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ADDITIVE ρ-FUNCTIONAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we solve the additive ρ-functional equations

$$\begin{aligned} & f(x+y+z)-f(x)-f(y)-f(z) \\ & = \rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right), \end{aligned} \tag{0.1}$$

where ρ is a fixed number with $\rho \neq 1, 2$, and

$$\begin{aligned} & f(x+y+z)-f(x)-f(y)-f(z) \\ & = \rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right), \end{aligned} \tag{0.2}$$

where ρ is a fixed number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive ρ-functional equations (0.1) and (0.2) in Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [5] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [3] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [2] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In Section 2, we solve the additive functional equation (0.1) and prove the Hyers-Ulam stability of the additive functional equation (0.1) in Banach spaces.

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In Section 3, we solve the additive functional equation (0.2) and prove the Hyers-Ulam stability of the additive functional equation (0.2) in Banach spaces.

Throughout this paper, assume that X is a normed space and that Y is a Banach space.

2. Additive ρ-functional Equation (0.1)

Let ρ be a number with $\rho \neq 1, 2$.

We solve and investigate the additive ρ-functional equation (0.1) in normed spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} & f(x+y+z) - f(x) - f(y) - f(z) \\ & = \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right) \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = z = 0$ in (2.1), we get $-2f(0) = -\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ and $z = 0$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y+z) - f(x) - f(y) - f(z) &= \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right) \\ &= \rho(f(x+y+z) - f(x) - f(y) - f(z)) \end{aligned}$$

and so $f(x+y+z) = f(x) + f(y) + f(z)$ for all $x, y, z \in X$. Since $f(0) = 0$,

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (2.1) in Banach spaces.

Theorem 2.2. *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y,z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \tag{2.3}$$

$$\begin{aligned} &\Bigg\| f(x+y+z) - f(x) - f(y) - f(z) \\ &\quad - \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right)\Bigg\| \le \varphi(x,y,z) \end{aligned} \tag{2.4}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x,x,0) \tag{2.5}$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = 0$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x,x,0) \tag{2.6}$$

for all $x \in X$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{y}{2}, 0\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\leq 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, 0\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (2.3) and (2.4) that

$$\begin{aligned} &\Bigg\|A(x+y+z) - A(x) - A(y) - A(z) - \rho\Big(2A\left(\frac{x+y+z}{2}\right) \\ &\qquad -A(x) - A(y) - A(z)\Big)\Bigg\| \\ &= \lim_{n\to\infty} \Bigg\|2^n \left(f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right. \\ &\qquad \left. -\rho\left(2f\left(\frac{x+y+z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right)\right)\Bigg\| \\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right) = 0 \end{aligned}$$

for all $x, y, z \in X$. So

$$A(x+y) - A(x) - A(y) - A(z) = \rho\left(2A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z)\right)$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\Bigg\|f(x+y+z) - f(x) - f(y) - f(z) \\ &-\rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right)\Bigg\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{2.8}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. Letting $\varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 2.2, we get the desired result. □

Theorem 2.4. *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2.4) and*

$$\Psi(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x,x,0) \tag{2.9}$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \le \frac{1}{2}\varphi(x,x,0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\| &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right) \right\| \\ &\le \sum_{j=l}^{m-1} \frac{1}{2^{j+1}}\varphi(2^j x, 2^j x, 0) \end{aligned} \tag{2.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.8). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. Letting $\varphi(x,y,z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 2.4, we get the desired result. □

3. Additive ρ-functional Equation (0.2)

Let ρ be a number with $\rho \neq 1$.

We solve and investigate the additive ρ-functional equation (0.2) in normed spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} & f(x+y+z) - f(x) - f(y) - f(z) \\ & = \rho\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z)\right) \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = z = 0$ in (2.1), we get $-2f(0) = -2\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ and $z = 0$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} f(x+y) - f(x) - f(y) &= \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \\ &= \rho(f(x+y) - f(x) - f(y)) \end{aligned}$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (3.1) in Banach spaces.

Theorem 3.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty,$$

$$\begin{aligned} & \Bigg\| f(x+y+z) - f(x) - f(y) - f(z) \\ & \quad - \rho\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z)\right) \Bigg\| \leq \varphi(x, y, z) \end{aligned} \tag{3.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x, x, 0)$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = 0$ in (3.3), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x, x, 0) \tag{3.4}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\Bigg\| f(x+y+z) - f(x) - f(y) - f(z) \\ &\quad - \rho\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z)\right)\Bigg\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{3.5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. Letting $\varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 3.2, we get the desired result. □

Theorem 3.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.3) and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (3.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{1}{2} \varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 2.4. □

Corollary 3.5. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.5). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^r} \|x\|^r$$

for all $x \in X$.

Proof. Letting $\varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 3.4, we get the desired result. □

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