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PARAMETER ESTIMATION AND SPECTRUM OF FRACTIONAL ARIMA PROCESS†

JOO-MOK KIM* AND YUN-KYONG KIM

Abstract. We consider fractional Brownian motion and FARIMA process with Gaussian innovations and show that the suitably scaled distributions of the FARIMA processes converge to fractional Brownian motion in the sense of finite dimensional distributions. We figure out ACF function and estimate the self-similarity parameter H of FARIMA$(0, d, 0)$ by using R/S method. Finally, we display power spectrum density of FARIMA process.

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1. Introduction

Traffic measurements in many network environments demonstrate long range dependent process(LRD) and self-similar processes which appear in many contexts, for example, in the analysis of traffic load in high speed networks [3, 14]. On the other hand, Self-similarity, long range dependence and heavy tailed process have been observed in many time series, i.e. signal processing and finance [5, 9].

The models based on self-similarity could reflect the features of LRD only by Hurst parameter [6, 13]. Because the traditional short range dependent process(SRD) could not reflect self-similar traffic's attention to the network, FARIMA model and superposition model in which the sojourn time complies with heavy-tailed distributed ON/OFF sources are required [10].

In particular, A fractional autoregressive integrated moving average process(FARIMA) is widely used in video and network traffic modeling [1, 2, 4, 7].

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*Corresponding author.
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In this paper, we use the fractionally integrated autoregressive moving average processes with Gaussian innovations to describe convergence to Fractional Brownian motion [8, 11].

On the other hand, various methods for estimating the self-similarity parameter H or the intensity of long-range dependence in a time series has investigated [12, 13]. In particular, we use R/S method for parameter estimation of self-similarity in FARIMA processes.

In section 2, we define long range dependence, self-similar process, fractional Brownian motion, fractional Gaussian noise and FARIMA processes with Gaussian innovations. In section 3, we prove the weak convergence of FARIMA processes to fractional Bronian motion. In section 4, we figure out the self-similarity H of FARIMA$(0, d, 0)$ and display the power spectrum density function of fractional autoregressive integrated moving average processes.

2. Definition and Preliminary

In this section we first define short range dependence and long range dependence. Let $\tau_X(k)$ be the covariance of stationary stochastic process $X(t)$.

Definition 2.1. A stationary stochastic process $X(t)$ exhibits short range dependence if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| < \infty$$

Definition 2.2. A stationary stochastic process $X(t)$ exhibits long range dependence if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| = \infty$$

A standard example of a long range dependent process is fractional Brownian motion, with Hurst parameter $H > \frac{1}{2}$.

Definition 2.3. A continuous process $X(t)$ is self-similar with self-similarity parameter $H \geq 0$ if it satisfies the condition:

$$X(t) \stackrel{d}{=} c^{-H} X(ct), \quad \forall\, t \geq 0, \forall c > 0,$$

where the equality is in the sense of finite-dimensional distributions.

Self-similar processes are invariant in distribution under scaling of time and space. Brownian motion is a Gaussian process with mean zero and autocovariance function

$$E[X(t_1)X(t_2)] = \min(t_1, t_2).$$

It is H self-similar with $H = 1/2$. And, Fractional Brownian motion is important example of self-similar process.

Definition 2.4. A stochastic process $\{B_H(t)\}$ is said to be a fractional Brownian motion(FBM) with Hurst parameter H if

1. $B_H(t)$ has stationary increments
3. $B_H(0) = 0$ $a.s.$
4. The increments of $B_H(t)$, $Z(j) = B_H(j+1) - B_H(j)$ satisfy

$$\rho_Z(k) = \frac{1}{2}\{|k+1|^{2H} + |k-1|^{2H} - 2k^{2H}\}$$

Definition 2.5. Let

$$G_j = B_H(j+1) - B_H(j), \quad j = \cdots, -1, 0, 1, \cdots.$$

The sequence $\{G_j, j \in Z\}$ is called fractional Gaussian noise (FGN).

Since fractional Brownian motion $\{B_H(t) : t \in R\}$ has stationary increments, its increments G_j form a stationary sequence. Fractional Gaussian noise is a mean zero and stationary Gaussian time series whose autocovariance function $\tau(h) = EG_iG_{i+h}$ is given by

$$\tau(h) = 2^{-1}\{(h+1)^{2H} - 2h^{2H} + |h-1|^{2H}\},$$

$h \geq 0$. As $h \to \infty$,

$$\tau(h) \sim H(2H-1)h^{2H-2}.$$

Since $\tau(h) = 0$ for $h \geq 1$ when $H = 1/2$. the G_i are white noise. When $1/2 < H < 1$, they display long-range dependence.

We introduce a FARIMA(p, d, q) which is both long range dependent and has heavy tails. FARIMA(p, d, q) processes are capable of modeling both short and long range dependence in traffic models since the effect of d on distant samples decays hyperbolically while the effects of p and q decay exponentially.

Definition 2.6. A stationary process X_t is called a *FARIMA*(p, d, q) *process* if

$$\phi(B)\nabla^d X_t = \theta(B)Z_t$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$, $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ and the coefficients $\phi_1, \cdots, \phi_p$ and $\theta_1, \cdots, \theta_q$ are constants,

$$\nabla^d = (1-B)^d = \sum_{i=0}^{\infty} b_i(-d)B^i$$

and B is the backward shift operator defined as $B^i X_t = X_{t-i}$ and

$$b_i(-d) = \prod_{k=1}^{i} \frac{k+d-1}{k} = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}.$$

For large lags d, the autocovariance function satisfies for $0 < d < 1/2$,

$$\tau(h) \sim C_d h^{2d-1} \text{ as } h \to \infty$$

where $C_d = \pi^{-1}\Gamma(1-2d)\sin(\pi d)$. Thus, for large lags d, the autocovariance function has the same power decay as the autocovariance of fractional Gaussian noise. Relating the exponents gives

$$d = H - \frac{1}{2}.$$

3. Weak Convergence of FARIMA to Fractional Brownian motion

Lemma 3.1. *Fix $1/2 < H < 1$ and let $\{Z_j, j = \cdots, -1, 0, 1, \cdots\}$ be a stationary Gaussian sequence with mean zero and autocovariance function $\tau(j) = EZ_0Z_j$ satisfying:*

$$\tau(j) \sim cj^{2H-2} \quad as \quad j \to \infty \quad with \quad c > 0;$$

Then the finite dimensional distributions of $\{N^{-H}\sum_{j=1}^{[Nt]} Z_j, 0 \le t \le 1\}$ converge to those of $\{\sigma_0 B_H(t), 0 \le t \le 1\}$ where

$$\sigma_0^2 = H^{-1}(2H-1)^{-1}c$$

Proof. Theorem 7.2.11 of [6]. □

Theorem 3.2.

$$\frac{1}{N^H}\frac{1}{M^{1/2}}\sum_{k=iN+1}^{(i+1)N}\sum_{j=0}^{M} b_j(-d)a_{k-j}$$

converges to $\sigma_0 G_i$ in the sense of finite dimensional distributions, as $M \to \infty$ and $N \to \infty$, where, $\sigma_0^2 = \dfrac{-\Gamma(2-2H)\cos(\pi H)}{\pi H(2H-1)}$.

Proof. By Lemma 2 of [8],

$$\lim_{M\to\infty}\frac{1}{M^{1/2}}\sum_{j=0}^{M} b_j(-d)a_{k-j} = G_H(k).$$

Here, $G_H(k)$ represents a stationary Gaussian process whose covariance function has the form $\tau(k) \sim ck^{2H-2}$ and $1/2 < H < 1$.
And, the covariance function of $\sum_{j=0}^{\infty} b_j(-d)a_{k-j}$,

$$\begin{aligned}\tau(k) &\sim \frac{\Gamma(1-2d)\sin(\pi d)}{\pi}k^{2d-1}\\ &= \frac{-\Gamma(2-2H)\cos(\pi H)}{\pi}k^{2H-2}\end{aligned}$$

where $H = d + 1/2$, has the same form as ck^{2H-2}. Therefore,

$$\lim_{M\to\infty}\frac{1}{N^H}\frac{1}{M^{1/2}}\sum_{k=iN+1}^{(i+1)N}\sum_{j=0}^{M} b_j(-d)a_{k-j} = \frac{1}{N^H}\sum_{k=iN+1}^{(i+1)N} G_H(k).$$

By Lemma 3.1, with $\sigma_0^2 = \frac{-\Gamma(2-2H)\cos(\pi H)}{\pi H(2H-1)}$,

$$N^{-H}\sum_{k=iN+1}^{(i+1)N} G_H(k) = N^{-H}\sum_{k=1}^{(i+1)N} G_H(k) - N^{-H}\sum_{k=1}^{iN} G_H(k)$$

converge to

$$\sigma_0 B_H(i+1) - \sigma_0 B_H(i) = \sigma_0 G_i.$$

□

Theorem 3.3.

$$\frac{1}{T^H M^{1/2}}\sum_{k=0}^{[Tt]}\sum_{j=0}^{M} b_j(-d)a_{k-j}$$

converges to $\sigma_0 B_H(t)$ *in the sense of finite dimensional distributions, as* $T \to \infty$ *and* $M \to \infty$, *where,* $\sigma_0^2 = \dfrac{-\Gamma(2-2H)\cos(\pi H)}{\pi H(2H-1)}$.

Proof. Consider the partial sum of

$$\frac{1}{T^H}\frac{1}{M^{1/2}}\sum_{k=iT+1}^{(i+1)T}\sum_{j=0}^{M} b_j(-d)a_{k-j}$$

which converges to fractional Brownian motion $\{B_H(t) : t \in R\}$ in the sense of finite dimensional distributions by Theorem 3.2 . □

4. Estimation of the self-similarity and PSD of FARIMA

When $d < 1/2$, the FARIMA process is stationary and the covariance function of a FARIMA$(0, d, 0)$ process with zero mean and unit variance Gaussian innovations has the form

$$\begin{aligned}\tau(k) &= \frac{(-1)^k(-2d)!}{(\pi-d)!(-k-d)!}\\ &\sim \frac{\Gamma(1-2d)\sin(\pi d)}{\pi}k^{2d-1} \quad as \;\; k\to\infty\end{aligned}$$

The covariance function of the generalized FARIMA(p, d, q) processes with Gaussian innovations has additional short-term components but follows the same asymptotic relation as the covariance function as FARIMA$(0, d, 0)$ processes. Hence, we consider FARIMA$(0, d, 0)$ in terms of $d = 0.3$ and estimate the self-similarity parameter H .

The R/S method which was used by Taqqu and Willinger([12,13]) is one of the better known method. For a time series $X = \{X_i : i \le 1\}$, with partial sum $Y(n) = \sum_{i=1}^n X_i$, and sample variance $S^2(n) = (1/n)\sum_{i=1}^n X_i^2 - (1/n)^2 Y(n)^2$, the R/S static, or the rescaled adjusted range, is given by

$$\frac{R}{S}(n) = \frac{1}{S(n)}\left[\max_{0\le t\le n}(Y(t) - \frac{t}{n}Y(n)) - \min_{0\le t\le n}(Y(t) - \frac{t}{n}Y(n))\right].$$

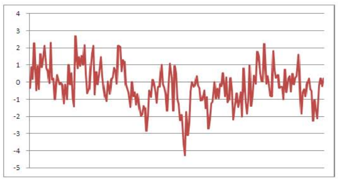

FIGURE 1. Simulated FARIMA$(0, 0.3, 0)$, $n = 1,000$

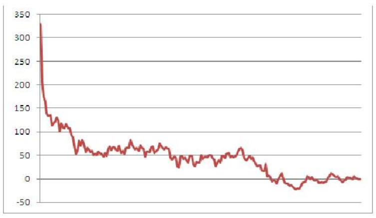

FIGURE 2. Autocorrelation Function

For fractional gaussian noise,

$$E[R/S(n)] \sim C_H n^H, \text{ as } n \to \infty,$$

where, C_H is positive and finite constant not depend on n.

The following Figure 1 and 2 illustrates simulated FARIMA process and autocorrelation function with $d = 0.3$ in the case $n = 200$.

To determine H using the R/S statistic, proceed as follows. For a time series of length N, subdivide the series into blocks. Then, for each lag n, compute $R(n)/S(n)$. Choosing logarithmically spaced values of n, plot $log[R(n)/S(n)]$ versus $\log(n)$ and get, for each n, several points on the plot. The spectrum density of a FARIMA(p, d, q) process is equal to

$$f_Y(\lambda) = \frac{\sigma^2 |\theta(exp(-i\lambda))|^2}{2\pi |\phi(exp(-i\lambda))|^2} |1 - exp(-i\lambda)|^{-2d}.$$

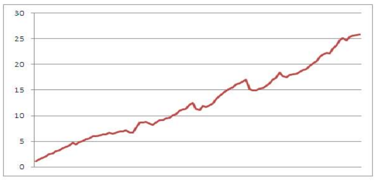

FIGURE 3. Estimating H

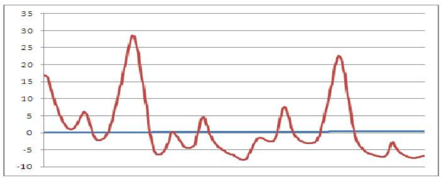

FIGURE 4. Power Spectrum Density

In Figure 3, we estimate H as 0.830126 by calculating the R/S statistic and Figure 4 display power spectrum density of FARIMA process with $d = 0.3$.

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Joo-Mok Kim recieved his Ph.D in Department of Mathematics from University of Utah in 1993 under the direction of Professor J. D. Mason. Since 1995, he has been at the Semyung University as a Professor. His research interests focus on the stochastic process, in particular, self-similar process, stable process and related application fields.

School of General Education, Semyung University, Jecheon 390-711, Korea.
E-mail:jmkim@semyung.ac.kr

Yun-Kyong Kim received his Ph.D degree in Department of Mathematics from Korea University in 1988 under the direction of Professor In-Suk Wee. From 1989 to 2002, he was a professor in Dept. of Mathematics, Dongshin University. Since 2002, he has been a professor in Department of Information and Communication Engineering, Dongshin University. His research interests center on Probability Theory and Stochastic Processes, in particular, Limit Theorems of Fuzzy Set Valued Random Variables.

Department of Information and Communication Engineering, Donghin University, Naju 520-714, Korea.
Email:ykkim@dsu.ac.kr