LIPSCHITZ AND ASYMPTOTIC STABILITY FOR PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS

SANG IL CHOI AND YOON HOE GOO*

ABSTRACT. In this paper, we investigate Lipschitz and asymptotic stability for perturbed functional differential systems

1. Introduction

The notion of uniformly Lipschitz stability (ULS) was introduced by Dannan and Elaydi [9]. This notion of ULS lies somewhere between uniformly stability on one side and the notions of asymptotic stability in variation of Brauer [4] and uniformly stability in variation of Brauer and Strauss [3] on the other side. An important feature of ULS is that for linear systems, the notion of uniformly Lipschitz stability and that of uniformly stability are equivalent. However, for nonlinear systems, the two notions are quite distinct. Also, Elaydi and Farran [10] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of ULS. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Gonzalez and Pinto [11] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems.

In this paper, we investigate Lipschitz and asymptotic stability for solutions of the functional differential systems. To do this we need some integral inequalities. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations. In the presence the method of integral inequalities is as efficient as the direct Lyapunov's method.

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2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, consider the perturbed differential system of (1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds + h(t, y(t), Ty(t)), \, y(t_0) = y_0, \tag{2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator .

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (3).

Before giving further details, we give some of the main definitions that we need in the sequel [9].

Definition 2.1. The system (1) (the zero solution $x = 0$ of (1)) is called
(S)*stable* if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \epsilon$ for all $t \geq t_0 \geq 0$,
(US)*uniformly stable* if the δ in (S) is independent of the time t_0,
(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$
(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0) \leq M$ for $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,
(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \leq K \, |x_0| e^{-c(t-t_0)}, 0 \leq t_0 \leq t$$

provided that $|x_0| < \delta$,

(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq K\,e^{-c(t-t_0)}, 0 \leq t_0 \leq t$$

provided that $|x_0| < \infty$.

Remark 2.1 ([11]). The last definition implies that for $|x_0| \leq \delta$

$$|x(t)| \leq K\,|x_0|e^{-c(t-t_0)}, 0 \leq t_0 \leq t.$$

We give some related properties that we need in the sequel. We need Alekseev formula to compare between the solutions of (1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.1. *Let x and y be a solution of (1) and (5), respectively. If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\,g(s, y(s))\,ds.$$

Lemma 2.2 ([8]). *Let $u, \lambda_1, \lambda_2, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. If , for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_1(s)\Big\{ \int_{t_0}^{s} \lambda_2(\tau)w(u(\tau))d\tau \Big\}ds,\ \ t \geq t_0 \geq 0,$$

then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda_2(s)ds\Big] \exp\Big(\int_{t_0}^{t} \lambda_1(s)ds \Big),\ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}, u > 0,\ u_0 > 0$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup\Big\{ t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda_2(s)ds \in \mathrm{dom}W^{-1} \Big\}.$$

Lemma 2.3 ([15]). *Let $u, p, q, w, and\ r \in C(\mathbb{R}^+)$ and suppose that, for some $c \geq 0$, we have*

$$u(t) \leq c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} [q(\tau)u(\tau) + w(\tau) \int_{t_0}^{\tau} r(a)u(a)da]d\tau ds,\ t \geq t_0.$$

Then

$$u(t) \le c \exp\left(\int_{t_0}^t p(s) \int_{t_0}^s [q(\tau) + w(\tau) \int_{t_0}^\tau r(a)da]d\tau ds\right), \ t \ge t_0.$$

Lemma 2.4 ([13]). *Let* $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s)\left(\int_{t_0}^s \lambda_3(\tau)u(\tau)d\tau\right)ds, \ \ 0 \le t_0 \le t.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau))ds\right], \ t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.2, and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau))ds \in \text{dom}W^{-1}\right\}.$$

Lemma 2.5 ([13]). *Let* $u, p, q, w, r \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c \ge 0$,

$$u(t) \le c + \int_{t_0}^t (p(s) \int_{t_0}^s (q(\tau)w(u(\tau)) + v(\tau) \int_{t_0}^\tau r(a)u(a)da)d\tau)ds, \ t \ge t_0.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^t (p(s) \int_{t_0}^s (q(\tau) + v(\tau) \int_{t_0}^\tau r(a)da)d\tau)ds\right], \ t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.2, and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^t (p(s) \int_{t_0}^s (q(\tau) + v(\tau) \int_{t_0}^\tau r(a)da)d\tau)ds \in \text{dom}W^{-1}\right\}.$$

Lemma 2.6 ([6]). *Let* $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s)\left(\int_{t_0}^s \lambda_3(\tau)w(u(\tau))d\tau\right)ds, \ \ 0 \le t_0 \le t.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau))ds\right], \ t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.2, and*

$$b_1 = \sup\left\{t \ge t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau))ds \in \text{dom}W^{-1}\right\}.$$

Lemma 2.7 ([5]). *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, w \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c \geq 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)\Big[(\int_{t_0}^s [\lambda_2(\tau)w(u(\tau)) + \lambda_3(\tau)\int_{t_0}^\tau k(r)u(r)dr]d\tau) + \lambda_4(s)w(u(s))\Big]ds,$$

for $t \geq t_0 \geq 0$ and for some $c \geq 0$. Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^t \lambda_1(s)(\int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^\tau k(r)dr)d\tau) + \lambda_4(s))ds\Big],$$

for $t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 2.2, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^t \lambda_1(s)(\int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^\tau k(r)dr)d\tau)$$
$$+ \lambda_4(s))ds \in \text{dom} W^{-1}\Big\}.$$

3. Main results

In this section, we investigate Lipschitz and asymptotic stability for solutions of the perturbed functional differential systems.

Theorem 3.1. *For the perturbed (2), we asssume that*

$$\int_{t_0}^t |g(s, y(s))|ds \leq a(t)w(|y(t)|) + b(t)\int_{t_0}^t k(s)|y(s)|ds, \tag{6}$$

and

$$|h(t, y(t), Ty(t))| \leq c(t)w(|y|), \tag{7}$$

where $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k \in L_1(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ is nondecreasing in u, $u \leq w(u)$, and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M\int_{t_0}^\infty (a(s) + c(s) + b(s)\int_{t_0}^s k(\tau)d\tau)ds\Big], \tag{8}$$

where $M(t_0) < \infty$ and $b_1 = \infty$. Then the zero solution of (2) is ULS whenever the zero solution of (1) is ULSV.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. Since $x = 0$ of (1) is ULSV, it is ULS ([9],Theorem 3.3). Applying Lemma 2.1, condition (6), and condition (7), we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))|(\int_{t_0}^s |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds$$

$$\leq M|y_0| + \int_{t_0}^t M|y_0|(a(s) + c(s))w(\frac{|y(s)|}{|y_0|})ds$$

$$+ \int_{t_0}^t M|y_0|b(s)\int_{t_0}^s k(\tau)\frac{|y(\tau)|}{|y_0|}d\tau ds.$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.4 yields

$$|y(t)| \leq |y_0|W^{-1}\Big[W(M) + M\int_{t_0}^t (a(s) + c(s) + b(s)\int_{t_0}^s k(\tau)d\tau)ds\Big].$$

By condition (8), we have $|y(t)| \leq M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. $\qquad\square$

Remark 3.1. Letting $c(t) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.6 in [12].

Theorem 3.2. *For the perturbed (2), we asssume that*

$$|g(t,y)| \leq a(t)w(|y(t)|) + b(t)\int_{t_0}^t k(s)|y(s)|ds, \qquad (9)$$

and

$$|h(t,y(t),Ty(t))| \leq c(t)w(|y|), \qquad (10)$$

where $a,b,c,k \in C(\mathbb{R}^+)$, $a,b,c,k \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *is nondecreasing in* u, $u \leq w(u)$, *and* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}(\int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau + c(s))ds\Big], \quad (11)$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *Then the zero solution of (2) is ULS whenever the zero solution of (1) is ULSV.*

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1) and (2), respectively. Since $x = 0$ of (1) is ULSV, it is ULS . Using the nonlinear variation of constants formula , condition (9), and condition (10), we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^t |\Phi(t,s,y(s))|(\int_{t_0}^s |g(\tau,y(\tau))|d\tau + |h(s,y(s),Ty(s))|)ds$$

$$\leq M|y_0| + \int_{t_0}^t M|y_0|\Big[\int_{t_0}^s (a(\tau)w(\frac{|y(\tau)|}{|y_0|}) + b(\tau)\int_{t_0}^\tau k(r)\frac{|y(r)|}{|y_0|}dr)d\tau$$

$$+ c(s)w(\frac{|y(s)|}{|y_0|})\Big]ds.$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, it follows from Lemma 2.7 that

$$|y(t)| \leq |y_0|W^{-1}\Big[W(M) + M\int_{t_0}^t (\int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau + c(s))ds\Big],$$

By condition (11), we have $|y(t)| \leq M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. $\qquad\square$

Remark 3.2. Letting $c(t) = 0$ in Theorem 3.2, we obtain the same result as that of Theorem 3.7 in [12].

Theorem 3.3. *For the perturbed (2), we asssume that*

$$|g(t,y)| \le a(t)w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)|y(s)|ds, \tag{12}$$

and

$$|h(t,y(t),Ty(t))| \le \int_{t_0}^{t} c(s)w(|y(s)|)ds, \tag{13}$$

where $a,b,c,k \in C(\mathbb{R}^+)$, $a,b,c,k \in L_1(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ is nondecreasing in u, $u \le w(u)$, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M \int_{t_0}^{\infty} \int_{t_0}^{s} (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau ds\Big], \tag{14}$$

where $M(t_0) < \infty$ and $b_1 = \infty$. Then the zero solution of (2) is ULS whenever the zero solution of (1) is ULSV.

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1) and (2), respectively. Since $x = 0$ of (1) is ULSV, it is ULS. Using the nonlinear variation of constants formula, condition (12), and condition (13), we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))|\Big(\int_{t_0}^{s} |g(\tau,y(\tau))|d\tau + |h(s,y(s),Ty(s))|\Big)ds$$

$$\le M|y_0| + \int_{t_0}^{t} M|y_0| \int_{t_0}^{s} [(a(\tau) + c(\tau))w(\frac{|y(\tau)|}{|y_0|})d\tau ds$$

$$+ \int_{t_0}^{t} M|y_0| \int_{t_0}^{s} b(\tau) \int_{t_0}^{\tau} k(r)\frac{|y(r)|}{|y_0|}dr]d\tau ds.$$

Set $u(t) = |y(t)||y_0|^{-1}$. Now an application of Lemma 2.5 yields

$$|y(t)| \le |y_0|W^{-1}\Big[W(M) + M \int_{t_0}^{t} \int_{t_0}^{s} (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau ds\Big],$$

$t \ge t_0$. From condition (14) we get $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

Remark 3.3. Letting $c(s) = 0$ in Theorem 3.3, we obtain the same result as that of Theorem 3.7 in [12].

Theorem 3.4. *Let the solution $x = 0$ of (1) be EASV. Suppose that the perturbing term $g(t,y)$ satisfies*

$$|g(t,y(t))| \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t) \int_{t_0}^{t} k(s)|y(s)|ds\Big), \tag{15}$$

and

$$|h(t,y(t),Ty(t))| \le \int_{t_0}^{t} e^{-\alpha s}c(s)|y(s)|ds, \tag{16}$$

where $\alpha > 0$, $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k \in L_1(\mathbb{R}^+)$, $w(u)$ is nondecreasing in u. If

$$M(t_0) = c\exp(\int_{t_0}^{\infty} Me^{\alpha s}\int_{t_0}^{s}[a(\tau)+c(\tau)+b(\tau)\int_{t_0}^{\tau}k(r)dr]d\tau ds) < \infty, \quad (17)$$

where $c = |y_0|Me^{\alpha t_0}$, then all solutions of (2) approch zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. Since the solution $x = 0$ of (1) is EASV, by remark 2.1, it is EAS. Using Lemma 2.1, condition (15), and condition (16), we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t}|\Phi(t,s,y(s))|(\int_{t_0}^{s}|g(\tau,y(\tau))|d\tau + |h(s,y(s),Ty(s))|)ds$$

$$\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t}Me^{-\alpha(t-s)}\int_{t_0}^{s}[e^{-\alpha\tau}(a(\tau)+c(\tau))|y(\tau)|$$

$$+e^{-\alpha\tau}b(\tau)\int_{t_0}^{\tau}k(r)|y(r)|drd\tau]ds,$$

$$\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t}Me^{-\alpha(t-s)}\int_{t_0}^{s}[(a(\tau)+c(\tau))|y(\tau)|e^{\alpha\tau}$$

$$+b(\tau)\int_{t_0}^{\tau}k(r)|y(r)|e^{\alpha r}drd\tau]ds.$$

Set $u(t) = |y(t)|e^{\alpha t}$. By Lemma 2.3, we obtain

$$|y(t)| \le ce^{-\alpha t}\exp(\int_{t_0}^{t}Me^{\alpha s}\int_{t_0}^{s}[a(\tau)+c(\tau)+b(\tau)\int_{t_0}^{\tau}k(r)dr]d\tau ds), \ t \ge t_0,$$

where $c = M|y_0|e^{\alpha t_0}$. By condition (17), we have $|y(t)| \le ce^{-\alpha t}M(t_0)$. This estimation yields the desired result. $\square$

Remark 3.4. Letting $c(s) = 0$ in Theorem 3.4, we obtain the same result as that of Theorem 3.8 in [12].

Theorem 3.5. *Let the solution $x = 0$ of (1) be EASV. Suppose that the perturbed term $g(t,y)$ satisfies*

$$\int_{t_0}^{t}|g(s,y(s))|ds \le e^{-\alpha t}\Big(a(t)w(|y(t)|)+b(t)\int_{t_0}^{t}k(s)w(|y(s)|)ds\Big), \quad (18)$$

and

$$|h(t,y(t),Ty(t))| \le e^{-\alpha t}c(t)w(|y|), \quad (19)$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k \in L_1(\mathbb{R}^+)$, $w(u)$ is nondecreasing in u, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. If

$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}(a(s)+c(s)+b(s)\int_{t_0}^{s}k(\tau)d\tau)ds\Big] < \infty, b_1 = \infty,$$
$$(20)$$

where $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2) approch zero as $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. Since the solution $x = 0$ of (1) is EASV, it is EAS by remark 2.1. Using Lemma 2.1, condition (18), and condition (19), we have

$$|y(t)| \le |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| (\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))|) ds$$

$$\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t Me^{-\alpha(t-s)}[e^{-\alpha s}a(s)w(|y(s)|)$$

$$+e^{-\alpha s}b(s)\int_{t_0}^s k(\tau)w(|y(\tau)|)d\tau + e^{-\alpha s}c(s)w(|y(s)|)]ds.$$

$$\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t Me^{-\alpha t}(a(s)+c(s))w(|y(s)|e^{\alpha s})ds$$

$$+\int_{t_0}^t Me^{-\alpha t}b(s)\int_{t_0}^s k(\tau)w(|y(\tau)|e^{\alpha\tau})d\tau]ds.$$

Set $u(t) = |y(t)|e^{\alpha t}$. Since $w(u)$ is nondecreasing, an application of Lemma 2.6 obtains

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^t (a(s) + c(s) + b(s)\int_{t_0}^s k(\tau)d\tau)ds\Big],\ t \ge t_0,$$

where $c = M|y_0|e^{\alpha t_0}$. By condition (20), we have $|y(t)| \le e^{-\alpha t}M(t_0)$. From this estimation, we obtain the desired result. □

Remark 3.5. Letting $c(t) = 0$ in Theorem 3.5, we obtain the same result as that of Theorem 3.9 in [12].

Acknowledgment

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Sang Il Choi received the BS from Korea University and Ph.D at North Carolina State University under the direction of J. Silverstein . Since 1995 he has been at Hanseo University as a professor. His research interests focus on Analysis and Probability theory.

Department of Mathematics, Hanseo University, Seasan 356-706, Republic of Korea.
`e-mail:schoi@hanseo.ac.kr`

Yoon Hoe Goo received the BS from Cheongju University and Ph.D at Chungnam National University under the direction of Chin-Ku Chu. Since 1993 he has been at Hanseo University as a professor. His research interests focus on topological dynamical systems and differential equations.

Department of Mathematics, Hanseo University, Seasan 356-706, Republic of Korea.
`e-mail:yhgoo@hanseo.ac.kr`