J. Appl. Math. & Informatics Vol. **33***(2015), No. 1 - 2, pp. 173 - 183*
http://dx.doi.org/10.14317/jami.2015.173

RADIO AND RADIO ANTIPODAL LABELINGS FOR CIRCULANT GRAPHS $G(4k+2;\{1,2\})$†

SAIMA NAZEER, IMRANA KOUSAR AND WAQAS NAZEER*

ABSTRACT. A radio k-labeling f of a graph G is a function f from $V(G)$ to $Z^+ \cup \{0\}$ such that $d(x,y)+|f(x)-f(y)| \geq k+1$ for every two distinct vertices x and y of G, where $d(x,y)$ is the distance between any two vertices $x,y \in G$. The span of a radio k-labeling f is denoted by $sp(f)$ and defined as $\max\{|f(x)-f(y)| : x,y \in V(G)\}$. The radio k-labeling is a radio labeling when $k = \operatorname{diam}(G)$. In other words, a radio labeling is an injective function $f : V(G) \to Z^+ \cup \{0\}$ such that

$$|f(x)-f(y)| \geq \operatorname{diam}(G)+1-d(x,y)$$

for any pair of vertices $x,y \in G$. The radio number of G denoted by $\operatorname{rn}(G)$, is the lowest span taken over all radio labelings of the graph. When $k = \operatorname{diam}(G)-1$, a radio k- labeling is called a radio antipodal labeling. An antipodal labeling for a graph G is a function $f : V(G) \to \{0,1,2,...\}$ such that $d(x,y)+|f(x)-f(y)| \geq \operatorname{diam}(G)$ holds for all $x,y \in G$. The radio antipodal number for G denoted by $\operatorname{an}(G)$, is the minimum span of an antipodal labeling admitted by G. In this paper, we investigate the exact value of the radio number and radio antipodal number for the circulant graphs $G(4k+2;\{1,2\})$.

AMS Mathematics Subject Classification : 05C12, 05C15, 05C78.
Key words and phrases : diameter, radio number, radio antipodal number, circulant graphs.

1. Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$ and let k be an integer, $k \geq 1$. A radio k-labeling f of G is an assignment of non negative integers to the vertices of G such that $|f(x)-f(y)| \geq k+1-d(x,y)$, where $d(x,y)$ denotes the distance for every two distinct vertices x and y of G. The span of the function f is $\max\{|f(x)-f(y)| : x,y \in V(G)\}$ and denoted by $\operatorname{sp}(f)$. The radio k-labeling number of G is the smallest span among all radio k-labelings of

Received March 24, 2014. Revised June 15, 2014. Accepted July 5, 2014. *Corresponding author. †This work was supported by the research grant of Higher Education Commission Pakistan

G. Chartrand et al. [1] was the first, who studied the radio k-labeling number for paths, where lower and upper bounds were given. These bounds have been improved by Kchikech et al. [7]. The radio k-labeling becomes a radio labeling for $k = \text{diam}(G)$. A radio labeling is a function from the vertices of the graph to some subset of non negative integers. The task of radio labeling is to assign to each station a non negative smallest integer such that the disturbance in the nearest channel should be minimized. In 1980 [5], Hale presented this channel assignment for the very first time by relating it to the theory of graphs.

Multilevel distance labeling problem was introduced by Chartrand et al. [4] in 2001. A radio labeling is an injective function $f : V(G) \to Z^{+} \cup \{0\}$ satisfying the condition

$$|f(x) - f(y)| \geq \text{diam}(G) + 1 - d(x, y)$$

for any pair of vertices x, y in G. Where $d(x, y)$ is the distance between any distinct pair of vertices in G, which is the length of the shortest path between them. The largest number that f maps to a vertex of a graph is the span of labeling f. Radio number of G is the minimum span taken over all radio labelings of G and is denoted by $\text{rn}(G)$. When $k = \text{diam}(G) - 1$, a radio k- labeling is referred to as a (radio) antipodal labeling, because only antipodal vertices can have the same label. The minimum span of an antipodal labeling is called the antipodal number, denoted by $\text{an}(G)$. In [1] and [2], Chartrand et al. were studied the radio antipodal labeling for path and cycle. In [3], Chartrand et al. gave general bounds for the antipodal number of a graph. The exact value of the radio antipodal number of path was found in [9]. Justic and Liu have computed the radio antipodal number of cycles. In [10], by using a generalization of binary Gray codes the radio antipodal number and the radio number of the hypercube are determined.

An undirected circulant graph denoted by $G(n; \pm\{1, 2, ..., j\})$ where $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $n \geq 3$ is defined as a graph with vertex set $V = \{0, 1, 2, ...n - 1\}$ and an edge set $E = \{(i, j) : |j - i| \equiv s \,(\text{mod}\, n), s \in \{1, 2, ..., j\}\}$. For the sake of simplicity, take the vertex set as $\{v_1, v_2, ...v_n\}$ in clockwise order.

Remark 1.1. The diameter of class of circulant graphs which are going to be discussed in this paper is:

$$\text{diam } (G(4k + 2; \{1, 2\}) = d = k + 1.$$

In this paper, radio and radio antipodal numbers for the class of circulant graphs $G(4k + 2 : \{1, 2\})$ are computed.

2. Main results

The main theorems of this paper are:

Theorem 2.1. *The radio number of the circulant graphs $G(4k + 2 : \{1, 2\})$ is given by*

$$rn\,(G(4k + 2; \{1, 2\}) = \begin{cases} k^2 + 5k + 1, & \text{if } k \text{ is odd;} \\ k^2 + 4k + 1, & \text{if } k \text{ is even.} \end{cases}$$

Theorem 2.2. *The radio antipodal number of the circulant graphs $G(4k+2;\{1,2\})$ is given by*

$$an\left(G(4k+2;\{1,2\}\right) = \begin{cases} k^2+k, & \text{if } k \text{ is odd;} \\ k^2+2k, & \text{if } k \text{ is even.} \end{cases}$$

3. Radio number for $G(4k+2;\{1,2\})$

In this section, we prove the Theorem 1 in two steps. First we provide a lower bound for $\mathrm{rn}(G(4k+2;\{1,2\}))$ then define a multilevel distance labeling of $(G(4k+2;\{1,2\}))$ with span equal to the lower bound, thus determining the radio number of $(G(4k+2;\{1,2\}))$.

3.1. Lower bound for $G(4k+2;\{1,2\})$. The lower bound for the radio number of $G(4k+2;\{1,2\})$ is determined in following way. First examine the maximum possible sum of the pairwise distance between any three vertices of $(G(4k+2;\{1,2\}))$ and use this maximum sum to compute a minimum possible gap between the i^{th} and $(i+2)^{nd}$ largest label. Then provides a lower bound for the span of any labeling by using 0 for the smallest label and considering the size of gap into account.

Lemma 3.1. *For each vertex on the graph $G(4k+2;\{1,2\})$ there is exactly one vertex at a distance diameter d, of the graph G.*

Proof. We show that $d(v_1, v_{2k+2}) = k+1 = d$. The path from v_1 to v_{2k+2} is of length $k+1$ as $v_1 \to v_{2(1)+1} \to v_{2(2)+1} \to \ldots \to v_{2(k)+1} \to v_{2(k)+1+1}$. □

The following Lemma provides a maximum possible sum of the pairwise distances between any three vertices of $G(4k+2;\{1,2\})$.

Lemma 3.2. *For any three vertices u, v, w on the graphs $G(4k+2;\{1,2\})$,*

$$d(u,v) + d(v,w) + d(w,u) \leq 2d.$$

Proof. By Lemma 3.1, $d(v_1, v_{2k+2}) = k+1 = d$. Case(i): For odd k.
$d(v_{2k+2}, v_{3k+3}) = \frac{k+1}{2}$ and a path of length $\frac{k+1}{2}$ between v_{2k+2} and v_{3k+3} is $v_{2k+1} \to v_{2k+2+1.2} \to v_{2k+2+2.2} \to \ldots \to v_{2k+2+\frac{k+1}{2}.2} = v_{3k+2}$ and $d(v_{3k+3}, v_1) = \frac{k+1}{2}$ as $v_{3k+3} \to v_{3k+3+1.2} \to v_{3k+3+2.2} \to \ldots \to v_{3k+3+\frac{k-1}{2}.2} v_{4k+2} \to v_{4k+3} = v_1$. This implies that $d(v_1, v_{2k+2}) + d(v_{2k+2}, v_{3k+3}) + d(v_{3k+3}, v_1) = k+1+\frac{k+1}{2}+\frac{k+1}{2} = 2(k+1) = 2d$.
Case (ii): For even k.
$d(v_{2k+2}, v_{3k+3}) = \frac{k}{2}+1$ and a path of length $\frac{k}{2}+1$ between v_{2k+2} and v_{3k+3} is $v_{2k+1} \to v_{2k+2+1.2} \to v_{2k+2+2.2} \to \ldots \to v_{2k+2+\frac{k}{2}.2} \to v_{2k+2+\frac{k}{2}+1} = v_{3k+2}$. Also, $d(v_{3k+3}, v_1) = \frac{k}{2}$ because $v_{3k+3} \to v_{3k+3+1.2} \to v_{3k+3+2.2} \to \ldots \to v_{3k+3+\frac{k}{2}.2} = v_{4k+3} = v_1$. Thus, $d(v_1, v_{2k+2}) + d(v_{2k+2}, v_{3k+3}) + d(v_{3k+3}, v_1) = k+1+\frac{k}{2}+$

$1 + \frac{k}{2} = 2(k+1) = 2d$. Therefore, for any three vertices u, v, w on the graphs $G(4k+2; \{1,2\})$,

$$d(u,v) + d(v,w) + d(w,u) \leq 2d.$$

□

The minimum distance between every other label (arranged in increasing order) in a multi-level distance labeling (or radio labeling) of $G(4k+2; \{1,2\})$ is determined by using this maximum possible sum of the pairwise distances between any three vertices of $G(4k+2; \{1,2\})$ together with the radio condition.

Lemma 3.3. *Let f be radio labeling for $V(G(4k+2; \{1,2\}))$, where $\{x_i : 1 \leq i \leq 4k+2\}$ be the ordering of $V(G(4k+2; \{1,2\}))$ such that $f(x_i) < f(x_{i+1})$ for all $1 \leq i \leq 4k+1$, then*

$$f(x_{i+2}) - f(x_i) = f_i + f_{i+1} \geq \begin{cases} \dfrac{k+4}{2}, & \textit{if } k \textit{ is even;} \\ \dfrac{k+5}{2}, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof. By definition,

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\geq d + 1 - d(x_{i+1}, x_i), \\ f(x_{i+2}) - f(x_{i+1}) &\geq d + 1 - d(x_{i+2}, x_{i+1}), \\ f(x_{i+2}) - f(x_i) &\geq d + 1 - d(x_{i+2}, x_i). \end{aligned}$$

Summing these inequalities yields

$$2(f(x_{i+2}) - f(x_i)) \geq 3d + 3 - [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_i, x_{i+2})].$$

Furthermore, by Lemma 4, $d(u,v) + d(v,w) + d(w,u) \leq 2d$, so we have

$$2(f(x_{i+2}) - f(x_i)) \geq 3d + 3 - 2d = d + 3.$$

As $d = \text{diam}(G(4k+2; \{1,2\})) = k+1$, it follows that

$$(f(x_{i+2}) - f(x_i)) \geq \frac{d+3}{2} = \frac{k+4}{2}.$$

Thus

$$f(x_{i+2}) - f(x_i) = f_i + f_{i+1} \geq \begin{cases} \dfrac{k+4}{2}, & \text{if } k \text{ is even;} \\ \dfrac{k+5}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

□

The above Lemma makes it possible to calculate the minimum possible span of a radio labeling of $G(4k+2; \{1,2\})$.

Theorem 3.4. *The radio number of the circulant graphs $G(4k+2; \{1,2\})$ satisfies*

$$rn\,(G(4k+2; \{1,2\}) \geq \begin{cases} k^2 + 5k + 1, & \textit{if } k \textit{ is odd;} \\ k^2 + 4k + 1, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. Let f be a distance labeling for $G(4k{+}2;\{1,2\})$ and $\{x_1, x_2, x_3, ..., x_{4k+2}\}$ be the ordering of vertices of $G(4k{+}2;\{1,2\})$, such that $f(x_i) < f(x_{i+1})$ defined by $f(x_1) = 0$ and, set $d_i = d(x_i, x_{i+1})$ and $f_i = f(x_{i+1}) - f(x_i)$. Then $f_i \geq d{+}1{-}d_i$ for all i. By Lemma 5, the span of a distance labeling for $G(4k{+}2;\{1,2\})$ is

$$\begin{aligned}
f(x_{4k+2}) &= \sum_{i=1}^{4k+1} f_i = f_1 + f_2 + f_3 + ... + f_{4k} + f_{4k+1} \\
&= [f(x_2) - f(x_1)] + [f(x_3) - f(x_2)] + ... + [f(x_{4k+1}) - f(x_{4k})] \\
&\quad + [f(x_{4k+2}) - f(x_{4k+1})] \\
&= (f_1 + f_2) + (f_3 + f_4) + (f_5 + f_6) + ... + (f_{4k-1} + f_{4k}) + f_{4k+1} \\
&= \sum_{i=1}^{\frac{4k}{2}} (f_{2i-1} + f_{2i}) + f_{4k+1}
\end{aligned}$$

Thus,

$$f(x_{4k+2}) \geq \begin{cases} \dfrac{4k}{2}\left(\dfrac{k+5}{2}\right) + 1, & \text{if } k \text{ is odd;} \\ \dfrac{4k}{2}\left(\dfrac{k+4}{2}\right) + 1, & \text{if } k \text{ is even.} \end{cases}$$

$$f(x_{4k+2}) \geq \begin{cases} k^2 + 5k + 1, & \text{if } k \text{ is odd;} \\ k^2 + 4k + 1, & \text{if } k \text{ is even.} \end{cases}$$

□

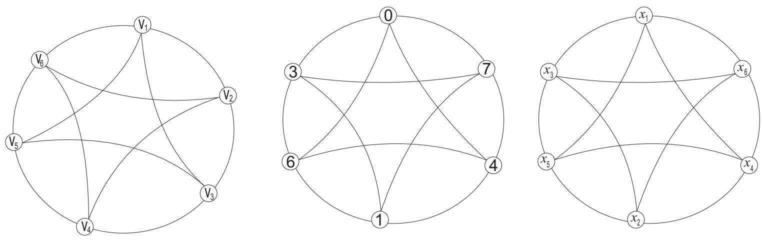

FIGURE 1. Radio labeling and ordinary labeling of $G(6;\{1,2\})$

3.2. Upper bound for rn$G(4k{+}2;\{1,2\})$. To complete the proof of Theorem 1, we find upper bound and show that this upper bound is equal to the lower bound for $G(4k+2;\{1,2\})$. The labeling is generated by three sequences, the distance gap sequence

$$D = (d_1, d_2, d_3, ..., d_{4k+1}),$$

the color gap sequence

$$F = (f_1, f_2, f_3, ..., f_{4k+1}),$$

and the vertex gap sequence T

$$T = (t_1, t_2, t_3, ..., t_{4k+1}).$$

For odd k. The distance gap sequence is given by:

$$d_i = \begin{cases} k+1, & \text{if } i \text{ is odd;} \\ \dfrac{k+1}{2}, & \text{if } i \text{ is even.} \end{cases}$$

The color gap sequence F is given by:

$$f_i = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ \dfrac{k+3}{2}, & \text{if } i \text{ is even.} \end{cases}$$

For even k. The distance gap sequence is given by:

$$d_i = \begin{cases} k+1, & \text{if } i \text{ is odd;} \\ \dfrac{k}{2}+1, & \text{if } i \text{ is even.} \end{cases}$$

The color gap sequence F is given by:

$$f_i = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ \dfrac{k+2}{2}, & \text{if } i \text{ is even.} \end{cases}$$

The vertex gap sequence for all values of k is:

$$t_i = \begin{cases} 2k, & \text{if } i \text{ is odd;} \\ k, & \text{if } i \equiv 0 (\text{mod} 4). \end{cases}$$

Where t_i denotes number of vertices between x_i and x_{i+1}.
Let $\pi : \{1, 2, 3, ..., 4k+2\} \to \{1, 2, 3, ..., 4k+2\}$ be defined by $\pi(1) = 1$ and

$$\pi(i+1) = \pi(i) + t_i + 1 \pmod{4k+2}$$

Let $x_i = u_{\pi(i)}$ for $i = 1, 2, 3, ..., 4k+2$. Then $x_1, x_2, x_3, ..., x_{4k+2}$ is an ordering of the vertices of G, assuming $f(x_1) = 0$, $f(x_{i+1}) = f(x_i) + f_i$. Then for $i = 1, 2, 3, ..., 2k+2$,

$$\pi(2i) = (3i-1)k + 2i \pmod{4k+2},$$

and for $i = 1, 2, ...2k+2$,

$$\pi(2i+1) = 3(i-1)k + 2i - 1 \pmod{4k+2}.$$

We will show that each of the sequences given above, the corresponding π are permutations. For odd k, g.c.d.$(4k+2, k) = 1$ and $3k+2 \equiv -k \pmod{4k+2}$ implies that $(3k+2)(i-i') \equiv k(i'-i) \not\equiv 0 \pmod{4k+2}$. Because if it does so then $k(i'-i) \equiv k.0 \pmod{4k+2}$ and $i'-i \equiv 0 \pmod{4k+2}$ which is impossible when $0 < i - i' < \frac{4k+2}{2}$. Therefore $\pi(2i-1) \neq \pi(2i'-1)$, if $i \neq i'$. Similarly $\pi(2i) \neq \pi(2i')$, if $i \neq i'$. If $\pi(2i) = \pi(2i'-1)$, then we get

$$(3i-1)k + 2i = 3(i'-1)k + 2i' - 1,$$

$$(i-i')(3k+2) = -2k-1 \equiv 2k+1 \pmod{4k+2},$$
$$2(i'-i)k \equiv 0 \pmod{4k+2}.$$

As k is odd and g.c.d.$(4k+2, k) = 1$ it follows that $i' - i \equiv 0 \pmod{4k+2}$. This implies that $4k+2$ divides $i' - i < 2k+1$, which is not possible.
When k is odd, then span of f is equal to:

$$\begin{aligned}
& f_1 + f_2 + f_3 +, ..., f_{4k} + f_{4k+1} \\
&= [(f_1 + f_3 + f_5 +, ..., + f_{4k+1})] + [(f_2 + f_4 + f_6 +, ..., + f_{4k})] \\
&= \frac{4k+2}{2}(1) + \frac{4k+2-2}{2}\left(\frac{k+3}{2}\right) \\
&= k^2 + 5k + 1.
\end{aligned}$$

For even k, g.c.d.$(4k+2, k) = 2$ and $3k+2 \equiv -k \pmod{4k+2}$ implies that $(3k+2)(i-i') \equiv k(i'-i) \not\equiv 0 \pmod{4k+2}$. Because if it does so then $k(i'-i) \equiv k.0 \pmod{4k+2}$ and $i' - i \equiv 0 \pmod{\frac{4k+2}{2}}$ which is impossible when $0 < i - i' < \frac{4k+2}{2}$. Therefore $\pi(2i-1) \neq \pi(2i'-1)$, if $i \neq i'$. Similarly $\pi(2i) \neq \pi(2i')$, if $i \neq i'$. If $\pi(2i) = \pi(2i'-1)$, then

$$(3i-1)k + 2i = 3(i'-1)k + 2i' - 1,$$
$$(i-i')(3k+2) = -2k-1 \equiv 2k+1 \pmod{4k+2},$$
$$2(i'-i)k \equiv 0 \pmod{4k+2}.$$

As k is even and g.c.d.$(4k+2, k) = 2$ it follows that $i' - i \equiv 0 \pmod{\frac{4k+2}{2}}$. Which is not possible.
When k is even, then span of f is equal to:

$$\begin{aligned}
& f_1 + f_2 + f_3 +, ..., f_{4k} + f_{4k+1} \\
&= [(f_1 + f_3 + f_5 +, ..., + f_{4k+1})] + [(f_2 + f_4 + f_6 +, ..., + f_{4k})] \\
&= \frac{4k+2}{2}(1) + \frac{4k+2-2}{2}\left(\frac{k+2}{2}\right) \\
&= k^2 + 4k + 1.
\end{aligned}$$

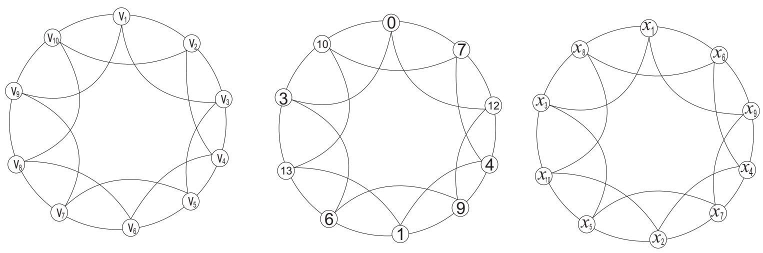

FIGURE 2. Radio labeling and ordinary labeling of $G(10;\{1,2\})$

4. Radio antipodal number for $G(4k+2;\{1,2\})$

In this section, the lower and upper bound for the radio antipodal number are determined and have shown that these bounds are equal.

4.1. Lower bound for an$(G(4k+2;\{1,2\})$. The technique for finding the lower bound for an$(G(4k+2;\{1,2\})$ is analogous to that of rn$(G(4k+2;\{1,2\})$.

Lemma 4.1. *Let f be radio antipodal labeling for $V(G(4k+2;\{1,2\}))$, where $\{x_i : 1 \le i \le 4k+2\}$ be the ordering of $V(G(4k+2;\{1,2\}))$ such that $f(x_i) \le f(x_{i+1})$ for all $1 \le i \le 4k+1$, then*

$$f(x_{i+2}) - f(x_i) = f_i + f_{i+1} \ge \begin{cases} \frac{k+2}{2}, & \text{if } k \text{ is even;} \\ \frac{k+1}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. By definition, $f(x_{i+1}) - f(x_i) \ge d - d(x_{i+1}, x_i)$, $f(x_{i+2}) - f(x_{i+1}) \ge d - d(x_{i+2}, x_{i+1})$ and $f(x_{i+2}) - f(x_i) \ge d - d(x_{i+2}, x_i)$. Summing up these three in-equalities and by Lemma 4, we get

$$2(f(x_{i+2}) - f(x_i)) \ge 3d - [d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_i, x_{i+2})]$$
$$2(f(x_{i+2}) - f(x_i)) \ge 3d - 2d = d$$
$$(f(x_{i+2}) - f(x_i)) \ge \frac{d}{2} = \frac{k+1}{2}$$

Thus,

$$f(x_{i+2}) - f(x_i) = f_i + f_{i+1} \ge \begin{cases} \frac{k+2}{2}, & \text{if } k \text{ is even;} \\ \frac{k+1}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

□

Theorem 4.2. *The radio antipodal number of the circulant graphs $G(4k+2;\{1,2\})$ is given by*

$$rn(G(4k+2;\{1,2\}) \ge \begin{cases} k^2+k, & \text{if } k \text{ is odd;} \\ k^2+2k, & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let f be a distance labeling for $G(4k+2;\{1,2\})$ and $\{x_1, x_2, x_3, ..., x_{4k+2}\}$ be the ordering of vertices of $G(4k+2;\{1,2\})$, such that $f(x_i) \le f(x_{i+1})$ defined by $f(x_1) = 0$ and, set $d_i = d(x_i, x_{i+1})$ and $f_i = f(x_{i+1}) - f(x_i)$. Then $f_i \ge d - d_i$ for all i. By Lemma 7, the span of a distance labeling for $G(4k+2;\{1,2\})$ is

$$\begin{aligned} f(x_{4k+2}) &= \sum_{i=1}^{n-1} f_i = f_1 + f_2 + f_3 + ... + f_{4k} + f_{4k+1} \\ &= [f(x_2) - f(x_1)] + [f(x_3) - f(x_2)] + ... + [f(x_{4k+1}) - f(x_{4k})] \\ &\quad + [f(x_{4k+2}) - f(x_{4k+1})] \end{aligned}$$

$$= (f_1+f_2)+(f_3+f_4)+(f_5+f_6)+...+(f_{4k-1}+f_{4k})+f_{4k+1}$$
$$= \sum_{i=1}^{\frac{4k}{2}}(f_{2i-1}+f_{2i})+f_{4k+1}$$

Thus,

$$f(x_{4k+2}) \geq \begin{cases} \frac{4k}{2}\left(\frac{k+1}{2}\right)+0, & \text{if } k \text{ is odd;} \\ \frac{4k}{2}\left(\frac{k+2}{2}\right)+0, & \text{if } k \text{ is even.} \end{cases} \Rightarrow f(x_{4k+2}) \geq \begin{cases} k^2+k, & \text{if } k \text{ is odd;} \\ k^2+2k, & \text{if } k \text{ is even.} \end{cases}$$

□

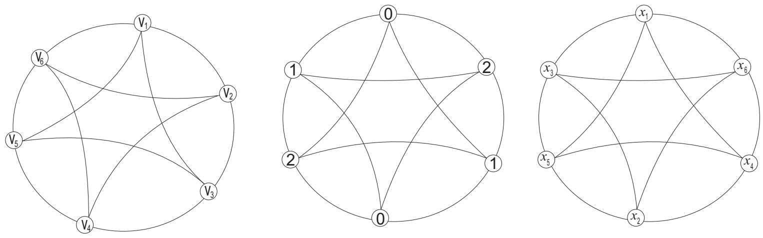

FIGURE 3. Radio antipodal labeling and ordinary labeling of $G(6;\{1,2\})$

4.2. Upper bound for an$(G(4k+2;\{1,2\})$**.** To complete the proof of Theorem 2, we find upper bound and show that this upper bound is same as the lower bound for $\text{an}(G(4k+2;\{1,2\}))$. The technique for an upper bound of $\text{an}(G(4k+2;\{1,2\})$ is analogous to that of $\text{rn}(G(4k+2;\{1,2\})$, with replacing the color gap sequence.

For odd k. The color gap sequence F is given by:

$$f_i = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ \frac{k+1}{2}, & \text{if } i \text{ is even.} \end{cases}$$

For even k. The color gap sequence F is given by:

$$f_i = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ \frac{k+2}{2}, & \text{if } i \text{ is even.} \end{cases}$$

When k is odd, then span of f is equal to:

$$f_1+f_2+f_3+,...,f_{4k}+f_{4k+1}$$
$$= [(f_1+f_3+f_5+,...,+f_{4k+1})]+[(f_2+f_4+f_6+,...,+f_{4k})]$$
$$= \frac{4k+2}{2}(0)+\frac{4k+2-2}{2}\left(\frac{k+1}{2}\right)$$

$$= k^2 + k.$$

When k is even, then span of f is equal to:

$$\begin{aligned} & f_1 + f_2 + f_3 +, ..., f_{4k} + f_{4k+1} \\ &= [(f_1 + f_3 + f_5 +, ..., + f_{4k+1})] + [(f_2 + f_4 + f_6 +, ..., + f_{4k})] \\ &= \frac{4k+2}{2}(0) + \frac{4k+2-2}{2}\left(\frac{k+2}{2}\right) \\ &= k^2 + 2k. \end{aligned}$$

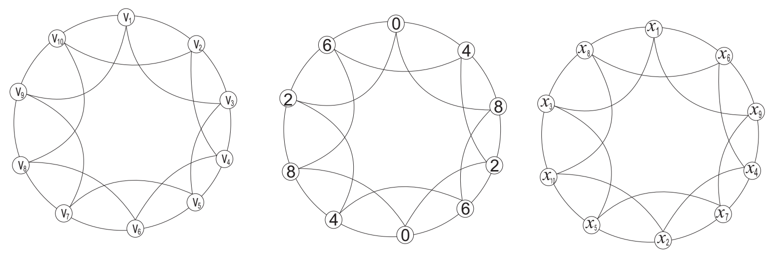

FIGURE 4. Radio antipodal labeling and ordinary labeling of $G(10; \{1, 2\})$

REFERENCES

1. G. Chartrand, L. Nebesk'y and P. Zhang, *Radio k-colorings of paths,* Discussiones Mathematicae Graph Theory, **24** (2004), 5-21.
2. G. Chartrand, D. Erwin, and P. Zhang, *Radio antipodal colourings of cycles,* Congressus Numerantium, **144** (2000), 129-141.
3. G. Chartrand, D. Erwin, and P. Zhang, *Radio antipodal colourings of graphs,* Math. Bohemica, **127** (2002), 57-69.
4. G. Chartrand, D. Erwin, and P. Zhang, and F. Harary, *Radio labelings of grphs,* Bull. Inst. Combin. appl., **33** (2001), 77-85.
5. W.K. Hale, *Frequency assignment: theory and application,* Proc IEEE, **68** (1980), 1497-1514.
6. J.S. Juan and D.F. Liu, *Antipodal labelings for cycles,* Dec. 12, 2006 (Manuscript).
7. M. Kchikech, R. Khenoufa and O. Togni, *Linear and cyclic radio k-labelings of trees,* Discussiones Mathematicae Graph Theory, **27** (2007), 105-123.
8. M. Kchikech, R. Khenoufa and O. Togni, *Radio k-labelings for cartesian products of graphs,* Discussion Mathematicae Graph Theory.
9. R. Khennoufa and O. Togni, *A note on radio antipodal colourings of paths,* Math. Bohemica, **130** (2005), 277-282.
10. R. Khennoufa and O. Togni, *The radio antipodal and radio numbers of the hypercube,* Ars Combinatoria, **102** (2011), 447-461.
11. D. Liu and X. Zhu, *Multi-level distance labelings for paths and cycles,* SIAM J. Disc. Math., **19** (2005), 610-621.

12. D. Liu and M. Xie, *Radio number for square cycles,* Congr. Numerantium, **169** (2004), 105-125.

Saima Nazeer received M.Sc. from University of the Punjab, Lahore-Pakistan, and Ph.D. at University of the Punjab, Lahore-Pakistan. She is currently assistant Professor at Lahore College for Women University, Lahore-Pakistan. Her research area is graph theory.

Department of Mathematics, Lahore College for Women University, Lahore-Pakistan.
e-mail: saimanazeer123@yahoo.com

Imrana Kousar received M.Sc. from University of the Punjab, Lahore-Pakistan, and Ph.D. from University of the Punjab, Lahore-Pakistan,. She is currently assistant Professor at Lahore College for Women University, Lahore-Pakistan. Her research area is graph theory.

Department of Mathematics, Lahore College for Women University, Lahore-Pakistan.
e-mail: imrana.kousar@hotmail.com

Waqas Nazeer received M.Sc. from University of the Punjab, Lahore-Pakistan, and Ph.D. from Abdus Salam School of Mathematical Sciences, GC University, Lahore-Pakistan. He is currently assistant Professor at University of Education Township Lahore. His research interests are functional analysis and graph theory.

Department of Mathematics, University of Education Township Lahore, Lahore-Pakistan.
e-mail: waqaster@yahoo.com