THE SET OF PRIORS IN THE REPRESENTATION OF CHOQUET EXPECTATION WHEN A CAPACITY IS SUBMODULAR

Ju Hong Kim

ABSTRACT. We show that the set of priors in the representation of Choquet expectation is the one of equivalent martingale measures under some conditions, when given capacity is submodular. It is proven via Peng's g-expectation and related topics.

1. INTRODUCTION

A starting point for a mathematical definition of risk is simply as standard deviation. The more risk we take, the more we stand to lose or gain. Standard deviation (or volatility) is a kind of simple risk measure. Different families of risk measures have been proposed in literature like coherent, convex, spectral risk measures, conditional value-at-risk etc. and discussed to measure or quantify the market risks in theoretical and practical perspectives. Risk measures are also linked to insurance premiums.

Markowitz [18] used the standard deviation to measure the market risk in his portfolio theory but his method doesn't tell the difference between the positive and the negative deviation. Artzer et al. [1, 2] proposed a coherent risk measure in an axiomatic approach, and formulated the representation theorems. Fritelli [11] proposed sublinear risk measure to weaken coherent axioms. Heath [14] firstly studied the convex risk measures and Föllmer & Schied [8, 9, 10] and Frittelli & Rosazza Gianin [12] extended them to general probability spaces. They had weakened the conditions of positive homogeneity and subadditivity by replacing them with convexity.

There exist stochastic phenomena like Allais paradox and Ellsberg paradox which can not be dealt with linear mathematical expectation in economics. So Choquet [4] introduced a nonlinear expectation called Choquet expectation which applied to many areas such as statistics, economics and finance. Choquet expectation is equivalent to the convex(or coherent) risk measure if given capacity is submodular. But Choquet expectation has a difficulty in defining a conditional expectation. Peng [21] introduced a nonlinear expectation, g-expectation which is a solution of a nonlinear backward stochastic differential equation. It's easy to define conditional expectation with Peng's g-expectation (see papers[5, 13, 15, 17, 20, 22] for related topics).

In this paper, we show that the set of priors in the representation of Choquet expectation is the one of equivalent martingale measures under some conditions, when the distortion is submodular. That is, if a capacity c is submodular, then we have the representation

$$\int X \, dc = \max_{Q \in \mathcal{Q}_c} E_Q[X] \quad \text{for } X \in L^2(\mathcal{F}_T),$$

where $\mathcal{Q}_c := \{Q \in \mathcal{M}_{1,f} : Q[A] \leq c(A) \, \forall A \in \mathcal{F}_T\}$. There is no specific explanation in the literature for the structure of the set $\mathcal{Q}_c$. It is worthy of examining it. By using g-expectation and related topics, we'll show that

$$(1.1) \quad \mathcal{Q}_c = \left\{ Q^\theta : \theta \in \Theta^g, \, \frac{dQ^\theta}{dP}\Big|_{\mathcal{F}_t} = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right) \right\}$$

for some density generator set Θ^g.

This paper consists of as follows. Introduction is given in section 1. Definitions of Choquet expectation(or integral) and risk measures are stated in section 2. Definition of Peng's g-expectation and related topics are given in section 3. The set of priors in the representation of Choquet expectation is discussed and the main Theorem 4.4 is given in section 4.

2. DEFINITIONS OF CHOQUET EXPECTATION(OR INTEGRAL) AND RISK MEASURES

In this section, we give definitions of Choquet expectation(or integral) and coherent(or convex) risk measures. Let $(\Omega, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be the given filtered probability space.

Definition 2.1. A set function $c : \mathcal{F} \to [0,1]$ is called *monotone* if

$$c(A) \leq c(B) \quad \text{for } A \subset B, A, B \in \mathcal{F}$$

and normalized if

$$c(\emptyset) = 0 \ \text{ and } \ c(\Omega) = 1.$$

The monotone and normalized set function is called a *capacity*. A monotone set function is called *submodular* or 2-*alternating* if

$$c(A \cup B) + c(A \cap B) \le c(A) + c(B).$$

Two real functions X and Y defined on Ω are called *comonotonic* if

$$[X(\omega_1) - X(\omega_2)][Y(\omega_1) - Y(\omega_2)] \ge 0, \quad \omega_1, \omega_2 \in \Omega.$$

A class of function $\mathcal{X}$ is said to be comonotonic if for every pair $(X, Y) \in \mathcal{X} \times \mathcal{X}$, X and Y are comonotonic.

Definition 2.2. Let $\psi : [0, 1] \to [0, 1]$ be increasing function with $\psi(0) = 0$ and $\psi(1) = 1$. The set function

$$c_\psi(A) := \psi(P(A)), \quad A \in \mathcal{F}$$

is called *distortion* of P with respect to the distortion function ψ.

The c_ψ defined in Definition 2.2 becomes normalized monotone function. The notion of integral with respect to a capacity is due to Choquet [4].

Definition 2.3. Let $c : \mathcal{F} \to [0, 1]$ be monotone and normalized set function. The *Choquet integral* or *concave distortion risk measure* of $X \in L^2(\mathcal{F}_T)$ with respect to c is defined as

$$\int_\Omega X \, dc := \int_{-\infty}^0 (c(X > x) - 1) \, dx + \int_0^\infty c(X > x) \, dx.$$

The following is the definition of coherent risk measure of which concept is borrowed from one of norm.

Definition 2.4. A *coherent risk measure* $\rho : \mathcal{X} \to \mathbb{R}$ is a mapping satisfying for $X, Y \in \mathcal{X}$

(1) $\rho(X) \ge \rho(Y)$ if $X \le Y$ (monotonicity),
(2) $\rho(X + m) = \rho(X) - m$ for $m \in \mathbb{R}$ (translation invariance),
(3) $\rho(X + Y) \le \rho(X) + \rho(Y)$ (subadditivity),
(4) $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \ge 0$ (positive homogeneity).

The subadditivity and the positive homogeneity can be relaxed to a weaker quantity, i.e. convexity

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y) \quad \forall \lambda \in [0, 1],$$

which means *diversification* should not increase the risk.

A *convex risk measure* $\rho :\to \mathbb{R}$ is a functional satisfying monotonicity, translation invariance and convexity.

Definition 2.5. Choquet integral of the loss is defined as

$$\rho(X) := \int (-X)dc,$$

where c is a capacity.

Choquet integral of the loss $\rho : \mathcal{X} \to \mathbb{R}$ satisfies monotonicity, cash invariance, positive homogeneity and the others.

(1) $\int \lambda dc = \lambda$ for constant λ (constant preserving).
(2) If $X \leq Y$, then $\int(-X)dc \geq \int(-Y)dc$ (monotonicity).
(3) For $\lambda \geq 0$, $\int \lambda(-X)dc = \lambda \int(-X)dc$ (positive homogeneity).
(4) If X and Y are comonotone functions, then

$$\int [(-X) + (-Y)]dc = \int (-X)dc + \int (-Y)dc \text{ (comonotone additivity).}$$

(5) If c is submodular or concave function, then

$$\int (X + Y)\, dc \leq \int X\, dc + \int Y\, dc \text{ (subadditivity).}$$

3. Peng's g-expectation

In this section, the definition of g-expectation is given. Let $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function that $g \mapsto g(t, y, z)$ is measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^n$ and satisfy the following conditions

$$(3.1a) \qquad |g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq K(|y - \bar{y}| + |z - \bar{z}|)$$

$$\forall t \in [0, T], \forall (y, z), (\bar{y}, \bar{z}) \in \mathbb{R} \times \mathbb{R}^n, \text{ for some } K > 0,$$

$$(3.1b) \qquad \int_0^T |g(t, 0, 0)|^2\, dt < \infty,$$

$$(3.1c) \qquad \text{For each } (t, y) \in [0, T] \times \mathbb{R}, \quad g(t, y, 0) = 0.$$

Theorem 3.1 ([21]). *For every terminal condition $\xi \in L^2(\mathcal{F}_T) := L^2(\Omega, \mathcal{F}_T, P)$ the following backward stochastic differential equation*

(3.2a) $$-dy_t = g(t, y_t, z_t)\, dt - z_t dB_t, \quad 0 \le t \le T,$$

(3.2b) $$y_T = \xi$$

has a unique solution

$$(y_t, z_t)_{t \in [0,T]} \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}) \times L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n).$$

Definition 3.2. For each $\xi \in L^2(\mathcal{F}_T)$ and for each $t \in [0,T]$ $g-$expectation of X and the conditional $g-$expectation of X under $\mathcal{F}_t$ is respectively defined by

$$\mathcal{E}_g[\xi] := y_0, \quad \mathcal{E}_g[\xi | \mathcal{F}_t] := y_t,$$

where y_t is the solution of the BSDE (3.2).

3.1. Two sets of probability measures, $\mathcal{S}_1^g$ and $\mathcal{S}_2^g$ Let g be independent of y and $g(t, y, 0) = 0$. We define two sets of probability measures on the measurable space $(\Omega, \mathcal{F}_T)$,

$$\mathcal{S}_1^g := \{ Q : E_Q[\xi] \le \mathcal{E}_g[\xi] \quad \forall \xi \in L^2(\Omega, \mathcal{F}_T, P) \},$$

$$\mathcal{S}_2^g := \left\{ Q^\theta : \theta \in \Theta^g, \left. \frac{dQ^\theta}{dP} \right|_{\mathcal{F}_t} = exp\left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right) \right\}$$

where $t \in [0,T]$ and Θ^g is defined as

$$\Theta^g = \{ (\theta_t)_{t \in [0,T]} : \quad \theta \text{ is } \mathbb{R}^d - \text{valued, progressively measurable,}$$

$$\theta_t \cdot z \le g(t, z) \ \forall z \in \mathbb{R}^d, \ dP \times dt - a.s. \}$$

Let's see properties of set of priors, $\mathcal{Q}_c$ as in (1.1). Set

$$z_t^\theta = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right), 0 \le t \le T.$$

Then $(z_t^\theta)_{0 \le t \le T}$ is a P-martingale since $dz_t^\theta = z_t^\theta \theta_t \cdot dB_t$. Also z_T^θ is a P-density on $\mathcal{F}_T$ since $1 = z_0^\theta = E[z_T^\theta]$. A probability measure Q^θ on $(\Omega, \mathcal{F})$ is equivalent to P, where Q^θ is defined as

$$Q^\theta(A) = E[1_A z_T^\theta], \quad A \in \mathcal{F}_T.$$

We can easily see that $\mathcal{Q}_c$ is convex and weakly compact in $L^1(\Omega, \mathcal{F}, P)$. For every deterministic $\tau \in [0,T]$ and every $B \in \mathcal{F}_\tau$,

$$\mathcal{Q}_c = \left\{ Q(\cdot) = \int \left[Q^1(\cdot | \mathcal{F}_\tau) 1_B + Q^2(\cdot | \mathcal{F}_\tau) 1_{B^c} \right] dQ_\tau^3 \Big| \{Q^i\}_{i=1}^3 \subset \mathcal{Q}_c \right\}$$

where Q_τ^3 denotes the restriction of Q^3 to $\mathcal{F}_\tau$(See the paper [23] for details).

If $\theta \in \Theta^g$, i.e. $\theta_t \cdot z \leq g(t,z)$, then we have $\theta_t \cdot z \leq |g(t,z)| \leq K|z|$ and so $|\theta_t| \leq K$ by taking $z = \theta_t$. The Girsanov transformation implies that there exists a probability measure Q^θ on the space $(\Omega, \mathcal{F}_t)$ such that

$$\frac{dQ^\theta}{dP}\Big|_{\mathcal{F}_t} = \exp\left(\int_0^t \theta_s \, dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right), \quad 0 \leq t \leq T,$$

and

$$B_t^\theta := B_t - \int_0^t \theta_s ds, \quad t \in [0,T] \text{ is a } Q^\theta\text{-Brownian motion.}$$

The two prior sets, $\mathcal{S}_1^g$ and $\mathcal{S}_2^g$ are the same set under some conditions.

Theorem 3.3 ([16]). *Let g be independent of y and satisfy the conditions (3.1a) and (3.1c). Then*

$$\mathcal{S}_1^g = \mathcal{S}_2^g.$$

Definition 3.4. Let g be independent of y and satisfy the conditions (3.1a) and (3.1c). The generator g is said to be *sublinear* with respect to z if for $a \geq 0$, $z_1, z_2 \in \mathbb{R}^d$

$$g(t, az_1) = ag(t, z_1) \quad dP \times dt - a.s.,$$
$$g(t, z_1 + z_2) \leq g(t, z_1) + g(t, z_2) \quad dP \times dt - a.s..$$

Theorem 3.5 ([16]). *Let g be independent of y and satisfy the conditions (3.1a) and (3.1c). Then*

$$\mathcal{E}_g[\xi] = \sup_{Q \in \mathcal{S}_1^g} E_Q[\xi] \quad \forall \xi \in L^2(\Omega, \mathcal{F}_t, P) \text{ if and only if } g \text{ is sublinear with respect to } z.$$

4. THE SET OF PRIORS IN THE REPRESENTATION OF CHOQUET EXPECTATION

The following theorem is about the equivalent properties on the Choquet integral with respect to a capacity c.

Theorem 4.1 ([7]). *For the Choquet integral with respect to a capacity c, the followings are equivalent.*

(1) $\rho(X) := \int(-X) \, dc$ is a convex risk measure on $L^2(\mathcal{F}_T)$.
(2) $\rho(X) := \int(-X) \, dc$ is a coherent risk measure on $L^2(\mathcal{F}_T)$.

(3) For $\mathcal{Q}_c := \{Q \in \mathcal{M}_{1,f} \ : \ Q[A] \leq c(A) \ \forall A \in \mathcal{F}_T\}$,

$$(4.1) \qquad \int X \, dc = \max_{Q \in \mathcal{Q}_c} E_Q[X] \quad \text{for } X \in L^2(\mathcal{F}_T),$$

where $M_{1,f} := M_{1,f}(\Omega, \mathcal{F})$ is the set of all finitely additive normalized set functions $Q : \mathcal{F} \to [0,1]$.

(4) The set function c is submodular. In this case, $\mathcal{Q}_c = \mathcal{Q}_{max}$, where $\mathcal{Q}_{max} := \{Q << P \ : \ \alpha_{min}(Q) = 0\}$ is in the representation of the convex risk measure

$$\rho(X) = \sup_{Q<<P} \left(E_Q[-X] - \alpha_{min}(Q) \right), X \in L^2(\mathcal{F}_T).$$

Peng's g-expectation provides various features. We will use the properties of g-expectation to investigate the set of prior, $\mathcal{Q}_c$. The classical mathematical expectation can be represented by the Choquet expectation if g is linear function of z. The following theorem deals with the one-dimensional Brownian motion case, and $y, z \in \mathbb{R}$.

Theorem 4.2 ([3]). *Suppose that g satisfies the conditions (3.1a), (3.1b) and (3.1c). Then there exists a Choquet expectation whose restriction to $L^2(\Omega, \mathcal{F}, P)$ is equal to a g-expectation if and only if g is independent of y and is linear in z, i.e. there exists a continuous function ν_t such that*

$$g(y, z, t) = \nu_t z.$$

Set $g(y, z, t) = \nu_t z$ through the last of this paper. Then Choquet expectation is equal to g-expectation by Theorem 4.2. I.e., there exist a capacity c_g such that

$$(4.2) \qquad \mathcal{E}_g[\xi] = \int_\Omega \xi \, dc_g \quad \forall \xi \in L^2(\Omega, \mathcal{F}, P).$$

If we take $\xi = I_A$ for $A \in \mathcal{F}$ in (4.2), then the capacity c_g satisfies

$$\mathcal{E}_g(I_A) = \int I_A \, dc_g = c_g(A) \text{ for } A \in \mathcal{F}.$$

Then we can prove that c_g is submodular.

Theorem 4.3. *The capacity c_g in (4.2) is submodular.*

Proof. By Dellacherie's theorem in Dellacherie [6], Choquet expectation on $L^2(\Omega, \mathcal{F}, P)$ is comonotonic additive. That is, if $\mathcal{E}_g$ is Choquet expectation, then we have

$$\mathcal{E}_g[\xi + \eta] = \mathcal{E}_g[\xi] + \mathcal{E}_g[\eta] \quad \text{whenever } \xi \text{ and } \eta \text{ are comonotonic.}$$

Note that $I_{A \cup B}$ and $I_{A \cap B}$ is a pair of comonotone functions for all $A, B \in \mathcal{F}$. Hence comonotonicity and subadditivity of $\mathcal{E}_g$ imply

$$
\begin{aligned}
c_g(A \cap B) + c_g(A \cup B) &= \mathcal{E}_g[I_{A \cap B}] + \mathcal{E}_g[I_{A \cup B}] = \mathcal{E}_g[I_{A \cap B} + I_{A \cup B}] \\
&= \mathcal{E}_g[I_A + I_B] \\
&\leq \mathcal{E}_g[I_A] + \mathcal{E}_g[I_B] = c_g(A) + c_g(B).
\end{aligned}
$$

So the proof is done. $\square$

Since c_g is submodular, by Theorem 4.1 we have the representation

$$
(4.3) \qquad \int X \, dc_g = \max_{Q \in \mathcal{Q}_{c_g}} E_Q[X] \quad \text{for } X \in L^2(\mathcal{F}_T),
$$

where $\mathcal{Q}_{c_g} := \{Q \in \mathcal{M}_{1,f} : Q[A] \leq c_g(A) \ \forall A \in \mathcal{F}_T\}$.

The following is the main theorem.

Theorem 4.4. $\mathcal{Q}_{c_g} = \mathcal{S}_2^g$ where $\mathcal{Q}_{c_g}$ is the prior set in the representation (4.3).

Notice that $\mathcal{S}_2^g$ is defined as

$$
\mathcal{S}_2^g := \left\{ Q^\theta : \theta \in \Theta^g, \left. \frac{dQ^\theta}{dP} \right|_{\mathcal{F}_t} = exp \left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right) \right\}
$$

where $t \in [0, T]$ and Θ^g is defined as

$$
\begin{aligned}
\Theta^g = \{(\theta_t)_{t \in [0,T]} : \quad & \theta \text{ is } \mathbb{R} - \text{valued, progressively measurable \&} \\
& |\theta_t| \leq \nu_t\}.
\end{aligned}
$$

Proof. Since $\mathcal{S}_1^g$ has the same expression as

$$
\mathcal{S}_1^g := \{Q : Q(A) \leq \mathcal{E}_g[I_A] = c_g(A) \quad \forall A \in \mathcal{F}_T\},
$$

$\mathcal{S}_1^g$ becomes $\mathcal{Q}_{c_g}$. Since $g(y, z, t) = \nu_t z$ is independent of y and satisfy the conditions (3.1a) and (3.1c), $\mathcal{S}_1^g = \mathcal{S}_2^g$ by Theorem 3.3. Therefore, we have $\mathcal{Q}_{c_g} = \mathcal{S}_2^g$. $\square$

In fact, for the linear function $g(t, y, z) = \nu_t z$, let us consider the BSDE

$$
(4.4) \qquad y_t = \xi + \int_t^T \nu_s z_s \, ds - \int_t^T z_s \, dB_s, \quad \xi \in L^2(\mathcal{F}_T).
$$

The above differential equation (4.4) is reduced to

$$
y_t = \xi - \int_t^T z_s d\tilde{B}_s, \quad \tilde{B}_t = B_t - \int_0^t \nu_s \, ds.
$$

By Girsanov's Theorem, $(\tilde{B}_t^\nu)_{0 \leq t \leq T}$ is a Q^ν-Brownian motion under Q^ν defined as

$$\frac{dQ^\nu}{dP} = \exp\left[-\frac{1}{2}\int_0^T \nu_s^2 ds + \int_0^T \nu_s dB_s\right].$$

Therefore we have the relations

$$\mathcal{E}_g[\xi] = E_{Q^\nu}[\xi], \quad \mathcal{E}_g[\xi|\mathcal{F}_t] = E_{Q^\nu}[\xi|\mathcal{F}_t] \quad \text{for } Q^\nu \in \mathcal{S}_2^g,$$

which means g-expectation is a classical mathematical expectation.

Acknowledgment

References

1. P. Artzner, F. Delbaen, J.-M. Eber & D. Heath: Thinking coherently. *Risk* **10** (1997), 68-71.

2. _____: Coherent measures of risk. *Mathematical Finance* **9** (1999), 203-223.

3. Z. Chen, T. Chen & M. Davison: Choquet expectation and Peng's g-expectation. *The Annals of Probability* **33** (2005), 1179-1199.

4. G. Choquet: Theory of capacities. *Ann. Inst. Fourier (Grenoble)* **5** (1953), 131-195.

5. F. Coquet, Y. Hu, J. Mémin & S. Peng: Filtration consistent nonlinear expectations and related g-expectations. *Probability Theory and Related Fields* **123** (2002), 1-27.

6. C. Dellacherie: *Quelques commentaires sur les prolongements de capacités. Séminaire de Probabilités V. Strasbourg. Lecture Notes in Math.* **191** Springer, 77-81, 1970.

7. H. Föllmer & A. Schied: *Stochastic Finance: An Introduction in Discrete Time.* Springer-Verlag, New York, 2002.

8. _____: Convex measures of risk and trading constraints. *Finance & Stochastics* **6** (2002), 429-447.

9. _____: Robust preferences and convex measures of risk, in: Advances in Finance and Stochastics, K. Sandmann and P.J. Schönbucher eds., Springer-Verlag.

10. _____: Stochastic Finance: An introduction in discrete time. Walter de Gruyter, Berlin, 2004.

11. M. Frittelli: Representing sublinear risk measures and pricing rules. http://www.mat.unimi.it/users/frittelli/pdf/Sublinear2000.pdf, 2000.

12. M. Frittelli & E. Rosazza Gianin: Putting order in risk measures. *Journal of Banking & Finance* **26** (2002), 1473-1486.

13. K. He, M. Hu & Z. Chen: The relationship between risk measures and Choquet expectations in the framework of g-expectations. *Statistics and Probability Letters* **79** (2009), 508-512.

14. D. Heath: Back to the future. Plenary lecture at the First World Congress of the Bachelier Society, Paris, June 2000.

15. L. Jiang: Convexity, translation invariance and subadditivity for g-expectation and related risk measures. *Annals of Applied Provability* **18** (2006), 245-258.

16. _____: A necessary and sufficient condition for probability measures dominated by g-expectation. *Statistics and Probability Letters* **79** (2009), 196-201.

17. N. El Karoui, S.G. Peng & M.-C. Quenez: Backward stochastic differential equations in finance. *Math. Finance* **7** (1997), 1-71.

18. H. Markowitz: Portfolio selection. *The Journal of Finance* **26** (1952), 1443-1471.

19. S. Peng: Backward SDE and related g-expectation, backward stochastic DEs. *Pitman* **364** (1997), 141-159.

20. E. Pardoux & S.G. Peng: Adapted solution of a backward stochastic differential equation. *Systems and Control Letters* **14** (1990), 55-61.

21. S. Peng: Backward SDE and related g-expectations, in: Backward stochastic differential equations, N. El Karoui and L. Mazliak eds., Pitman Res. Notes Math. Ser., **364** (1997), 141-159.

22. E. Rosazza Gianin: Some examples of risk measures via g-expectations. *Insurance: Mathematics and Economics* **39** (2006), 19-34.

23. Zengjing Chen & Larry Epstein: Ambiguity, risk and asset returns in continuous time. *Econometrica* **70** (2002), 1403-1443.

Department of Mathematics, Sungshin Women's University, Seoul 136-742, Republic of Korea
Email address: jhkkim@sungshin.ac.kr