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ORTHOGONALITY AND LINEAR MAPPINGS IN BANACH MODULES

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of linear mappings in Banach modules over a unital C^*-algebra and in non-Archimedean Banach modules over a unital C^*-algebra associated with the orthogonally Cauchy-Jensen additive functional equation.

1. Introduction and Preliminaries

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x + y) = f(x) + f(y), \langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

G. Pinsker [40] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundaresan [50] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \qquad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [18]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, J. Rätz [47] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover,

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he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [48] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of J. Rätz; cf. [47].

Suppose X is a real vector space (algebraic module) with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:
(O_1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X - \{0\}, x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O_4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space (module). By an orthogonality normed space (normed module) we mean an orthogonality space (module) having a normed (normed module) structure.

Assume that if A is a C^*-algebra and X is a module over A and if $x, y \in X, x \perp y$, then $ax \perp by$ for all $a, b \in A$.

Some interesting examples are
(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.
(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.
(iii) The Birkhoff-James orthogonality on a normed space $(X, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [1]–[3], [7, 14, 23, 24, 36]).

The stability problem of functional equations was originated from the following question of Ulam [52]: *Under what condition does there is an additive mapping near an approximately additive mapping?* In 1941, Hyers [20] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [42] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x+y)-f(x)-f(y)\| \le \varepsilon(\|x\|^p+\|y\|^p)$, $(\varepsilon>0, p\in[0,1))$. The result of Rassias has provided a lot of influence in the development of what we now call *generalized Hyers-Ulam stability* or *Hyers-Ulam stability* of functional equations. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. The reader is referred to [11, 21, 25, 46] and references therein for detailed information on stability of functional equations.

R. Ger and J. Sikorska [17] investigated the orthogonal stability of the Cauchy functional equation $f(x+y)=f(x)+f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x+y)-f(x)-f(y)\|\le\varepsilon$ for all $x,y\in X$ with $x\perp y$ and some $\varepsilon>0$, then there exists exactly one orthogonally additive mapping $g: X\to Y$ such that $\|f(x)-g(x)\|\le\frac{16}{3}\varepsilon$ for all $x\in X$.

The first author treating the stability of the quadratic equation was F. Skof [49] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\le\varepsilon$ for some $\varepsilon>0$, then there is a unique quadratic mapping $g: X\to Y$ such that $\|f(x)-g(x)\|\le\frac{\varepsilon}{2}$. P.W. Cholewa [8] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by S. Czerwik [9] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [10, 39], [43]–[45]).

The orthogonally quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y),\ x\perp y$$

was first investigated by F. Vajzović [53] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, H. Drljević [15], M. Fochi [16], M.S. Moslehian [31, 32] and Gy. Szabó [51] generalized this result.

In 1897, Hensel [19] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [12, 27, 28, 35]).

Definition 1.1. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|:\mathbb{K}\to[0,\infty)$ such that for all $r,s\in\mathbb{K}$, the following conditions hold:

(1) $|r|=0$ if and only if $r=0$;

(2) $|rs|=|r||s|$;

(3) $|r+s| \leq \max\{|r|, |s|\}$.

Definition 1.2 ([34])**.** Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot|| : X \to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(1) $||x|| = 0$ if and only if $x = 0$;

(2) $||rx|| = |r|||x||$ ($r \in \mathbb{K}$, $x \in X$);

(3) The strong triangle inequality (ultrametric); namely,

$$||x+y|| \leq \max\{||x||, ||y||\}, \qquad x, y \in X.$$

Then $(X, ||.||)$ is called a non-Archimedean space.

Assume that if A is a C^*-algebra and X is a module over A, which is a non-Archimedean space, and if $x, y \in X, x \perp y$, then $ax \perp by$ for all $a, b \in A$. Then $(X, ||.||)$ is called an *orthogonality non-Archimedean module.*

Due to the fact that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\} \qquad (n > m).$$

Definition 1.3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.4 ([4, 13])**.** *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [22] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 30, 37, 38, 41]).

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the orthogonally Cauchy-Jensen additive functional equation in Banach modules over a unital C^*-algebra. In Section 3, we prove the Hyers-Ulam stability of the orthogonally Cauchy-Jensen additive functional equation in non-Archimedean Banach modules over a unital C^*-algebra.

2. Stability of the Orthogonally Cauchy-Jensen Additive Functional Equation in Banach Modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with unit e and unitary group $U(A) := \{u \in A \mid u^*u = uu^* = e\}$, $(X, \perp)$ is an orthogonality normed module over A and $(Y, \|.\|_Y)$ is a Banach module over A.

In this section, applying some ideas from [17, 21], we deal with the stability problem for the orthogonally Cauchy-Jensen additive functional equation

$$2f\left(\frac{x+y}{2}+z\right) = f(x) + f(y) + 2f(z)$$

for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$.

Theorem 2.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y,z) \le 2\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{2.1}$$

for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\|2uf\left(\frac{x+y}{2}+z\right) - f(ux) - f(uy) - 2f(uz)\right\|_Y \le \varphi(x,y,z) \tag{2.2}$$

for all $u \in U(A)$ and all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally Cauchy-Jensen additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{\alpha}{1-\alpha}\varphi(x,0,0) \tag{2.3}$$

for all $x \in X$.

Proof. Putting $y = z = 0$ and $u = e$ in (2.2), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_Y \leq \varphi(x,0,0) \tag{2.4}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \leq \frac{1}{2}\varphi(2x,0,0) \leq \alpha \cdot \varphi(x,0,0) \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq \mu\varphi\left(x,0,0\right),\ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [29]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g\left(2x\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\|_Y \leq \varphi\left(x,0,0\right)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\|\frac{1}{2}g\left(2x\right) - \frac{1}{2}h\left(2x\right)\right\|_Y \leq \alpha\varphi\left(x,0,0\right)$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that $d(f, Jf) \leq \alpha$.

By Theorem 1.4, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$L\left(2x\right) = 2L(x) \tag{2.6}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set

$$M = \{g \in S : d(h,g) < \infty\}.$$

This implies that L is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - L(x)\|_Y \quad \leq \quad \mu\varphi(x, 0, 0)$$

for all $x \in X$;

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = L(x)$$

for all $x \in X$;

(3) $d(f, L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, L) \leq \frac{\alpha}{1-\alpha}.$$

This implies that the inequalities (2.3) holds.

Let $u = e$ in (2.2). It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\left\|2L\left(\frac{x+y}{2} + z\right) - L(x) - L(y) + 2L(z)\right\|_Y \\
&= \lim_{n\to\infty} \frac{1}{2^n} \|2f(2^{n-1}(x+y) + 2^n z) - f(2^n x) - f(2^n y) - 2f(2^n z)\|_Y \\
&\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) \leq \lim_{n\to\infty} \frac{2^n \alpha^n}{2^n} \varphi(x, y, z) = 0
\end{aligned}$$

for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. So

$$2L\left(\frac{x+y}{2} + z\right) - L(x) - L(y) - 2L(z) = 0$$

for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. Hence $L : X \to Y$ is an orthogonally Cauchy-Jensen additive mapping.

Let $y = z = 0$ in (2.2). It follows from (2.1) and (2.2) that

$$\begin{aligned}
\left\|2uL\left(\frac{x}{2}\right) - L(ux)\right\|_Y &= \lim_{n\to\infty} \frac{1}{2^n} \|2uf(2^{n-1}x) - f(2^n ux)\|_Y \\
&\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 0, 0) \leq \lim_{n\to\infty} \frac{2^n \alpha^n}{2^n} \varphi(x, 0, 0) = 0
\end{aligned}$$

for all $x \in X$. So

$$2uL\left(\frac{x}{2}\right) - L(ux) = 0$$

for all $x \in X$. Hence

$$L(ux) = 2uL\left(\frac{x}{2}\right) = uL(x) \tag{2.7}$$

for all $u \in U(A)$ and all $x \in X$.

By the same reasoning as in the proof of [42, Theorem], we can show that $L : X \to Y$ is $\mathbb{R}$-linear, since the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x \in X$ and $L : X \to Y$ is additive.

Since L is $\mathbb{R}$-linear and each $a \in A$ is a finite linear combination of unitary elements (see [26, Theorem 4.1.7]), i.e., $a = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$), it follows from (2.7) that

$$\begin{aligned} L(ax) &= L(\sum_{j=1}^{m} \lambda_j u_j x) = L\left(\sum_{j=1}^{m} |\lambda_j| \cdot \frac{\lambda_j}{|\lambda_j|} u_j x\right) = \sum_{j=1}^{m} |\lambda_j| L\left(\frac{\lambda_j}{|\lambda_j|} u_j x\right) \\ &= \sum_{j=1}^{m} |\lambda_j| \cdot \frac{\lambda_j}{|\lambda_j|} u_j L(x) = \sum_{j=1}^{m} \lambda_j u_j L(x) = aL(x) \end{aligned}$$

for all $x \in X$. It is obvious that $\frac{\lambda_j}{|\lambda_j|} u_j \in U(A)$. Thus $L : X \to Y$ is a unique orthogonally Cauchy-Jensen additive and A-linear mapping satisfying (2.3). □

Corollary 2.2. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\left\| 2uf\left(\frac{x+y}{2} + z\right) - f(ux) - f(uy) - 2f(uz) \right\|_Y \le \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.8}$$

for all $u \in U(A)$ and all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally Cauchy-Jensen additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{2^p \theta}{2 - 2^p} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. Then we can choose $\alpha = 2^{p-1}$ and we get the desired result. □

Theorem 2.3. *Let $f : X \to Y$ be a mapping satisfying (2.2) and $f(0) = 0$ for which there exists a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\varphi(x, y, z) \le \frac{\alpha}{2} \varphi(2x, 2y, 2z)$$

for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally Cauchy-Jensen additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{1}{1-\alpha}\varphi(x, 0, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.4) that $d(f, Jf) \leq 1$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying* (2.8). *If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally Cauchy-Jensen additive and A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{2^p\theta}{2^p - 2}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. Then we can choose $\alpha = 2^{1-p}$ and we get the desired result. □

3. Stability of the Orthogonally Cauchy-Jensen Additive Functional Equation in Non-Archimedean Banach Modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with unit e and unitary group $U(A) := \{u \in A \mid u^*u = uu^* = e\}$, $(X, \perp)$ is an orthogonality non-Archimedean normed module over A and $(Y, \|.\|_Y)$ is a non-Archimedean Banach module over A. Assume that $|2| \neq 1$.

In this section, applying some ideas from [17, 21], we deal with the stability problem for the orthogonally Cauchy-Jensen additive functional equation.

Theorem 3.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y, z) \le |2|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{3.1}$$

for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally Cauchy-Jensen additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{\alpha}{1-\alpha}\varphi(x, 0, 0) \tag{3.2}$$

for all $x \in X$.

Proof. It follows from (2.4) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \le \frac{1}{|2|}\varphi(2x, 0, 0) \le \alpha \cdot \varphi(x, 0, 0) \tag{3.3}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (3.3) that $d(f, Jf) \le \alpha$.

By Theorem 1.4, there exists a mapping $L : X \to Y$ satisfying the following:

(1) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = L(x)$$

for all $x \in X$;

(2) $d(f, L) \le \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, L) \le \frac{\alpha}{1-\alpha}.$$

This implies that the inequality (3.2) holds.

It follows from (3.1) and (2.2) that

$$\begin{aligned}&\left\|2uL\left(\frac{x+y}{2}+z\right)-L(ux)-L(uy)-2L(uz)\right\|_Y\\&=\lim_{n\to\infty}\frac{1}{|2|^n}\|2uf(2^{n-1}(x+y)+2^nz)-f(2^nux)-f(2^nuy)-2f(2^nuz)\|_Y\\&\le\lim_{n\to\infty}\frac{1}{|2|^n}\varphi(2^nx,2^ny,2^nz)\le\lim_{n\to\infty}\frac{|2|^n\alpha^n}{|2|^n}\varphi(x,y,z)=0\end{aligned}$$

for all $u\in U(A)$ and all $x,y,z\in X$ with $x\perp y$, $x\perp z$ and $y\perp z$. So

$$2uL\left(\frac{x+y}{2}+z\right)-L(ux)-L(uy)-2L(uz)=0$$

for all $u\in U(A)$ and all $x,y,z\in X$ with $x\perp y$, $x\perp z$ and $y\perp z$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let θ be a positive real number and p a real number with $p>1$. Let $f:X\to Y$ be a mapping satisfying (2.8). If for each $x\in X$ the mapping $f(tx)$ is continuous in $t\in\mathbb{R}$, then there exists a unique orthogonally Cauchy-Jensen additive and A-linear mapping $L:X\to Y$ such that*

$$\|f(x)-L(x)\|_Y\le\frac{|2|^p\theta}{|2|-|2|^p}\|x\|^p$$

for all $x\in X$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x,y,z)=\theta(\|x\|^p+\|y\|^p+\|z\|^p)$ for all $x,y,z\in X$ with $x\perp y$, $x\perp z$ and $y\perp z$. Then we can choose $\alpha=|2|^{p-1}$ and we get the desired result. □

Theorem 3.3. *Let $f:X\to Y$ be a mapping satisfying (2.2) and $f(0)=0$ for which there exists a function $\varphi:X^3\to[0,\infty)$ such that*

$$\varphi(x,y,z)\le\frac{\alpha}{|2|}\varphi\left(2x,2y,2z\right)$$

for all $x,y,z\in X$ with $x\perp y$, $x\perp z$ and $y\perp z$. If for each $x\in X$ the mapping $f(tx)$ is continuous in $t\in\mathbb{R}$, then there exists a unique orthogonally Cauchy-Jensen additive and A-linear mapping $L:X\to Y$ such that

$$\|f(x)-L(x)\|_Y\le\frac{1}{1-\alpha}\varphi\left(x,0\right)\qquad(3.4)$$

for all $x\in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.4) that $d(f, Jf) \leq 1$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

Corollary 3.4. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying (2.8). If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally Cauchy-Jensen additive and A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{|2|^p\theta}{|2|^p - |2|}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by by taking $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in X$ with $x \perp y$, $x \perp z$ and $y \perp z$. Then we can choose $\alpha = |2|^{1-p}$ and we get the desired result. □

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