J. Appl. Math. & Informatics Vol. **33***(2015), No. 1 - 2, pp. 193 - 201*
http://dx.doi.org/10.14317/jami.2015.193

IMPROVING COMPARISON RESULTS ON PRECONDITIONED GENERALIZED ACCELERATED OVERRELAXATION METHODS†

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Abstract. In this paper, we present preconditioned generalized accelerated overrelaxation (GAOR) methods for solving weighted linear least square problems. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned GAOR methods converge faster than the GAOR method whenever the GAOR method is convergent. Finally, we give a numerical example to confirm our theoretical results.

AMS Mathematics Subject Classification : 65F10.
Key words and phrases : Preconditioned GAOR method, Linear system, Convergence, Comparison.

1. Introduction

Consider the weighted linear least squares problem

$$\min_{x\in R^n}(Ax-b)^T W^{-1}(Ax-b), \tag{1.1}$$

where W is the variance-covariance matrix. The problem has many scientific applications. A typical source is parameter estimation in mathematical modeling. This problem has been discussed in many books and articles. In order to solve it, one has to solve a nonsingular linear system as

$$Hy = f,$$

where

$$H = \begin{pmatrix} I-B & E \\ D & I-C \end{pmatrix}$$

Received March 19, 2014. Revised May 2, 2014. Accepted June 5, 2014. *Corresponding author. †This work was supported by the National Natural Science Foundation of China (Grant No.11001144) and the Natural Science Foundation of Shandong Province of China (ZR2012AL09).

is an invertible matrix with

$$B = (b_{ij})_{p\times p},\ C = (c_{ij})_{q\times q},\ D = (d_{ij})_{q\times p},\ \ p+q=n.$$

In order to solve the linear system using the GAOR method, we split H as

$$H = I - \begin{pmatrix} 0 & 0 \\ -D & 0 \end{pmatrix} - \begin{pmatrix} B & -E \\ 0 & C \end{pmatrix}.$$

Then, for $\omega \neq 0$, one GAOR method can be defined by

$$y^{(k+1)} = L_{r,\omega} y^{(k)} + \omega g, \quad k = 0, 1, 2, \cdots,$$

where

$$\begin{aligned} L_{r,\omega} &= \begin{pmatrix} I & 0 \\ rD & I \end{pmatrix}^{-1} \left\{ (1-\omega)I + (\omega - r) \begin{pmatrix} 0 & 0 \\ -D & 0 \end{pmatrix} + \omega \begin{pmatrix} B & -E \\ 0 & C \end{pmatrix} \right\} \\ &= \begin{pmatrix} (1-\omega)I + \omega B & -\omega E \\ \omega(r-1)D - \omega r DB & (1-\omega)I + \omega C + \omega r DE \end{pmatrix} \end{aligned}$$

is the iteration matrix and

$$g = \begin{pmatrix} I & 0 \\ -rD & I \end{pmatrix} f.$$

In order to decrease the spectral radius of the iteration matrix, an effective method is to precondition the linear system (1.1), namely,

$$PH = \begin{pmatrix} I - B^* & E^* \\ D^* & I - C^* \end{pmatrix},$$

then the preconditioned GAOR (PGAOR) method can be defined by

$$y^{(k+1)} = L^*_{r,\omega} y^{(k)} + \omega g^*, \quad k = 0, 1, 2, \cdots,$$

where

$$\begin{aligned} L^*_{r,\omega} &= \begin{pmatrix} (1-\omega)I + \omega B^* & -\omega E^* \\ \omega(1-r)D^* - \omega r D^* B^* & (1-\omega)I + \omega C^* + \omega r D^* E^* \end{pmatrix}, \\ g^* &= \begin{pmatrix} I & 0 \\ -rD^* & I \end{pmatrix} Pf. \end{aligned}$$

This paper is organized as follows. In Section 2, we propose three preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GAOR methods converge faster than the GAOR method whenever the GAOR method is convergent. In Section 3, we give one example to confirm our theoretical results.

2. Comparison results

In paper [5], the preconditioners introduced by Zhou et al. are of the form

$$P_i = \begin{pmatrix} I + S_i & 0 \\ 0 & I \end{pmatrix}.$$

In paper [3], the following preconditioned linear system was considered

$$\widetilde{H} y = \widetilde{f}, \tag{2.1}$$

where $\widetilde{H} = (I + \widetilde{S})H$ and $\widetilde{f} = (I + \widetilde{S})f$ with

$$\widetilde{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},$$

S is a $p \times p$ matrix with $1 < p < n$. And S was taken as follows:

$$S_1 = \begin{pmatrix} 0 & b_{12} & \cdots & 0 & 0 \\ b_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & b_{p-1,p} \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_{21} & 0 & \ddots & 0 & 0 \\ 0 & b_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{p,p-1} & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & b_{12} & 0 & \cdots & 0 \\ 0 & 0 & b_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{p-1,p} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The preconditioned GAOR methods for solving (2.1) are

$$y^{(k+1)} = L_{r,\omega}^{(i)} y^{(k)} + \omega \widetilde{g}, \quad k = 0, 1, 2, \cdots, \tag{2.2}$$

where

$$L_{r,\omega}^{i} = \begin{pmatrix} (1-\omega)I + \omega[B - S_i(I-B)] & -\omega(I+S_i)E \\ \omega(r-1)D - \omega r D[B - S_i(I-B)] & (1-\omega)I + \omega C + \omega r D(I+S_i)E \end{pmatrix}$$

are iteration matrices for $i = 1, 2, 3$.

In paper [4], the preconditioners introduced by Yun are of the form

$$P_i^* = \begin{pmatrix} I + S_i & 0 \\ 0 & I + V_i \end{pmatrix}.$$

In this paper, we will consider new preconditioners P_i^*

$$P_i^* = \begin{pmatrix} I + S_i & 0 \\ 0 & I + V_i \end{pmatrix}, \quad i = 1, 2, 3$$

where S_i are defined as above and

$$V_1 = \begin{pmatrix} 0 & c_{12} & \cdots & 0 & 0 \\ c_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & c_{p-1,p} \\ 0 & 0 & \cdots & c_{p,p-1} & 0 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_{21} & 0 & \ddots & 0 & 0 \\ 0 & c_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{p,p-1} & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & c_{12} & 0 & \cdots & 0 \\ 0 & 0 & c_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{p-1,p} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \widetilde{H}_i^* &= P_i^* H \\ &= \begin{pmatrix} I - [B - S_i(I-B)] & (I+S_i)E \\ (I+V_I)D & I - [C - V_i(I-C)] \end{pmatrix} \\ &= I - \begin{pmatrix} 0 & 0 \\ -(I+V_I)D & 0 \end{pmatrix} - \begin{pmatrix} [B - S_i(I-B)] & -(I+S_i)E \\ 0 & [C - V_i(I-C)] \end{pmatrix}. \end{aligned}$$

The preconditioned GAOR methods for solving $P_i^* H y = P_i^* f$ are defined as follows

$$y^{(k+1)} = L_{r,\omega}^{*(i)} y^{(k)} + \omega \widetilde{g}^*, \quad k = 0, 1, 2, \cdots,$$

where for $i = 1, 2, 3$,

$$L_{r,\omega}^{*(i)} = \begin{pmatrix} (1-\omega)I + \omega[B - S_i(I-B)] & -\omega(I+S_i)E \\ \omega(r-1)(I+V_i)D & (1-\omega)I + \omega[C - V_i(I-C)] \\ -\omega r(I+V_i)D[B - S_i(I-B)] & +\omega r(I+V_i)D(I+S_i)E \end{pmatrix},$$

$$\widetilde{g}_i^* = \begin{pmatrix} I & 0 \\ -r(I+V_i)D & I \end{pmatrix} \widetilde{f}.$$

Lemma 2.1 ([1, 2]). *Let $A \in R^{n\times n}$ be nonnegative and irreducible. Then*

- **(a):** *A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.*
- **(b):** *for $\rho(A)$,there corresponds an eigenvector $x > 0$.*
- **(c):** *if $0 \neq \alpha x \leq Ax \leq \beta x, \alpha x \neq Ax, Ax \neq \beta x$ for some nonnegative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.*

Theorem 2.1. *Let $L_{r,\omega}, L_{r,\omega}^{*(1)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix H is irreducible* *with* $D \leq 0,\ E \leq 0,\ B \geq 0,\ C \geq 0,\ b_{i,i+1} > 0,\ b_{i+1,i} > 0,\ c_{i,i+1} > 0,\ c_{i+1,i} > 0$ *for some* $i \in \{1, 2, \cdots, p-1\},\ 0 < \omega \leq 1,\ 0 \leq r < 1,$ *then either*

$$\rho(L^{*(1)}_{r,\omega}) < \rho(L_{r,\omega}) < 1 \tag{2.3}$$

or

$$\rho(L^{*(1)}_{r,\omega}) > \rho(L_{r,\omega}) > 1. \tag{2.4}$$

Proof. Since $0 < \omega \leq 1,\ 0 \leq r < 1,\ D \leq 0,\ E \leq 0,\ B \geq 0,\ C \geq 0$, it is easy to prove that both $L^{*(1)}_{r,\omega}$ and $L_{r,\omega}$ are irreducible and non-negative. By Lemma 2.1, there is a positive vector x such that$L_{r,\omega}x = \lambda x$, where $\lambda = \rho(L_{r,\omega})$. Then

$$\left\{(1-\omega)I - (\omega - r)\begin{pmatrix} 0 & 0 \\ -D & 0 \end{pmatrix} + \omega \begin{pmatrix} B & -E \\ 0 & C \end{pmatrix}\right\} x = \lambda \begin{pmatrix} I & 0 \\ rD & I \end{pmatrix} x$$

$$\begin{aligned}
& L^{*(1)}_{r,\omega}x - \lambda x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \left\{(1-\omega)I + (\omega - r)\begin{pmatrix} 0 & 0 \\ -(I+V_1)D & 0 \end{pmatrix}\right\} x \\
&\quad - \lambda \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix} x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \begin{pmatrix} -\omega S_1(I-B) & -\omega S_1 E \\ (-\omega + r - \lambda r)V_1 D & -\omega V_1(I-C) \end{pmatrix} x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} \begin{pmatrix} -\omega(I-B) & -\omega E \\ (-\omega + r - \lambda r)D & -\omega(I-C) \end{pmatrix} x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} \left\{ \begin{pmatrix} (\lambda - \omega)I + \omega B & -\omega E \\ -(\omega - r)D & (\lambda - \omega)I + \omega C \end{pmatrix} \right. \\
&\quad \left. -\lambda \begin{pmatrix} I & 0 \\ rD & I \end{pmatrix}\right\} x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} \begin{pmatrix} (\lambda - 1)I & 0 \\ 0 & (\lambda - 1)I \end{pmatrix} x \\
&= (\lambda - 1) \begin{pmatrix} I & 0 \\ -r(I+V_1)D & I \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} x
\end{aligned}$$

Since $b_{i,i+1} > 0,\ b_{i+1,i} > 0,\ c_{i,i+1} > 0,\ c_{i+1,i} > 0$ then $S_1 > 0,\ V_1 > 0$ and

$$\begin{pmatrix} I & 0 \\ -r(I+V_1)D & I \end{pmatrix} > 0, \quad \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} > 0.$$

If $\lambda < 1$, then $L^{*(1)}_{r,\omega}x - \lambda x < 0$. By Lemma 2.1, we get $\rho(L^{*(1)}_{r,\omega}) < \rho(L_{r,\omega\omega}) < 1$.
If $\lambda > 1$, then $L^{*(1)}_{r,\omega}x - \lambda x > 0$. By Lemma 2.1, we get $\rho(L^{*(1)}_{r,\omega}) > \rho(L_{r,\omega\omega}) > 1$. □

By the analogous proof of Theorem 2.1, we can prove the following two theorems.

Theorem 2.2. *Let* $L_{r,\omega}, L^{*(2)}_{r,\omega}$ *be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix* H *is irreducible* *with* $D \le 0,\ E \le 0,\ B \ge 0,\ C \ge 0,\ b_{i,i+1} > 0,\ b_{i+1,i} > 0,\ c_{i,i+1} > 0,\ c_{i+1,i} > 0$ *for some* $i \in \{1, 2, \cdots, p-1\},\ 0 < \omega \le 1,\ 0 \le r < 1,$ *then either*

$$\rho(L^{*(2)}_{r,\omega}) < \rho(L_{r,\omega}) < 1 \quad or \quad \rho(L^{*(2)}_{r,\omega}) > \rho(L_{r,\omega}) > 1.$$

Theorem 2.3. *Let* $L_{r,\omega}, L^{*(3)}_{r,\omega}$ *be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix* H *is irreducible with* $D \le 0,\ E \le 0,\ B \ge 0,\ C \ge 0,\ b_{i,i+1} > 0,\ b_{i+1,i} > 0,\ c_{i,i+1} > 0,\ c_{i+1,i} > 0$ *for some* $i \in \{1, 2, \cdots, p-1\},\ 0 < \omega \le 1,\ 0 \le r < 1,$ *then either*

$$\rho(L^{*(3)}_{r,\omega}) < \rho(L_{r,\omega}) < 1 \quad or \quad \rho(L^{*(3)}_{r,\omega}) > \rho(L_{r,\omega}) > 1.$$

Theorem 2.4. *Under the assumptions of Theorem 2.1, then either*

$$\rho(L^{*(1)}_{r,\omega}) < \rho(L^{*(2)}_{r,\omega}) < 1, if\ \rho(L_{r,\omega}) < 1$$

or

$$\rho(L^{*(1)}_{r,\omega}) > \rho(L^{*(2)}_{r,\omega}) > 1, if\ \rho(L_{r,\omega}) > 1.$$

Proof. By Lemma 2.1, there is a positive vector x ,such that

$$L_{r,\omega} x = \lambda x \tag{2.5}$$

where $\lambda = \rho(L_{r,\omega})$. Then

$$\begin{aligned}
& L^{*(1)}_{r,\omega} x - L^{*(2)}_{r,\omega} x \\
&= (L^{*(1)}_{r,\omega} - \lambda x) - (L^{*(2)}_{r,\omega} - \lambda x) \\
&= (\lambda - 1) \begin{pmatrix} I & 0 \\ -r(I+V_1)D & I \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix} x \\
&\quad - (\lambda - 1) \begin{pmatrix} I & 0 \\ -r(I+V_2)D & I \end{pmatrix} \begin{pmatrix} S_2 & 0 \\ 0 & V_2 \end{pmatrix} x \\
&= (\lambda - 1) \begin{pmatrix} S_1 - S_2 & 0 \\ r(I+V_2)DS_2 - r(I+V_1)DS_1 & V_1 - V_2 \end{pmatrix} x \\
&= (\lambda - 1) \begin{pmatrix} S_1 - S_2 & 0 \\ rD(S_2 - S_1) + rV_2 D(S_2 - S_1) + r(V_2 - V_1)DS_1 & V_1 - V_2 \end{pmatrix} x
\end{aligned}$$

Under the conditions of Theorem 2.1, we know that

$$D < 0, S_1 > S_2 > 0, V_1 > V_2 > 0.$$

Thus

$$rD(S_2 - S_1) + rV_2 D(S_2 - S_1) + r(V_2 - V_1)DS_1 > 0, S_1 - S_2 > 0, V_1 - V_2 > 0.$$

Then

(1) If $\lambda < 1$, then $L^{*(1)}_{r,\omega} x - L^{*(2)}_{r,\omega} x < 0$. By Lemma 2.1, we get

$$\rho(L^{*(1)}_{r,\omega}) < \rho(L^{*(2)}_{r,\omega}) < 1.$$

(2) If $\lambda > 1$, then $L^{*(1)}_{r,\omega} x - L^{*(2)}_{r,\omega} x > 0$. By Lemma 2.1, we get

$$\rho(L^{*(1)}_{r,\omega}) > \rho(L^{*(2)}_{r,\omega}) > 1.$$

□

By the analogous proof of Theorem 2.4, we can prove the following one theorem.

Theorem 2.5. *Under the assumptions of Theorem 2.1, then either*

$$\rho(L^{*(1)}_{r,\omega}) < \rho(L^{*(3)}_{r,\omega}) < 1, if\ \rho(L_{r,\omega}) < 1$$

or

$$\rho(L^{*(1)}_{r,\omega}) > \rho(L^{*(3)}_{r,\omega}) > 1, if\ \rho(L_{r,\omega}) > 1.$$

Theorem 2.6. *Under the assumptions of Theorem 2.1, then either*

$$\rho(L^{*(1)}_{r,\omega}) < \rho(L^{(1)}_{r,\omega}) < 1, if\ \rho(L^{(1)}_{r,\omega}) < 1$$

or

$$\rho(L^{*(1)}_{r,\omega}) > \rho(L^{(1)}_{r,\omega}) > 1, if\ \rho(L^{(1)}_{r,\omega}) > 1.$$

Proof. By Lemma 2.1, there is a positive vector x ,such that

$$L^{(1)}_{r,\omega}x = \lambda x \tag{2.6}$$

where $\lambda = \rho(L^{(1)}_{r,\omega})$. Then

$$\left\{(1-\omega)I - (\omega - r)\begin{pmatrix} 0 & 0 \\ -D & 0 \end{pmatrix} + \omega \begin{pmatrix} B - S_1(I-B) & -(I+S_1)E \\ 0 & C \end{pmatrix}\right\} x$$
$$= \lambda \begin{pmatrix} I & 0 \\ rD & I \end{pmatrix} x$$

$$\begin{aligned}
&L^{*(1)}_{r,\omega}x - \lambda x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \left\{(1-\omega)I + (\omega - r)\begin{pmatrix} 0 & 0 \\ -(I+V_1)D & 0 \end{pmatrix}\right\} x \\
&- \lambda \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix} x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ (-\omega + r - \lambda r)V_1 D & -\omega V_1(I-C) \end{pmatrix} x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V_1 \end{pmatrix} \begin{pmatrix} -\omega(I-(B-S_1(I-B))) & -\omega(I+S_1)E \\ (-\omega + r - \lambda r)D & -\omega(I-C) \end{pmatrix} x \\
&= \begin{pmatrix} I & 0 \\ r(I+V_1)D & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V_1 \end{pmatrix} \begin{pmatrix} (\lambda-1)I & 0 \\ 0 & (\lambda-1)I \end{pmatrix} x \\
&= (\lambda - 1)\begin{pmatrix} 0 & 0 \\ 0 & V_1 \end{pmatrix} x.
\end{aligned}$$

By assumptions, $V_1 > 0$. Hence we obtain the following results.
If $\lambda < 1$, then $L^{*(1)}_{r,\omega}x - \lambda x < 0$. By Lemma 2.1, we get $\rho(L^{*(1)}_{r,\omega}) < \rho(L^{(1)}_{r,\omega}) < 1$.
If $\lambda > 1$, then $L^{*(1)}_{r,\omega}x - \lambda x > 0$. By Lemma 2.1, we get $\rho(L^{*(1)}_{r,\omega}) > \rho(L^{(1)}_{r,\omega}) > 1$. □

By the analogous proof of Theorem 2.6, we can prove the following two theorems.

Theorem 2.7. *Let $L_{r,\omega}^{(2)},\ L_{r,\omega}^{*(2)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix H is irreducible with $D \le 0,\ E \le 0,\ B \ge 0,\ C \ge 0,\ b_{i,i+1} > 0,\ b_{i+1,i} > 0,\ c_{i,i+1} > 0,\ c_{i+1,i} > 0$ for some $i \in \{1, 2, \cdots, p-1\},\ 0 < \omega \le 1,\ 0 \le r < 1$, then either*

$$\rho(L_{r,\omega}^{*(2)}) < \rho(L_{r,\omega}^{(2)}) < 1 \quad or \quad \rho(L_{r,\omega}^{*(2)}) > \rho(L_{r,\omega}^{(2)}) > 1.$$

Theorem 2.8. *Let $L_{r,\omega}^{(3)},\ L_{r,\omega}^{*(3)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix H is irreducible with $D \le 0,\ E \le 0,\ B \ge 0,\ C \ge 0,\ b_{i,i+1} > 0,\ b_{i+1,i} > 0,\ c_{i,i+1} > 0,\ c_{i+1,i} > 0$ for some $i \in \{1, 2, \cdots, p-1\},\ 0 < \omega \le 1,\ 0 \le r < 1$, then either*

$$\rho(L_{r,\omega}^{*(3)}) < \rho(L_{r,\omega}^{(3)}) < 1 \quad or \quad \rho(L_{r,\omega}^{*(3)}) > \rho(L_{r,\omega}^{(3)}) > 1.$$

3. Numerical example

Now, we present an example to illustrate our theoretical results.

Example 3.1. The coefficient matrix H in (1.1) is given by

$$H = \begin{pmatrix} I - B_1 & U \\ C & I - B_2 \end{pmatrix}$$

where $B_1 = (b_{ij})_{p\times p}, B_2 = (b_{ij})_{(n-p)\times(n-p)}, C = (c_{ij})_{(n-p)\times p}, U = (u_{ij})_{p\times(n-p)}$
with $b_{ii} = \frac{1}{10\times(i+1)},\ \ i = 1, 2, \cdots, p$
$b_{ij} = \frac{1}{30} - \frac{1}{30\times j+i},\ \ i < j,\ \ i = 1, 2, \cdots, p-1,\ \ j = 2, \cdots, p$
$b_{ij} = \frac{1}{30} - \frac{1}{30\times i-j+1},\ \ i > j,\ \ i = 2, \cdots, p,\ \ j = 1, 2, \cdots, p-1$
$b'_{ii} = \frac{1}{10\times(p+i+1)},\ \ i = 1, 2, \cdots, n-p$
$b'_{ij} = \frac{1}{30} - \frac{1}{30\times(p+j)+p+i},\ \ i < j,\ \ i = 1, 2, \cdots, n-p-1,\ \ j = 2, \cdots, n-p$
$b'_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+p+i},\ \ i > j,\ \ i = 2, \cdots, n-p,\ \ j = 1, 2, \cdots, n-p-1$
$c_{ij} = \frac{1}{30\times(p+i-j+1)+p+i} - \frac{1}{30},\ \ i = 1, 2, \cdots, n-p,\ \ j = 1, 2, \cdots, p$
$u_{ij} = \frac{1}{30\times(p+j)+i} - \frac{1}{30},\ \ i = 1, 2, \cdots, p,\ \ j = 1, 2, \cdots, n-p.$

Table 1 displays the spectral radii of the corresponding iteration matrices with some randomly chosen parameters r, ω, p. From Table 1, we see that these results accord with Theorems 2.1-2.8.

Remark: In this paper, we propose three preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GAOR methods converge faster than the GAOR method whenever the GAOR method is convergent.

Table 1. The spectral radii of the GAOR and preconditioned GAOR iteration matrices

n	ω	r	p	ρ	ρ_1^*	ρ_2^*	ρ_3^*	ρ_1	ρ_2	ρ_3
5	0.95	0.7	3	0.1450	0.1272	0.1356	0.1338	0.1330	0.1384	0.1376
10	0.9	0.85	5	0.2782	0.2509	0.2665	0.2620	0.2664	0.2726	0.2711
15	0.95	0.8	5	0.3834	0.3553	0.3720	0.3670	0.3777	0.3808	0.3800
20	0.75	0.65	10	0.6350	0.6172	0.6278	0.6248	0.6271	0.6317	0.6303
25	0.7	0.55	8	0.7872	0.7764	0.7829	0.7815	0.7846	0.7861	0.7856
30	0.65	0.55	16	0.9145	0.9099	0.9126	0.9126	0.9122	0.9136	0.9131
40	0.6	0.5	10	1.1426	1.1505	1.1458	1.1489	1.1442	1.1433	1.1436
50	0.6	0.5	10	1.3668	1.3877	1.3753	1.3815	1.3705	1.3683	1.3690
500	0.05	0.1	100	1.9831	2.0564	2.0168	2.0283	2.0028	1.9915	1.9944
1000	0.05	0.05	100	2.8492	2.9810	2.9115	2.9256	2.8710	2.8584	2.8617
2000	0.05	0.05	100	4.7927	5.0690	4.9252	4.9522	4.8298	4.8085	4.8140

Here $\rho = \rho(L_{r,\omega})$, $\rho_1^* = \rho(L_{r,\omega}^{*(1)})$, $\rho_2^* = \rho(L_{r,\omega}^{*(2)})$, $\rho_3^* = \rho(L_{r,\omega}^{*(3)})$, $\rho_1 = \rho(L_{r,\omega}^{(1)})$, $\rho_2 = \rho(L_{r,\omega}^{(2)})$, $\rho_3 = \rho(L_{r,\omega}^{(3)})$.

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