-p}{(1-a)^q})^{1-r} = 1$$

of r with respect to $0 < q < \infty$. Further a is the critical point of the solution $r = r(q)$ for the slope equation for the Hölder derivative sets of Hölder order q to have full Lebesgue measure, leading to the definition of the singularity order for an increasing continuous function.

We note that the multifractal characterization for the particular Hölder order, namely $q = 1$, is a generalization of recent results ([4, 17]) for the RNT singular function.

We define the singularity order for a strictly increasing function on the unit interval using the Hölder orders whose the null Hölder derivative set has full Lebesgue measure and the infinite Hölder derivative set has full Lebesgue measure. Finally we show that the RNT function $\Psi_{a,p}$ has the singularity order $g(a,p) = \frac{a \log p + (1-a) \log(1-p)}{a \log a + (1-a) \log(1-a)} \geq 1$.

2. Preliminaries

We ([2, 4]) recall the unit interval $(0, 1]$ having the generalized dyadic expansion with a base a where $0 < a < 1$. Let $\mathbb{N}$ be the set of the positive integers. We define a fundamental interval $I_{i_1 \cdots i_k} = f_{i_1} \circ \cdots \circ f_{i_k}(I)$ where $f_0(x) = ax$ and $f_1(x) = (1-a)x + a$ on $I = (0, 1]$, $i_j \in \{0, 1\}$ and $1 \leq j \leq k$. If $x \in (0, 1]$, then there is a unique code $\sigma \in \{0, 1\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1 i_2 \cdots i_k$ where $\sigma = i_1 i_2 \cdots i_k i_{k+1} \cdots$). We ([4]) called a code $\sigma \in \{0, 1\}^{\mathbb{N}}$ where $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ the generalized dyadic expansion with a base a of x and identify x with the code σ without confusion.

If $x \in (0,1]$ and $x \in I_\nu$ where $\nu \in \{0,1\}^k$, a cylinder $c_k(x)$ denotes the fundamental interval I_ν and $|c_k(x)|$ denotes the diameter of $c_k(x)$ for each $k = 0, 1, 2, \ldots$.

Given $0 < a, p < 1$ and $x \in (0,1]$, the Riesz-Nágy-Takács (RNT) function $\Psi_{a,p}$ ([17]) satisfies

$$\Psi_{a,p}(x) = \sum_{j=1}^{\infty} \frac{(1/p - 1)^{j-1}}{1/p^{a_j}}$$

for

$$x = \sum_{j=1}^{\infty} \frac{(1/a - 1)^{j-1}}{1/a^{a_j}}$$

with integers $1 \le a_1 < a_2 < \cdots < a_n < \cdots$. If $a = p$, then the RNT function is the identity function. We note that J. Paradís *et al.* used the non-terminating expansion and the above x can be represented by

$$x = \overbrace{0 \cdots 0}^{a_1 - 1} 1 \overbrace{0 \cdots 0}^{a_2 - a_1 - 1} 1 \cdots \overbrace{0 \cdots 0}^{a_n - a_{n-1} - 1} 1 \cdots .$$

We note that their expression for $x \in (0,1]$ is exactly the same as its corresponding code essentially.

From now on $\dim(E)$ denotes the Hausdorff dimension of E and $\mathrm{Dim}(E)$ denotes the packing dimension of E ([11]). We note that $\dim(E) \le \mathrm{Dim}(E)$ for every set E ([11]). We denote $n_0(x|k)$ the number of times the digit 0 occurs in the first k places of $x = \sigma$ (cf. [1]).

For $r \in [0,1]$, we define the lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ containing the digit 0 in proportion r by

$$\underline{F}(r) = \{x \in (0,1] : \liminf_{k \to \infty} \frac{n_0(x|k)}{k} = r\},$$

$$\overline{F}(r) = \{x \in (0,1] : \limsup_{k \to \infty} \frac{n_0(x|k)}{k} = r\}.$$

We write $\underline{F}(r) \cap \overline{F}(r) = F(r)$ and call it the distribution set containing the digit 0 in proportion r. Let $p \in (0,1)$ and denote γ_p a self-similar Borel probability measure on $(0,1]$ satisfying $\gamma_p(I_0) = p$ (cf. [1, 11]). We write $\underline{E}_\alpha^{(p)}(\overline{E}_\alpha^{(p)})$ for the set of points at which the lower(upper) local cylinder density of γ_p on $(0,1]$ is exactly α, so that

$$\underline{E}_\alpha^{(p)} = \{x \in (0,1] : \liminf_{k \to \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = \alpha\},$$

$$\overline{E}_\alpha^{(p)} = \{x \in (0,1] : \limsup_{k \to \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = \alpha\}.$$

We write $\underline{E}_\alpha^{(p)} \cap \overline{E}_\alpha^{(p)} = E_\alpha^{(p)}$ and call it the local dimension set having local dimension α by a self-similar measure γ_p. In this paper, we assume that

$0 \log 0 = 0$ for convenience. We introduce the relation between the distribution sets and the local dimension sets, which is an essential result from [1].

Proposition 2.1 ([4])**.** *Let $p \in (0, 1)$, $r \in [0, 1]$ and*

$$g(r, p) = \frac{r \log p + (1 - r) \log(1 - p)}{r \log a + (1 - r) \log(1 - a)}.$$

Then

(1) $\underline{F}(r) = \underline{E}_{g(r,p)}^{(p)}$ *if $0 < p < a$,*

(2) $\underline{F}(r) = \overline{E}_{g(r,p)}^{(p)}$ *if $a < p < 1$,*

(3) $\overline{F}(r) = \overline{E}_{g(r,p)}^{(p)}$ *if $0 < p < a$,*

(4) $\overline{F}(r) = \underline{E}_{g(r,p)}^{(p)}$ *if $a < p < 1$.*

From now on, we will continue to use $g(r, p)$ as above. To study the multifractal spectra of the derivative sets and the set of non-differentiability points, the following proposition is necessary. The results can be obtained easily from [1, 2, 5, 15].

Proposition 2.2 ([3])**.** *For $0 < s < 1$,*

(1) $\qquad \dim([\cup_{s < r \leq 1} \overline{F}(r)] \cap [\cup_{0 \leq r < s} \underline{F}(r)]) = g(s, s),$

(2) $\qquad \mathrm{Dim}([\cup_{s < r \leq 1} \overline{F}(r)] \cap [\cup_{0 \leq r < s} \underline{F}(r)]) = 1.$

For $a \leq s \leq 1$,

(3) $\qquad \dim(\cup_{s \leq r \leq 1} \underline{F}(r)) = \mathrm{Dim}(\cup_{s \leq r \leq 1} \underline{F}(r)) = g(s, s).$

For $0 \leq s \leq a$,

(4) $\qquad \dim(\cup_{0 \leq r \leq s} \overline{F}(r)) = \mathrm{Dim}(\cup_{0 \leq r \leq s} \overline{F}(r)) = g(s, s).$

For $s \neq a$,

(5) $\qquad \dim(F(s)) = \mathrm{Dim}(F(s)) = g(s, s) < 1,$

while

(6) $\qquad \dim(F(a)) = \mathrm{Dim}(F(a)) = 1.$

3. Characterization of the Hölder derivative sets and the Hölder non-differentiability set

We define *the Hölder derivative $D^q f(x)$ of a real valued function f at $x \in (0, 1]$ of Hölder order $0 < q < \infty$* by

$$D^q f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h^q},$$

where $h^q = \mathrm{sgn}(h)|h|^q$, if it exists in $[-\infty, \infty]$. We remark that this definition is the same as that of [12]. We note that $D^q f(x) = f'(x)$ for $q = 1$. Since we include the infinite values in its derivative, we sometimes use the terminology that $D^q f(x)$ exists *in a wide sense* (cf. [17]).

The following proposition is a general theorem related to the Vitali covering lemma ([18]) for the Hölder differentiability of the increasing function.

Proposition 3.1. *Let f be an increasing real valued function on the unit interval $(0, 1]$. Then $D^q f(x)$ of f at x of Hölder order $0 < q < \infty$ exists in a wide sense almost everywhere. In particular, for $q = 1$, $0 \le D^q f(x) < \infty$.*

Proof. It follows from the similar arguments of the proof of Theorem 3 in the section 5 of [18]. For this, it is only necessary to change small intervals $[x - h, x]$ and $(y, y + k)$ into $[x - h^q, x]$ and $(y, y + k^q)$ using $f(x) - f(x - h) < vh^q$ and $f(y+k)-f(y) > uk^q$ instead of using $f(x)-f(x-h) < vh$ and $f(y+k)-f(y) > uk$ in the arguments of the proof. In particular, $0 \le f'(x) < \infty$ from Theorem 3 in the section 5 of [18]. $\qquad\square$

Lemma 3.2. *Let α and β be positive real numbers. Then for $0 < q < \infty$,*

$$(\alpha + \beta)^q \le 2^q(\alpha^q + \beta^q).$$

For $0 < q \le 1$,

$$\alpha^q + \beta^q \le 2^{1-q}(\alpha + \beta)^q.$$

For $1 \le q < \infty$,

$$\alpha^q + \beta^q \le (\alpha + \beta)^q.$$

Proof. Let α and β be positive real numbers. For $0 < q < \infty$, $(\frac{\alpha+\beta}{2})^q \le \alpha^q$ or $(\frac{\alpha+\beta}{2})^q \le \beta^q$ since $\frac{\alpha+\beta}{2} \le \alpha$ or $\frac{\alpha+\beta}{2} \le \beta$. This gives $(\alpha + \beta)^q \le 2^q(\alpha^q + \beta^q)$. For $0 < q < 1$, the concavity of the function $y = x^q$ gives $(\frac{\alpha+\beta}{2})^q \ge \frac{\alpha^q+\beta^q}{2}$. For $1 < q < \infty$, $\alpha^q + \beta^q \le (\alpha + \beta)^q$ follows from that $\int_\alpha^{\alpha+\beta} y'dx \ge \int_0^\beta y'dx$ gives $(\alpha + \beta)^q - \alpha^q \ge \beta^q$ where $y = x^q$. $\qquad\square$

The following lemma is a variation of Theorems 6 and 7 in [4] which are useful for the study of the concrete examples of the differentiable points and the non-differentiability points even though the proof of the theorems is rather complicate. The proof of this lemma is simple even though this lemma can be applied to all $0 < q < \infty$.

Lemma 3.3. *Let x be not an end point of the fundamental intervals $c_n(x)$. For $0 < q < \infty$, if $0 < D^q f(x) = l < \infty$, then*

$$(7) \qquad 0 < 2^{-q}l \le \liminf_{n\to\infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} \le \limsup_{n\to\infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} \le 2l < \infty.$$

In particular, if $0 < f'(x) = l < \infty$, then

$$(8) \qquad 0 < \lim_{n\to\infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|} = l < \infty.$$

Consequently if $D^q f(x) = \infty$, then

$$(9) \qquad \lim_{n\to\infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} = \infty.$$

Similarly if $D^q f(x) = 0$, then

$$(10) \qquad\qquad \lim_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} = 0.$$

Proof. We note that for $0 < q < \infty$ and $0 < D^q f(x) = l < \infty$,

$$0 < \lim_{y \uparrow x} \frac{f(x) - f(y)}{(x-y)^q} = l = \lim_{z \downarrow x} \frac{f(z) - f(x)}{(z-x)^q} = l < \infty.$$

Given $\epsilon > 0$, there is $\delta > 0$ such that $x - \delta < y < x < z < x + \delta$ satisfying

$$(l - \epsilon)(x-y)^q < f(x) - f(y) < (l + \epsilon)(x-y)^q$$

and

$$(l - \epsilon)(z-x)^q < f(z) - f(x) < (l + \epsilon)(z-x)^q.$$

From the above lemma, for the same ϵ and δ with $x - \delta < y < x < z < x + \delta$, we have

$$2^{-q}(l - \epsilon)(z-y)^q < f(z) - f(y) < 2(l + \epsilon)(z-y)^q,$$

which gives, for $c_n(x) = (y_n, z_n]$ therefore $y_n < x < z_n$ since x is not an end point of $c_n(x)$,

$$0 < 2^{-q}l \le \liminf_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} \le \limsup_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} \le 2l < \infty.$$

In particular, for $q=1$, we have $(l - \epsilon)(z-y)^q < f(z) - f(y) < (l + \epsilon)(z-y)^q$, which gives, for $c_n(x) = (y_n, z_n]$ therefore $y_n < x < z_n$ since x is not an end point of $c_n(x)$,

$$0 < l \le \liminf_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} \le \limsup_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} \le l < \infty.$$

The assertions for the values $l = \infty$ and $l = 0$ follow from the similar arguments with the above ones. $\square$

Remark 3.4. From now on, in this section, f will be used as the RNT singular function $\Psi_{a,p}$ in the preliminaries if there is no particular mention of f. In this section, for simplicity, we fix $0 < a < p < 1$. We consider $q \in (\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)})$ without any particular mention. We note that the dual results hold for $0 < p < a < 1$, which are left as an exercise. Let $N^q = (0,1] - (D_0^q \cup D_\infty^q \cup D_1^q)$ where

$$\begin{aligned} D_0^q &= \{x \in (0,1] : D^q f(x) = 0\}, \\ D_\infty^q &= \{x \in (0,1] : D^q f(x) = \infty\}, \\ D_1^q &= \{x \in (0,1] : 0 < D^q f(x) < \infty\}. \end{aligned}$$

Then N^q is the set of the points x at which the derivatives $D^q f(x)$ of f do not exist. We note that if x is an end point of the fundamental intervals $c_n(x)$, then $D^q f(x)$ does not exist for every $0 < q < 1$, therefore $x \in N^q$. This means that we can apply (7)-(10) of the above lemma to the relation between

D_0^q, D_1^q, D_∞^q and the local dimension set. From now on, we will continue to use $D_0^q, D_1^q, D_\infty^q, N^q$ as above.

The following proposition and remark were shown already in [4], but we introduce them for the comparison with the next proposition.

Proposition 3.5 ([4]). *For $x \in (0, 1]$ at which $f'(x)$ exists and $0 < f'(x) < \infty$,*

$$\lim_{n \to \infty} \frac{\log \gamma_p(c_n(x))}{\log |c_n(x)|} = 1.$$

Remark 3.6 ([4]). From the above proposition, $D_1^1 \subset E_1^{(p)}$. Now, $E_1^{(p)} = F(s)$ for some s from (2) and (4) of Proposition 2.1. Then $s \neq a$. For, if $s = a$, then $g(a, p) \neq 1$ giving a contradiction. Hence the Hausdorff and packing dimension of $E_1^{(p)}$ is less than 1 from (5), which implies D_1^1 has null Lebesgue measure. Further, from (8), we see that

$$D_1^1 \subset \{x \in (0, 1] : 0 < \lim_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|} < \infty\}.$$

P. Billingsley ([6]) showed that $\{x \in (0, 1] : 0 < \lim_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|} < \infty\} = \phi$ using the fact that

$$\lim_{n \to \infty} \left| \frac{\gamma_p(c_{n+1}(x))}{\gamma_p(c_n(x))} - \frac{|c_{n+1}(x)|}{|c_n(x)|} \right| = 0$$

for $x \in \{x \in (0, 1] : 0 < \lim_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|} < \infty\}$ whereas

$$\left| \frac{\gamma_p(c_{n+1}(x))}{\gamma_p(c_n(x))} - \frac{|c_{n+1}(x)|}{|c_n(x)|} \right| = |p - a| \neq 0,$$

which implies that $D_1^1 = \phi$. This together with Proposition 3.1 gives that f is a singular function. However this can be shown also from Corollary 3.13.

Unlike $q = 1$, we cannot argue that $D_1^q = \phi$ for $q \neq 1$. However the set having the essential property of D_1^q is empty.

Proposition 3.7. *For $q \in (\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)})$,*

$$\{x \in (0, 1] : 0 < \lim_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} < \infty\} = \phi.$$

Proof. If there is an $x \in (0, 1]$ such that $0 < \lim_{n \to \infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} = l < \infty$, then

$$\lim_{n \to \infty} \left| \frac{\gamma_p(c_{n+1}(x))}{\gamma_p(c_n(x))} - \frac{l|c_{n+1}(x)|^q}{l|c_n(x)|^q} \right| = 0.$$

However

$$\left| \frac{\gamma_p(c_{n+1}(x))}{\gamma_p(c_n(x))} - \frac{l|c_{n+1}(x)|^q}{l|c_n(x)|^q} \right| = \begin{cases} |p - a^q| \neq 0 \\ \text{or} \\ |(1-p) - (1-a)^q| \neq 0 \end{cases}$$

since $p \neq a^q \Leftrightarrow q \neq \frac{\log p}{\log a}$, and $1 - p \neq (1-a)^q \Leftrightarrow q \neq \frac{\log(1-p)}{\log(1-a)}$. $\square$

Remark 3.8. By the above remark, we see that $N^1 = (0,1] - (D_0^1 \cup D_\infty^1)$. However we cannot guarantee that $N^q = (0,1] - (D_0^q \cup D_\infty^q)$ since we cannot assure that $D_1^q = \phi$ for $q(\neq 1) \in (\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)})$.

We define $r(q)$ by the solution r of the equation $g(r,p) = q$. We note that the function $g(r,p)$ is a strictly decreasing function for $r \in [0,1]$ having its range $[g(1,p), g(0,p)] = [\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)}]$ from $0 < a < p < 1$. We note that $0 < a < r(1) < 1$.

Theorem 3.9. *For* $q \in (\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)})$, *we have*

(11) $$[\cup_{r(q)<r\leq 1}\overline{F}(r)] \cap [\cup_{0\leq r<r(q)}\underline{F}(r)] \subset N^q.$$

For $q \in (\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)})$, *we have*

(12) $$\cup_{0\leq r\leq r(q)}\overline{F}(r) - F(r(q)) \subset D_0^q \cup N^q,$$

(13) $$\cup_{r(q)\leq r\leq 1}\underline{F}(r) - F(r(q)) \subset D_\infty^q \cup N^q,$$

further

(14) $$D_0^q \subset \cup_{0\leq r\leq r(q)}\overline{F}(r),$$

(15) $$D_\infty^q \subset \cup_{r(q)\leq r\leq 1}\underline{F}(r),$$

(16) $$D_1^q \subset F(r(q)).$$

Proof. For (12), if $x \in \cup_{0\leq r\leq r(q)}\overline{F}(r) - F(r(q))$, then $x \in \cup_{0\leq r<r(q)}\underline{F}(r)$. Therefore

$$x \in \cup_{0\leq r<r(q)}\overline{E}_{g(r,p)}^{(p)}.$$

That is, $\limsup_{n\to\infty} \frac{\log \gamma_p(c_n(x))}{\log|c_n(x)|} = g(r,p) > q = g(r(q),p)$ since $g(r,p)$ is a strictly decreasing function for r. This gives $\limsup_{n\to\infty} \frac{\log \gamma_p(c_n(x))}{\log|c_n(x)|^q} > 1$. This implies that $\liminf_{n\to\infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} = 0$. From (9), we see that $x \notin D_\infty^q$, which means $x \in D_0^q \cup N^q \cup D_1^q$. Assume that $x \in D_1^q$. Then a contradiction arises from (7) also. So $x \in D_0^q \cup N^q$.

For (13), if $x \in \cup_{r(q)\leq r\leq 1}\underline{F}(r) - F(r(q))$, then $\liminf_{n\to\infty} \frac{\log \gamma_p(c_n(x))}{\log|c_n(x)|^q} < 1$. This implies that $\limsup_{n\to\infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} = \infty$, which means $x \in D_\infty^q \cup N^q$ from the similar arguments above with (10) and (7).

For (11), if $x \in [\cup_{r(q)<r\leq 1}\overline{F}(r)] \cap [\cup_{0\leq r<r(q)}\underline{F}(r)]$, then from the same arguments above,

$$0 = \liminf_{n\to\infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} < \limsup_{n\to\infty} \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} = \infty.$$

From (9) and (10), $x \notin D_0^q \cup D_\infty^q$. Further we also see that $x \notin D_1^q$ from (7).

For (14), assume that $x \in D_0^q$. Then from (10),

$$x \notin [[\cup_{r(q)<r\leq 1}\overline{F}(r)] \cap [\cup_{0\leq r<r(q)}\underline{F}(r)]] \cup [\cup_{r(q)\leq r\leq 1}\underline{F}(r) - F(r(q))],$$

which implies $x \in \cup_{0\leq r\leq r(q)}\overline{F}(r)$. Similarly (15) holds.

For (16), since log function is a continuous increasing function, from (7)

$$\log 2^{-q}D^q f(x) \leq \liminf_{n\to\infty} \log \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} \leq \limsup_{n\to\infty} \log \frac{\gamma_p(c_n(x))}{|c_n(x)|^q} \leq \log 2D^q f(x),$$

which gives

$$\lim_{n\to\infty} \frac{\log \gamma_p(c_n(x)) - q\log |c_n(x)|}{\log |c_n(x)|} = 0,$$

from $0 < D^q f(x) < \infty$. This means that $D_1^q \subset E_q^{(p)}$. From Proposition 2.1(2) and (4), $E_q^{(p)} = F(r(q))$. $\square$

Corollary 3.10. *For $q \in (\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)})$, we have*

$$0 < g(r(q), r(q)) \leq \dim(N^q) \leq \mathrm{Dim}(N^q) = 1.$$

In particular, if $r(q) = a$, then $\dim(N^q) = \mathrm{Dim}(N^q) = 1$.

Proof. It follows from (11) with (1) and (2). $\square$

Remark 3.11. Since $\underline{F}(0) \cap \overline{F}(1)(\subset N^q)$ is comeager in $(0, 1]$ ([4, 14]), the RNT singular function is nowhere Hölder differentiable in the sense of topological magnitude for $q \in (\frac{\log p}{\log a}, \frac{\log(1-p)}{\log(1-a)})$. We note that if x is an end point of a fundamental interval, then $x \in F(0) \subset \underline{F}(0)$ and $x \in N^q$ for $0 < q < \infty$.

Corollary 3.12. *Let $0 < r(q) < 1$. Then, for $r(q) \neq a$,*

$$\dim(D_1^q) \leq \mathrm{Dim}(D_1^q) \leq g(r(q), r(q)) < 1,$$

while, for $r(q) = a$,

$$\dim(D_1^q) \leq \mathrm{Dim}(D_1^q) \leq g(r(q), r(q)) = 1.$$

Proof. It is immediate from (16) with (5) and (6). $\square$

Before going into the two corollaries, we note that $D_0^q \cup D_1^q \cup D_\infty^q$ has full Lebesgue measure from Proposition 3.1.

Corollary 3.13. *Let $a < r(q) < 1$. Then*

$$\dim(D_\infty^q) \leq \mathrm{Dim}(D_\infty^q) \leq g(r(q), r(q)) < 1,$$

therefore D_0^q has full Lebesgue measure.

Proof. Let $a < r(q) < 1$. From (15) with (3),

$$\dim(D_\infty^q) \leq \mathrm{Dim}(D_\infty^q) \leq \mathrm{Dim}(\cup_{r(q)\leq r\leq 1}\underline{F}(r)) = g(r(q), r(q)) < 1.$$

This means that D_∞^q has null Lebesgue measure. Further, D_1^q has null Lebesgue measure from the above corollary. Hence Proposition 3.1 gives that D_0^q has full Lebesgue measure. $\square$

Corollary 3.14. *Let* $0 < r(q) < a$. *Then*

$$\dim(D_0^q) \leq \mathrm{Dim}(D_0^q) \leq g(r(q), r(q)) < 1,$$

therefore D_∞^q *has full Lebesgue measure.*

Proof. Let $0 < r(q) < a$. From (14) with (4),

$$\dim(D_0^q) \leq \mathrm{Dim}(D_0^q) \leq \mathrm{Dim}(\cup_{0 \leq r \leq r(q)} \overline{F}(r)) = g(r(q), r(q)) < 1.$$

This means that D_0^q has null Lebesgue measure. Further, D_1^q has null Lebesgue measure from Corollary 3.12. Hence Proposition 3.1 gives that D_∞^q has full Lebesgue measure. $\qquad\square$

Remark 3.15. We note that $g(a, p) < q < g(0, p) \Leftrightarrow 0 < r(q) < a$, and $g(1, p) < q < g(a, p) \Leftrightarrow a < r(q) < 1$. Therefore $g(a, p)$ is the critical point of q for which one of D_0^q and D_∞^q has full Lebesgue measure. That is, D_0^q has full Lebesgue measure for $0 < q < g(a, p)$ while D_∞^q has full Lebesgue measure for $g(a, p) < q < \infty$. However, for $q = g(a, p) \Leftrightarrow r(q) = a$, we do not have any information about which one of D_0^q, D_1^q, and D_∞^q has full Lebesgue measure.

4. Application of the characterization to the metric number theory

In the above section, we used f for the RNT singular function, but in this section we will use $\Psi_{a,p}$ instead of f for comparison of our results with those of [17]. J. Paradís *et al.* ([17]) studied the RNT singular function $\Phi_{\alpha,\tau}(x) = \Psi_{a,p}(x)$ where $\alpha = 1/a$, $\tau = 1/p$ and gave a critical point to check the existence of derivative of the singular function. In this section we assume that $\alpha = 1/a$, $\tau = 1/p$. The critical point is $K = K(\alpha, \tau) = \frac{\log(\frac{\alpha-1}{\tau-1})}{\log(\alpha/\tau)}$. It is not difficult to show that $K = \frac{1}{1-r(1)}$. Similarly for the metric number theoretical study with respect to the Hölder derivative, we define $K_q = \frac{1}{1-r(q)}$, where q is between $\frac{\log p}{\log a}$ and $\frac{\log(1-p)}{\log(1-a)}$. Then

$$K_q = K_q(\alpha, \tau) = \frac{\log(\frac{(\alpha-1)^q}{\tau-1})}{\log(\frac{\alpha(\alpha-1)^{q-1}}{\tau})}.$$

We easily see that $K_q(\alpha, \tau) \geq 1$.

From now on, we assume that $x = \sum_{j=1}^{\infty} \frac{(\alpha-1)^{j-1}}{\alpha^{a_j}}$ with positive integers a_j such that $1 \leq a_1 < a_2 < \cdots$ as in the preliminaries.

Lemma 4.1 ([4]). *For an extended real number* $1 \leq A \leq \infty$ *we have*

$$\text{(17)} \qquad \liminf_{n \to \infty} \frac{a_n}{n} \geq A \Leftrightarrow \liminf_{m \to \infty} \frac{n_0(x|m)}{m} \geq 1 - \frac{1}{A},$$

and

$$\text{(18)} \qquad \limsup_{n \to \infty} \frac{a_n}{n} \leq A \Leftrightarrow \limsup_{m \to \infty} \frac{n_0(x|m)}{m} \leq 1 - \frac{1}{A}.$$

As was shown in [4], from the above lemma, we easily see that

$$(19) \qquad \liminf_{m \to \infty} \frac{n_0(x|m)}{m} = 1 - \frac{1}{\liminf_{n \to \infty} \frac{a_n}{n}}$$

and

$$(20) \qquad \limsup_{m \to \infty} \frac{n_0(x|m)}{m} = 1 - \frac{1}{\limsup_{n \to \infty} \frac{a_n}{n}}.$$

As was shown in [4], the following theorem for the particular value $q = 1$ is a generalization of Theorem 4.2 in [17] which is the main result of [17].

Theorem 4.2. *If $\lim_{n \to \infty} \frac{a_n}{n} \neq K_q$, then we have:*
  (i) *Case $1 < \tau < \alpha$. If*

$$\limsup_{n \to \infty} \frac{a_n}{n} \leq K_q,$$

then, if $D^q \Psi_{a,p}(x)$ exists in a wide sense, it has to be 0. If

$$\liminf_{n \to \infty} \frac{a_n}{n} \geq K_q,$$

then, if $D^q \Psi_{a,p}(x)$ exists in a wide sense, it has to be ∞.
  (ii) *Case $\tau > \alpha > 1$.*

$$\liminf_{n \to \infty} \frac{a_n}{n} \geq K_q,$$

then, if $D^q \Psi_{a,p}(x)$ exists in a wide sense, it has to be 0. If

$$\limsup_{n \to \infty} \frac{a_n}{n} \leq K_q,$$

then, if $D^q \Psi_{a,p}(x)$ exists in a wide sense, it has to be ∞.

Proof. For Case $1 < \tau < \alpha (\Leftrightarrow 0 < a < p < 1)$, assume that $\liminf_{n \to \infty} \frac{a_n}{n} \geq K_q$. (17) gives that

$$\liminf_{m \to \infty} \frac{n_0(x|m)}{m} \geq 1 - \frac{1}{K_q} = r(q),$$

which means

$$x \in \cup_{r(q) \leq r \leq 1} \underline{F}(r).$$

Since $\lim_{n \to \infty} \frac{a_n}{n} \neq K_q$, we easily see that $x \notin F(r(q))$ from (17) and (18). From (13): $\cup_{r(q) \leq r \leq 1} \underline{F}(r) - F(r(q)) \subset D_\infty^q \cup N^q$ for $0 < a < p < 1$, we immediately have $\overline{D^q \Psi_{a,p}(x)} = \infty$ if $D^q \Psi_{a,p}(x)$ exists in a wide sense. Similarly the rest holds from the above lemma with (12). For Case $1 < \tau < \alpha$, it follows from the dual arguments for $0 < p < a < 1$. $\qquad\square$

The following theorem is the converse of the above theorem in some sense.

Theorem 4.3. *We have:*
  (i) *Case $1 < \tau < \alpha$. If $D^q \Psi_{a,p}(x) = 0$, then*

$$\limsup_{n \to \infty} \frac{a_n}{n} \leq K_q.$$

If $D^q\Psi_{a,p}(x) = \infty$, then

$$\liminf_{n\to\infty} \frac{a_n}{n} \geq K_q.$$

(ii) *Case $\tau > \alpha > 1$. If $D^q\Psi_{a,p}(x) = 0$, then*

$$\liminf_{n\to\infty} \frac{a_n}{n} \geq K_q.$$

If $D^q\Psi_{a,p}(x) = \infty$, then

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq K_q.$$

Proof. For Case $1 < \tau < \alpha$, it follows from the above lemma with (14) and (15). For Case $\tau > \alpha > 1$, it follows from the above lemma with the dual results of (14) and (15) for $0 < p < a < 1$. $\qquad\square$

The following is a sufficient condition for the Hölder non-differentiability points of the RNT singular function.

Theorem 4.4. *If*

$$\liminf_{n\to\infty} \frac{a_n}{n} < K_q < \limsup_{n\to\infty} \frac{a_n}{n},$$

then $D^q\Psi_{a,p}(x)$ does not exist.

Proof. From (17) and (18), if $\liminf_{n\to\infty} \frac{a_n}{n} < K_q < \limsup_{n\to\infty} \frac{a_n}{n}$, then we easily see that

$$x \in [\cup_{r(q)<r\leq 1}\overline{F}(r)] \cap [\cup_{0\leq r<r(q)}\underline{F}(r)].$$

It follows from (11). $\qquad\square$

As was shown in [4], the following theorem for $q = 1$ is also an essential generalization of Theorem 4.1 in [17]. Noting that $r(q) = 1 - \frac{1}{K_q}$, we have $\alpha(1 - \frac{1}{K_q}) > 1 \Leftrightarrow r(q) > a$ when $0 < a < p < 1$. For $\alpha(1 - \frac{1}{K_q}) > 1$, in particular $q = 1$, it assures that $D^q\Psi_{a,p}(x) = 0$ for a normal point $x \in F_a$ where $a = \frac{1}{\alpha}$ when $D^q\Psi_{a,p}(x)$ exists in a wide sense.

Theorem 4.5. *If $\lim_{n\to\infty} \frac{a_n}{n} = \frac{\alpha}{\alpha-1}$, then*
(1) *for $\alpha(1 - \frac{1}{K_q}) > 1$, if $D^q\Psi_{a,p}(x)$ exists in a wide sense, $D^q\Psi_{a,p}(x) = 0$.*
(2) *for $\alpha(1 - \frac{1}{K_q}) < 1$, if $D^q\Psi_{a,p}(x)$ exists in a wide sense, $D^q\Psi_{a,p}(x) = \infty$.*

Proof. We note that $\alpha(1 - \frac{1}{K_q}) = \alpha r(q)$. If $\lim_{n\to\infty} \frac{a_n}{n} = \frac{\alpha}{\alpha-1}$, then $x \in F(\frac{1}{\alpha})$ from (19) and (20).

Let $0 < a < p < 1$. Since $F(\frac{1}{\alpha}) = F(a) \subset D_0^q \cup N^q$ from (12) if $a < r(q) \Leftrightarrow \alpha r(q) > 1$, it follows. Since $F(\frac{1}{\alpha}) = F(a) \subset D_\infty^q \cup N^q$ from (13) if $a > r(q) \Leftrightarrow \alpha r(q) < 1$, it follows. Dually it holds for $0 < p < a < 1$, namely, $F(\frac{1}{\alpha}) = F(a) \subset D_0^q \cup N^q$ if $a > r(q) \Leftrightarrow \alpha r(q) > 1$, and $F(\frac{1}{\alpha}) = F(a) \subset D_\infty^q \cup N^q$ if $a < r(q) \Leftrightarrow \alpha r(q) < 1$. $\qquad\square$

5. Application of the characterization to the general singularity

In this section, we define a general singularity and its dual singularity for a strictly increasing continuous function f having $f(0) = 0$ and $f(1) = 1$. We clearly see that the RNT function f assuming $f(0) = 0$ and $f(1) = 1$ is a strictly increasing continuous function on the unit interval [0,1]. We used the concept of the singularity for a non-constant increasing continuous function f whose derivative $f'(x)$ is zero for full Lebesgue measure or *a.e.*. Now it is natural for us to extend its concept to the general case in the view of Corollaries 3.13 and 3.14. For $0 < q < \infty$, we say that a strictly increasing continuous function f on the unit interval is *a singular function of order q* if $D^q f(x) = 0$ for Lebesgue almost all point x. We exclude the singular function having some constant part from our concern, for example the Cantor function, since a non-strictly increasing singular function gives a degeneration case for some meaningful definition. Further we also say that a strictly increasing continuous function f on the unit interval is *a dual singular function of order q* if $D^q f(x) = \infty$ for Lebesgue almost all point x. We write S_q for the set of the singular functions of order q and T_q for the set of the dual singular functions of order q. Then we clearly see that for $q_1 \geq q_2$

$$S_{q_1} \subset S_{q_2}$$

and

$$T_{q_2} \subset T_{q_1}.$$

Further we also see that $S_q \cap T_q = \phi$ for every $q \in (0, \infty)$.

Corollary 5.1. (i) *Assume that $0 < a \neq p < 1$. If $0 < q < g(a,p)$, then $\Psi_{a,p} \in S_q$. If $g(a,p) < q < \infty$, then $\Psi_{a,p} \in T_q$. In particular, $\Psi_{a,p} \in S_1$.*
  (ii) *Assume that $0 < a = p < 1$. If $0 < q < 1$, then $\Psi_{a,p} \in S_q$. If $1 < q < \infty$, then $\Psi_{a,p} \in T_q$.*

Proof. For $0 < a < p < 1$, assume that $0 < q < g(a,p)$. Since $g(1,p) < q < g(a,p) \Leftrightarrow a < r(q) < 1$ in Remark 5, Corollary 3.13 gives that the RNT function $\Psi_{a,p}$ is a singular function of order q. Clearly for $0 < q \leq g(1,p)$, the RNT function $\Psi_{a,p}$ is a singular function of order q since $S_{q_1} \subset S_{q_2}$ for $q_1 \geq q_2$.
  Assume that $g(a,p) < q < \infty$. Since $g(a,p) < q < g(0,p) \Leftrightarrow 0 < r(q) < a$, in Remark 3.15, Corollary 3.14 gives that the RNT function $\Psi_{a,p}$ is a dual singular function of order q. Clearly for $g(0,p) \leq q < \infty$, the RNT function $\Psi_{a,p}$ is a dual singular function of order q since $T_{q_2} \subset T_{q_1}$ for $q_1 \geq q_2$.
  Dually it also holds for $0 < p < a < 1$ from the exercise which was mentioned in Remark 3.4.
  In particular, if $a \neq p$, then $0 < 1 < g(a,p)$. $\Psi_{a,p} \in S_1$ from the above.
  For $0 < a = p < 1$, $\Psi_{a,p}$ is the identity function. It follows immediately from the simple arguments which can be used to show $S_{q_1} \subset S_{q_2}$ and $T_{q_2} \subset T_{q_1}$ for $q_1 \geq q_2$. $\qquad\square$

Proposition 5.2. *For any strictly increasing continuous function f,*

$$0 \leq \sup\{q \in (0, \infty) : D^q f(x) = 0 \ a.e.\}$$
$$\leq \inf\{q \in (0, \infty) : D^q f(x) = \infty \ a.e.\} \leq \infty.$$

Proof. Suppose that it does not hold. That is, we assume that

$$\inf\{q \in (0, \infty) : D^q f(x) = \infty \text{ a.e.}\} < \sup\{q \in (0, \infty) : D^q f(x) = 0 \text{ a.e.}\}.$$

Then there are positive real numbers $q_1 < q_2$ between them and $D^{q_1} f(x) = \infty$ a.e. and $D^{q_2} f(x) = 0$ a.e., which gives a contradiction. $\square$

For a strictly increasing function f on the unit interval, we define *the lower singularity order of f* by

$$\sup\{q \in (0, \infty) : D^q f(x) = 0 \text{ a.e.}\} = L(f),$$

and *the upper singularity order of f* by

$$\inf\{q \in (0, \infty) : D^q f(x) = \infty \text{ a.e.}\} = U(f).$$

If $L(f) = U(f)$, we call such $L(f)(= U(f))$ *the singularity order of f*. The following theorem shows that the RNT function $\Psi_{a,p}$ has the singularity order $g(a, p)$.

Corollary 5.3. *For the RNT function $f = \Psi_{a,p}$,*

$$L(f) = g(a, p) = U(f).$$

Proof. It follows from the above corollary. $\square$

Proposition 5.4. *Let f be strictly increasing and $f'(x)$ exist in a wide sense on the unit interval and $f'(x) > 0$ a.e.. Then*

$$L(f) = 1 = U(f).$$

Proof. Assume that f is a strictly increasing smooth function on the unit interval and $f'(x) > 0$ a.e.. Since f is an increasing function, $0 \leq f'(x) < \infty$ a.e. by Proposition 3.1. Hence $0 < f'(x) < \infty$ a.e., from which it follows easily. $\square$

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