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ON POSITIVE SEMIDEFINITE PRESERVING STEIN TRANSFORMATION

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Abstract. In the setting of semidefinite linear complementarity problems on $\mathbf{S}^n$, we focus on the Stein Transformation $S_A(X) := X - AXA^T$ for $A \in R^{n\times n}$ that is positive semidefinite preserving (i.e., $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$) and show that such transformation is strictly monotone if and only if it is nondegenerate. We also show that a positive semidefinite preserving S_A has the Ultra-GUS property if and only if $1 \notin \sigma(A)\sigma(A)$.

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1. Introduction

In this paper, we focus on the so-called *semidefinite linear complementarity problem* (SDLCP) introduced by Gowda and Song [4]: Let $\mathbf{S}^n$ denote the space of all real symmetric $n \times n$ matrices, and $\mathbf{S}^n_+$ be the set of symmetric positive semidefinite matrices in $\mathbf{S}^n$. With the inner product defined by $\langle Z, W\rangle := tr(Z\,W)$, $\forall Z, W \in \mathbf{S}^n$, the space $\mathbf{S}^n$ becomes a Hilbert space. Clearly, $\mathbf{S}^n_+$ is a closed convex cone in $\mathbf{S}^n$. Given a linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ and a matrix $Q \in \mathbf{S}^n$, the *semidefinite linear complementarity problem*, denoted by SDLCP(L, Q), is the problem of finding a matrix $X \in \mathbf{S}^n$ such that

$$X \in \mathbf{S}^n_+,\ Y := L(X) + Q \in \mathbf{S}^n_+, \quad \text{and} \quad \langle X, Y\rangle = 0. \tag{1}$$

Specializing L to the Stein transformation $S_A(X) := X - AXA^T$, various authors tried to characterize **GUS**-property in terms of the matrix $A \in R^{n\times n}$. The most recent result is by Balaji [1] when A is a 2×2 matrix. When we translate the statements of Theorem 6 of [1] to $S_A : S^2 \to S^2$, then S_A is **GUS** if and only if $I \pm A$ is positive semidefinite. However, Tao [14] showed that this is not true in general (see Example 4.1 of [14]). The results of this paper states that when S_A

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is positive semidefinite preserving, then S_A is Ultra-GUS if and only if $I \pm A$ is positive definite.

We list out needed definitions below.

(a) A matrix $M \in R^{n\times n}$ is called
   - *positive semidefinite* if $\langle Mx, x\rangle \geq 0$ for all $x \in R^n$. If M is symmetric positive semidefinite, we use the notation $M \succeq 0$. The notation $M \preceq 0$ means $-M \succeq 0$. Note that a nonsymmetric matrix M is positive semidefinite if and only if the symmetric matrix $M + M^T$ is positive semidefinite.
   - *positive definite* if $\langle Mx, x\rangle > 0$ for all nonzero $x \in R^n$. If M is symmetric positive definite, we use the notation $M \succ 0$.

Definition of various properties below are from [4], [13], [14], [2], [8], [7], [9], [6]. A linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ has the

(b) **P**-property if $[XL(X) = L(X)X \preceq 0] \quad \Rightarrow \quad X = 0$

(c) **Globally Uniquely Solvable (GUS)**-property if for all $Q \in \mathbf{S}^n$, SDLCP(L, Q) in (1) has a unique solution.

(d) A linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ is called *monotone* if $\langle L(X), X\rangle \geq 0 \quad \forall X \in \mathbf{S}^n$; *strictly monotone* if $\langle L(X), X\rangle > 0$ for all nonzero $X \in \mathbf{S}^n$.

(e) A linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ is called *copositve on* $\mathbf{S}^n_+$ if $\langle L(X), X\rangle \geq 0 \quad \forall X \succeq 0$; *strictly copositive on* $\mathbf{S}^n_+$ if $\langle L(X), X\rangle > 0$ for all nonzero $X \succeq 0$.

(f) A linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ is said to have the **Cone-Gus**-*property* if for all $Q \succeq 0$, SDLCP(L, Q) has a unique solution.

(g) $\mathbf{P_2'}$-*property* if $[X \succeq 0,\ XL(X)X \preceq 0] \Rightarrow X = 0$.

(h) $\mathbf{P_2}$-*property* if $[X, Y \succeq 0,\ (X - Y)L(X - Y)(X + Y) \preceq 0] \Rightarrow X = 0$.

(i) *nondegenrate* if $[XL(X) = L(X)X = 0] \quad \Rightarrow \quad X = 0$.

(j) **Z**-*property* if $[X, Y \succeq 0, \langle X, Y\rangle = 0] \quad \Rightarrow \langle X, L(X)\rangle \leq 0$.

(k) *Lyapunov-like* if both L and $-L$ have the **Z**-*property.*

(l) *positive semidefinite preserving* if $L(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$.

(m) Ultra-**T**-property if and only if $\widehat{L}$ and all its principal subtransformations have the **T** properties where
$\widehat{L}(X) := P^T L(PXP^T)P,\ P \in R^{n\times n}$ invertible $(X \in \mathbf{S}^n)$.

(n) Corresponding to any $\alpha \subseteq \{1, 2, \cdots, n\}$, we define a linear transformation $L_{\alpha\alpha} : S^{|\alpha|} \to S^{|\alpha|}$ by

$$L_{\alpha\alpha}(Z) = [L(X)]_{\alpha\alpha} \quad (Z \in S^{|\alpha|})$$

where, corresponding to $Z \in S^{|\alpha|}$, $X \in S^n$ is the unique matrix such that $X_{\alpha\alpha} = Z$ and $x_{ij} = 0$ for all $(i, j) \notin \alpha\times\alpha$. *We call* $L_{\alpha\alpha}$ *the principal subtransformation of* L *corresponding to* α.

Next, we list out some well known matrix theoretic properties that are needed in the paper [10].

(a) $X \succeq 0 \Rightarrow UXU^T \succeq 0$ for any orthogonal matrix U.

(b) $X \succeq 0,\, Y \succeq 0 \Rightarrow \langle X, Y\rangle \geq 0$.
(c) $X \succeq 0,\, Y \succeq 0, \langle X, Y\rangle = 0 \Rightarrow XY = YX = 0$.
(d) $X \in \mathbf{S}^n$, $\langle X, Y\rangle \geq 0\ \forall Y \succeq 0 \Rightarrow X \succeq 0$. This says that the cone $\mathbf{S}^n_+$ is self-dual.
(e) Given X and Y in $\mathbf{S}^n$ with $XY = YX$, there exist an orthogonal matrix U, diagonal matrices D and E such that $X = UDU^T$ and $Y = UEU^T$.

Finally, we state the known results (interpreting for the case of $L = S_A$) that are necessary for the paper. In the following and throughout the paper, $\sigma(A)$ denotes the spectrum of A, the set of all eigenvalues of an $n \times n$ matrix A; and $\rho(A)$ denotes the spectral radius of A, the maximum distance from the origin to an eigenvalue of A in the complex plane.

(a) Example 3 of [8]: For $A \in R^{n\times n}$, S_A has the **Z**-property .
(b) Theorem 11 of [3]: $\rho(A) < 1 \Leftrightarrow S_A \in \mathbf{Q} \Leftrightarrow S_A \in \mathbf{P}$
(c) Theorem 28 of [11]: S_A is nondegenerate if and only if $1 \notin \sigma(A)\sigma(A)$.
(d) Theorem 5 of [6]:$S_A \in \mathbf{P_2}$ if and only if S_A is Ultra-GUS.
(e) Theorem 3.3 of [14]: $S_A \in \mathbf{P_2'}$ if and only if S_A is Ultra Cone-Gus.
(f) Table on p56 of [11]: For S_A,
strictly monotone $\Rightarrow$ $\mathbf{P_2}$ $\Rightarrow$ **GUS** $\Rightarrow$ **P** $\Rightarrow$nondegenerate.
(g) Theorem 2.1 of [12]: S_A is (strictly) monotone if and only if for all orthogonal matrices U, $\nu_r(UAU^T \circ UAU^T)\,(<)\ \leq 1$ where $\nu_r(A) := \max\{|x^T Ax| : \|x\| = 1, x \in R^n\}$ and $\circ$ denotes the Hadamard product.

2. Characterization of Ultra-GUS property of a positive semidefinite preserving S_A

We start with a Lemma.

Lemma 2.1. *For $A \in R^{n\times n}$, suppose S_A is nondegenerate and copositive on $\mathbf{S}^n_+$. Then S_A is Cone-Gus.*

Proof. Let X be a solution to SDLCP$(S_A, -Q)$ where $Q \preceq 0$. It suffices to show $X = 0$ to prove S_A is Cone-Gus. Since
$X(S_A(X) - Q) = 0$, we have $XS_A(X) = XQ$, and $S_A(X)X = QX$. Since S_A is copositive on $\mathbf{S}^n_+$, $\langle X, S_A(X)\rangle \geq 0$, but $\langle X, Q\rangle \leq 0$, and hence $\langle X, Q\rangle = 0 = \langle X, -Q\rangle$. Since both $X, -Q \succeq 0$, $XQ = 0 = QX$. Then $X = 0$ follows from the nondegeneracy of S_A. □

Note that if S_A is positive semidefinite preserving, then S_A is Lyapunov-like. (This is because $S_A \in \mathbf{Z}$ and $\langle X, S_A(X)\rangle \geq 0$ for all $X \succeq 0$.) Then by Theorem 3.5 [13], S_A is Cone-Gus if and only if S_A is GUS. Since every positive semidefinite preserving S_A is copositive on $\mathbf{S}^n_+$, we get the following

Theorem 2.2. *If $1 \notin \sigma(A)\sigma(A)$ and $S_A(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$, then S_A is GUS.*

We now show that if S_A is nondegenerate and positive semidefinite preserving, then S_A is not only GUS, but also Ultra-GUS.

Theorem 2.3. *For $A \in R^{n\times n}$, suppose S_A is nondegenerate and positive semidefinite preserving. Then S_A is Ultra-GUS.*

Proof. First we show that $S_A \in \mathbf{P_2'}$. Assume the contrary and let $0 \neq X \succeq 0$ be such that $XS_A(X)X \preceq 0$. But S_A is positive semidefinite preserving, so $tr(XS_A(X)X) = 0$. Let $X = UDU^T$ where $D = \begin{bmatrix} D^+ & 0 \\ 0 & 0 \end{bmatrix}$ with $D^+ \succ 0$ diagonal and U orthogonal. Then

$$0 = tr(XS_A(X)X) = tr(U^T XS_A(X)XU) = tr(DU^T S_A(X)UD).$$

Let $U^T S_A(X)U = \begin{bmatrix} M & N \\ N^T & R \end{bmatrix} \succeq 0$. Note that $M \succeq 0$. Then the matrix product $DU^T S_A(X)UD = \begin{bmatrix} D^+MD^+ & 0 \\ 0 & 0 \end{bmatrix}$. Thus,
$0 = tr(XS_A(X)X) = tr(D^+MD^+) = tr(M(D^+)^2)$
with D^+ nonsingular, so $M = 0$, which implies $N = 0$. Therefore,
$U^T S_A(X)U = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}$. So D and $U^T S_A(X)U$ commute with the product 0 where both are in $\mathbf{S}^n_+$. Hence $XS_A(X) = 0 = S_A(X)X$. Then $X = 0$ by nondegeneracy of S_A.
As we noted earlier (right after Lemma 1), S_A is Lyapunov-like. So by Theorem 6.1 [14], $\mathbf{P_2'} = \mathbf{P_2}$, that is, Ultra Cone-Gus = Ultra GUS. This completes the proof. □

Now we characterize Ultra-GUS property of a positive semidefinite preserving S_A.

Theorem 2.4. *For $A \in R^{n\times n}$, let S_A be positive semidefinite preserving. Then the following are equivalent.*

(a) $1 \notin \sigma(A)\sigma(A)$.
(b) *S_A is Ultra-Gus.*
(c) *S_A is strictly monotone.*

Proof. The statement (a) ⇒ (b) is exactly Theorem 3.
Assume (b). Since $\mathbf{P_2} \Rightarrow$ nondegeneracy of S_A, we get (a).
Finally, (b) and (c) are equivalent because S_A is Lyapunov-like, and so by Theorem 6.1 of [14], $S_A \in \mathbf{P_2}$ if and only if S_A is strictly monotone. This completes the proof. □

Remark 2.1. In our previous paper [12], the strict monotonicity of S_A was characterized in terms of its real numerical radius (Theorem 2.1 of [12]). Hence if S_A is positive semidefinite preserving, then $1 \notin \sigma(A)\sigma(A)$ if and only if $\nu_r(UAU^T \circ UAU^T) < 1$ for all U orthogonal. We now show that under the assumption of positive semidefinite preservedness, both of these are equivalent to the (easier-to-check) statement, $I \pm A$ positive definite.

Theorem 2.5. *If S_A is positive semidefinite preserving, then the following are all true or all false:*

(a) *$I \pm A$ is positive definite.*
(b) $\rho(A) < 1$
(c) $1 \notin \sigma(A)\sigma(A)$
(d) *$\nu_r(UAU^T \circ UAU^T) < 1$ for all U orthogonal.*

Proof. Assume (a). Note that $I \pm U^TAU = U^T(I \pm A)U$ is also positive definite for all orthogonal matrices U, and hence the (k,k)-entry of U^TAU ($[U^TAU]_{kk}$) is less than 1 in absolute value. We will show first that S_A is strictly copositive on $\mathbf{S}^n_+$. Suppose there exists $0 \neq X \succeq 0$ with $\langle X, S_A(X)\rangle = 0$. Let $X = UDU^T = U(d_1E_{11} + \cdots + d_nE_{nn})U^T$, where $d_i \geq 0$ for all i and $d_k > 0$ for some k. The matrix E_{ii} is a diagonal matrix with all entries being 0 except the unit (i,i)-entry. Then

$$0 = \langle X, S_A(X)\rangle = \langle D, S_{U^TAU}(D)\rangle = \sum_{i,j} d_id_j\langle S_{U^TAU}(E_{ii}), E_{jj}\rangle.$$

Since S_A is positive semidefinite preserving, so is S_{U^TAU}, then $d_id_j\langle S_{U^TAU}(E_{ii}), E_{jj}\rangle \geq 0$ for each i and j. In particular, $\sum_{i,j} d_id_j\langle S_{U^TAU}(E_{ii}), E_{jj}\rangle \geq {d_k}^2\langle S_{U^TAU}(E_{kk}), E_{kk}\rangle$, but the last term is positive because $\langle S_{U^TAU}(E_{kk}), E_{kk}\rangle = 1 - ([U^TAU]_{kk})^2 > 0$. Then $\langle X, S_A(X)\rangle > 0$ which is a contradiction. Hence S_A is strictly copositive on $\mathbf{S}^n_+$. Then by Theorem 3.2 of [14], $S_A \in \mathbf{P_2'}$. Since $\mathbf{P_2'} = \mathbf{P_2}$ for this S_A (see the proof of Theorem 3) and $\mathbf{P_2} \Rightarrow \mathbf{P}$, we get (b).
Since $\mathbf{P} \Rightarrow$ nondegenerate, we have (b) $\Rightarrow$ (c).
The statement (c) $\Leftrightarrow$ (d) is done in Theorem 4.
Finally, assume (d). Then $\langle X, S_A(X)\rangle > 0$ for all $0 \neq X \in \mathbf{S}^n$. So, without loss of generality, $0 < \langle uu^T, S_A(uu^T)\rangle$ for all $0 \neq u \in R^n$ with $\|u\| = 1$. Then, $\langle uu^T, S_A(uu^T)\rangle = 1 - (\langle u, Au\rangle)^2 > 0$. So, $I \pm A$ is positive definite and the proof is complete. □

Remark 2.2. Theorem 6 offers a way of checking when S_A is not positive semidefinite perserving. For example,

$$A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix},$$

satisfies (b), but not (a) of Theorem 6, so S_A is not positive semidefinite preserving.

3. Conclusion

In an attempt to find a characterization of **GUS**-property of the Stein transformation, Balaji showed that for $S_A : S^2 \to S^2$, S_A is **GUS** if and only if $I \pm A$ is positive semidefinite (Theorem 6 [1]). Nevertheless, this does not generalize to $\mathbf{S}^n$ as Tao showed in his Example 4.1 [13]. In this paper, we showed $S_A : \mathbf{S}^n \to \mathbf{S}^n$ that is positive semidefinite preserving is Ultra-GUS if and only if

$I \pm A$ is positive definite. Still much to be done to characterize the **GUS**-property of a general Stein transformation and that is the author's future work.

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