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# SIMPLY CONNECTED MANIFOLDS OF DIMENSION $4k$ WITH TWO SYMPLECTIC DEFORMATION EQUIVALENCE CLASSES

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Abstract. We present smooth simply connected closed $4k$-dimensional manifolds $N := N_k$, for each $k \in \{2, 3, \cdots\}$, with distinct symplectic deformation equivalence classes $[[\omega_i]]$, $i = 1, 2$. To distinguish $[[\omega_i]]$'s, we used the symplectic $Z$ invariant in [4] which depends only on the symplectic deformation equivalence class. We have computed that $Z(N, [[\omega_1]]) = \infty$ and $Z(N, [[\omega_2]]) < 0$.

## 1. Introduction

An *almost-Kähler* metric on a smooth manifold $M^{2n}$ of real dimension $2n$ is a Riemannian metric $g$ compatible with a symplectic structure $\omega$, i.e. $\omega(X, Y) = g(X, JY)$ for an almost complex structure $J$, where $X, Y$ are tangent vectors at a point of the manifold. Two symplectic forms $\omega_0$ and $\omega_1$ on $M$ are called *deformation equivalent*, if there exists a diffeomorphism $\psi$ of $M$ such that $\psi^*\omega_1$ and $\omega_0$ can be joined by a smooth homotopy of sympelctic forms, [5]. For a symplectic form $\omega$, its deformation equivalence class shall be denoted by $[[\omega]]$. We denote by $\Omega_{[[\omega]]}$ the set of all almost Kähler metrics compatible with a symplectic form in $[[\omega]]$. Examples of smooth manifolds with more than one symplectic deformation class have been an interesting subject to study; refer to [6], [7] or [8].

For a smooth closed manifold $M$ of dimension $2n \geq 4$ which admits a symplectic structure $\omega$, we have defined a symplectic invariant $Z$ in [4];

$$Z(M, [[\omega]]) = \sup_{g \in \Omega_{[[\omega]]}} \frac{\int_M s_g d\mathrm{vol}_g}{(\mathrm{Vol}_g)^{\frac{n-1}{n}}},$$

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where $\mathrm{dvol}_g$, $s_g$, $\mathrm{Vol}_g$ are the volume form, the scalar curvature and the volume of $g$ respectively.

In [4], we presented a six dimensional non-simply connected closed manifold which admits two symplectic deformation classes $[[\omega_i]]$, $i = 1, 2$, such that their $Z$ values have distinct signs. Then in [3], we showed an eight dimensional simply connected closed manifold with the same property.

The main result in this article is to present a simply connected manifold of dimension $4k$, for each $k \in \{2, 3, \cdots\}$, with the above property.

## 2. EXAMPLES IN DIMENSION $4k$

Here we shall prove the following;

**Theorem 2.1.** *For each integer $k \geq 2$, there exists a smooth closed simply connected $4k$-dimensional manifold $N$ with symplectic deformation equivalence classes $[[\omega_i]]$, $i = 1, 2$ such that $Z(N, [[\omega_1]]) = \infty$ and $Z(N, [[\omega_2]]) < 0$.*

The manifold $N$ is (diffeomorphic to) the product of $k$ copies of a complex surface of general type with ample canonical line bundle which is homeomorphic to $R_8$, the blow up of the complex projective plane $\mathbb{CP}_2$ at 8 points in general position. This general type complex surface may be obtained as a small deformation of Barlow's explicit complex surfaces [1]. When $k = 2$, the manifold $N$ in the theorem can be the one studied by Catanese and LeBrun [2].

To prove this theorem, we need the following;

**Proposition 1.** *Let $W$ be a complex surface of general type with ample canonical line bundle, homeomorphic to $R_8$. Consider a Kähler Einstein metric of negative scalar curvature on $W$ with Kähler form $\omega_W$ on $W$. Set $N := W \times \cdots \times W$, the $k$-fold product of $W$.*

*Then $Z(N, [[\omega_W + \cdots + \omega_W]]) = -4\sqrt{2}\pi k$, and it is attained by a Kähler Einstein metric.*

*Proof.* The argument here follows the scheme in [4, Section 3] and is similar to that in [3]. We recall one known fact about $W$ from [7, Section 4]; there is a homeomorphism of $W$ onto $R_8$ which preserves the Chern class $c_1$. And there is a diffeomorphism of $N$ onto $R_8^{(k)} := R_8 \times \cdots \times R_8$, the $k$-fold product of $R_8$ [2, Section 4].

Note that $R_8$ is well known to admit a Kähler Einstein metric of positive scalar curvature obtained by Calabi-Yau solution.

Then, the first Chern class of $W$ can be written as $c_1(W) = 3E_0 - \sum_{i=1}^{8} E_i \in H^2(W, \mathbb{R}) \cong \mathbb{R}^9$, where $E_i$, $i = 0, \cdots 8$, is the Poincare dual of a homology class $\tilde{E}_i$, $i = 0, \cdots 8$ so that $\tilde{E}_i$, $i = 0, \cdots 8$, form a basis of $H_2(W, \mathbb{Z}) \cong \mathbb{Z}^9$ and their intersections satisfy $\tilde{E}_i \cdot \tilde{E}_j = \epsilon_i \delta_{ij}$, where $\epsilon_0 = 1$ and $\epsilon_i = -1$ for $i \geq 1$. So, in this basis the intersection form becomes

$$I = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & -1 & \cdot & \cdot & 0 \\ . & . & \cdot & \cdot & 0 \\ . & . & \cdot & \cdot & 0 \\ 0 & 0 & & 0 & -1 \end{bmatrix}.$$

We have the orientation of $W$ induced by the complex structure and the fundamental class $[W] \in H_4(W, \mathbb{Z}) \cong \mathbb{Z}$. As $\omega_W$ is the Kähler form of a Kähler Einstein metric $g_W$ of negative scalar curvature, we may get $[\omega_W] = -3E_0 + \sum_{i=1}^{8} E_i$ by scaling if necessary.

By Künneth theorem $H^2(N, \mathbb{R}) \cong \sum_{j=1}^{k} \pi_j^* H^2(W) \cong \mathbb{R}^9 \oplus \cdots \oplus \mathbb{R}^9$, where $\pi_j$ is the projection of $N$ onto the j-th factor. Then,

$$c_1(N) = \sum_{j=1}^{k} \pi_j^* c_1(W) = \sum_{j=1}^{k} \pi_j^* (3E_0 - \sum_{i=1}^{8} E_i).$$

Consider any smooth path of symplectic forms $\omega_t$, $0 \leq t \leq \delta$, on $N$ such that $\omega_0 = \omega_W + \cdots + \omega_W$. We may write

$$[\omega_t] = \sum_{j=1}^{k} \sum_{i=0}^{8} n_i^j(t) \pi_j^* E_i \in H^2(N, \mathbb{R})$$

for some continuous functions $n_i^j(t)$ in $t$, $i = 0, \cdots, 8$. As $\{\omega_t\}$ is connected, their first Chern class $c_1(\omega_t) = c_1(N)$ does not depend on $t$. Using the intersection form we do a combinatorial computation;

$$\begin{aligned} [\omega_t]^{2k}([N]) &= [\sum_{j=1}^{k} \sum_{i=0}^{8} n_i^j(t) \pi_j^* E_i]^{2k}([W \times \cdots \times W]) \\ &= C_2^{2k} C_2^{2k-2} \cdots C_2^2 \prod_{j=1}^{k} \{n_0^j(t)^2 - \sum_{i=1}^{8} n_i^j(t)^2\} > 0, \end{aligned} \tag{2.1}$$

where $C_k^n = \frac{n!}{(n-k)!k!}$.

Set $[\omega^j(t)] = \sum_{i=0}^{8} n_i^j(t) E_i \in H^2(W, \mathbb{R})$, so that $[\omega_t] = \sum_{j=1}^{k} \pi_j^* [\omega^j(t)]$. We put $A_j := A_j(t) = [\omega^j(t)]^2[W] = n_0^j(t)^2 - \sum_{i=1}^{8} n_i^j(t)^2$. As $A_j(0) = [\omega_W]^2[W] > 0$ and

$\prod_{j=1}^{k} A_j(t) > 0$ from (2.1), we have $A_j(t) > 0$. Then $n_0^j(t)^2 > \sum_{i=1}^{l} n_i^j(t)^2$ and as $n_0^j(0) = -3 < 0$, so $n_0^j(t) < 0$.

We also put $B_j := B_j(t) = (c_1(W) \cdot [\omega^j(t)])[W] = 3n_0^j(t) + \sum_{i=1}^{8} n_i^j(t)$. Since $n_0^j(t)^2 > \sum_{i=1}^{8} n_i^j(t)^2$ and $|\sum_{i=1}^{8} n_i^j(t)| \leq \sqrt{8}\sqrt{\sum_{i=1}^{8} n_i^j(t)^2}$, we get

$$
\begin{aligned}
3n_0^j(t) + \sum_{i=1}^{8} n_i^j(t) &\leq 3n_0^j(t) + 2\sqrt{2}\sqrt{\sum_{i=1}^{8} n_i^j(t)^2} \\
&< 3n_0^j(t) + 2\sqrt{2}\sqrt{n_0^j(t)^2} = (3 - 2\sqrt{2})n_0^j(t) < 0.
\end{aligned} \tag{2.2}
$$

As $c_1(\omega_t) = \pi_1^* c_1(W) + \cdots + \pi_k^* c_1(W)$, by combinatorial computation we obtain;

$$
c_1(\omega_t) \cdot [\omega_t]^{2k-1}([N]) = \sum_{j=1}^{k} (2k-1) C_2^{2k-2} C_2^{2k-4} \cdots C_2^2 (A_1 A_2 \cdots A_k) \cdot \frac{B_j}{A_j}. \tag{2.3}
$$

Putting $A = A_1 \cdots A_k$ and $C = C_2^{2k} C_2^{2k-2} \cdots C_2^2$, from (2.1) and (2.3) we have;

$$
\frac{c_1(\omega_t) \cdot [\omega_t]^{2k-1}}{[\omega_t^{2k}]^{\frac{2k-1}{2k}}} = \frac{\sum_{j=1}^{k} (CA) \cdot \frac{B_j}{A_j}}{k(CA)^{\frac{2k-1}{2k}}} = \frac{(CA)^{\frac{1}{2k}}}{k} \sum_{j=1}^{k} \frac{B_j}{A_j}.
$$

From the AM-GM (Arithmetic Mean - Geometric Mean) inequality; $\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$, setting $x_j = -\frac{B_j}{A_j} > 0$, we get

$$
\sum_{j=1}^{k} \frac{B_j}{A_j} \leq -k \frac{((-1)^k B_1 \cdots B_k)^{\frac{1}{k}}}{A^{\frac{1}{k}}}. \tag{2.4}
$$

So,

$$
\frac{c_1(\omega_t) \cdot [\omega_t]^{2k-1}}{[\omega_t^{2k}]^{\frac{2k-1}{2k}}} \leq -C^{\frac{1}{2k}} \frac{((-1)^k B_1 \cdots B_k)^{\frac{1}{k}}}{(A_1 \cdots A_k)^{\frac{1}{2k}}}.
$$

From (2.2),

$$
\frac{B_j^2}{A_j} \geq \frac{\{3n_0^j(t) + 2\sqrt{2}\sqrt{\sum_{i=1}^{8} n_i^j(t)^2}\}^2}{n_0^j(t)^2 - \sum_{i=1}^{8} n_i^j(t)^2} = \frac{(3 - 2\sqrt{2}\sqrt{y})^2}{1 - y}, \tag{2.5}
$$

where $y = \sum_{i=1}^{8} \frac{n_i^j(t)^2}{n_0^j(t)^2}$. By calculus, $\frac{(3-2\sqrt{2}\sqrt{y})^2}{1-y} \geq 1$ for $y \in [0, 1)$ with equality at $y = \frac{8}{9}$. So, we get $\frac{B_j^2}{A_j} \geq 1$ and $\frac{-B_j}{\sqrt{A_j}} \geq 1$.

From this we have

$$
\frac{c_1(\omega_t) \cdot [\omega_t]^{2k-1}}{[\omega_t^{2k}]^{\frac{2k-1}{2k}}} \leq -C^{\frac{1}{2k}}. \tag{2.6}
$$

There is a basic inequality for any symplectic structure $\omega$ on a closed manifold $M$ of dimension $2n$ [4];

$$Z(M,[[\omega]]) \leq \sup_{\omega\in[[\omega]]} \frac{4\pi c_1(\omega)\cdot\frac{[\omega]^{n-1}}{(n-1)!}}{(\frac{[\omega]^n}{n!})^{\frac{n-1}{n}}}. \tag{2.7}$$

As the expression $\frac{4\pi c_1(\omega)\cdot\frac{[\omega]^{n-1}}{(n-1)!}}{(\frac{[\omega]^n}{n!})^{\frac{n-1}{n}}}$ is invariant under a change $\omega \mapsto \phi^*(\omega)$ by any diffeomorphism $\phi$, so from (2.6) and the definition of $Z$, we get

$$Z(N,[[\omega_W+\cdots+\omega_W]]) \leq -4\pi\frac{((2k)!)^{\frac{2k-1}{2k}}}{(2k-1)!}C^{\frac{1}{2k}} = -4\sqrt{2}\pi k.$$

We consider the Kähler form $\omega_W+\cdots+\omega_W$ of the product Kähler Einstein metric $g_W+\cdots+g_W$ of negative scalar curvature on $N = W\times\cdots\times W$. One can readily check that this symplectic form satisfies the equality of both (2.6) and (2.7). So, we conclude $Z(N,[[\omega_W+\cdots+\omega_W]]) = -4\sqrt{2}\pi k$. □

*Proof of Theorem 2.1.* Consider the positive Kähler Einstein metric on $R_8$ and let $\omega_1$ be the Kähler form of the product positive Kähler Einstein metric on $R_8\times\cdots\times R_8$, which is diffeomorphic to $N$. We have $Z(N,[[\omega_1]]) = \infty$ (scaling by different constants on each factor gives $\infty$). And let $\omega_2$ be $\omega_W+\cdots+\omega_W$. Then $Z(N,[[\omega_2]]) < 0$ from Proposition 1. From the fact that these values are different, we conclude that $[[\omega_1]]$ and $[[\omega_2]]$ are distinct symplectic deformation equivalence classes. This proves Theorem 2.1. □

In this article I demonstrated examples in $4k$ dimension. But by refining the argument of [4], one may try to get, for each $k\geq 1$, examples of closed symplectic $(4k+2)$-dimensional manifolds admitting two symplectic deformation equivalence classes with distinct signs of $Z(\ \ ,[[\cdot]])$ invariants.

So far we only used the Catanese-LeBrun manifold as building blocks. But one may use other 4-dimensional closed simply connected symplectic manifolds of smaller Euler characteristic.

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