REFINEMENTS OF HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS[†]

RUIYIN XIANG

ABSTRACT. In this note, two new mappings associated with convexity are propoesd, by which we obtain some new Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals. We conclude that the results obtained in this work are the refinements of the earlier results.

AMS Mathematics Subject Classification : 26D15, 26A51.
Key words and phrases : Hermite-Hadamard inequality, convex function, Riemann-Liouville fractional integrals.

1. Introduction

If $f : I \to R$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been extensively obtained by a number of authors (e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9] and [10]).

In [4], S. S. Dragomir proposed the following Hermite-Hadamard type inequalities which refine the first inequality of (1).

Theorem 1.1 ([4])**.** *Let f is convex on $[a, b]$. Then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have*

$$f\Big(\frac{a+b}{2}\Big) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x)dx, \tag{2}$$

where

$$H(t) = \frac{1}{b-a} \int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)dx.$$

An analogous result for convex functions which refines the second inequality of (1) is obtained by G. S. Yang and M. C. Hong in [13] as follows.

Theorem 1.2 ([13])**.** *Let f is convex on $[a, b]$. Then P is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have*

$$\frac{1}{b-a} \int_a^b f(x)dx = P(0) \leq P(t) \leq P(1) = \frac{f(a) + f(b)}{2}, \tag{3}$$

where

$$P(t) = \frac{1}{2(b-a)} \int_a^b \Big[f\Big(\Big(\frac{1+t}{2}\Big)a + \Big(\frac{1-t}{2}\Big)x\Big) + f\Big(\Big(\frac{1+t}{2}\Big)b + \Big(\frac{1-t}{2}\Big)x\Big)\Big]dx.$$

G. S. Yang and K. L. Tseng in [12] established some generalizations of (2) and (3) based on the following results.

Theorem 1.3 ([12])**.** *Let $f : [a, b] \to R$ be a convex function, $0 < \alpha < 1$, $0 < \beta < 1$, $A = \alpha a + (1 - \alpha)b$, $u_0 = (b - a)\min\{\frac{\alpha}{1-\beta}, \frac{1-\alpha}{\beta}\}$, and let h be defined by $h(t) = (1 - \beta)f(A - \beta t) + \beta f(A + (1 - \beta)t)$, $t \in [0, u_0]$. Then h is convex, increasing on $[0, u_0]$ and for all $t \in [0, u_0]$,*

$$f(\alpha a + (1 - \alpha)b) \leq h(t) \leq \alpha f(a) + (1 - \alpha)f(b).$$

It is remarkable that M. Z. Sarikaya et al. [11] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.4 ([11])**.** *Let $f : [a, b] \to R$ be a positive function with $a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}, \tag{4}$$

with $\alpha > 0$.

We remark that the symbols J_{a+}^α and $J_{b-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \geq 0$ with $a \geq 0$ which are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)dt, \qquad x > a,$$

and

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \qquad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

In this paper, we establish some new Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals which refine the inequalities of (4).

2. Main results

Lemma 2.1. *Let $f : [a,b] \to R$ be a convex function and h be defined by*

$$h(t) = \frac{1}{2}\Big[f\Big(\Big(\frac{a+b}{2}\Big) - \frac{t}{2}\Big) + f\Big(\Big(\frac{a+b}{2}\Big) + \frac{t}{2}\Big)\Big].$$

Then $h(t)$ is convex, increasing on $[0, b-a]$ and for all $t \in [0, b-a]$,

$$f\Big(\frac{a+b}{2}\Big) \leq h(t) \leq \frac{f(a) + f(b)}{2}.$$

Proof. We can obtain the result by taking $\alpha = \beta = \frac{1}{2}$ in Theorem 1.3. $\qquad\square$

Theorem 2.2. *Let $f : [a,b] \to R$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then WH is convex and monotonically increasing on $[0,1]$ and*

$$f\Big(\frac{a+b}{2}\Big) \quad = \quad WH(0) \leq WH(t) \leq WH(1)$$

$$= \quad \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)],$$

with $\alpha > 0$, where

$$WH(t) = \frac{\alpha}{2(b-a)^\alpha} \int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)\Big((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\Big) dx.$$

Proof. Firstly, let $t_1, t_2, \beta \in [0,1]$, then

$$WH[(1-\beta)t_1 + \beta t_2]$$

$$= \frac{\alpha}{2(b-a)^\alpha} \int_a^b f\Big(\Big(x - \frac{a+b}{2}\Big)[(1-\beta)t_1 + \beta t_2] + \frac{a+b}{2}\Big)\Big((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\Big) dx$$

$$= \frac{\alpha}{2(b-a)^\alpha} \int_a^b f\Big\{(1-\beta)\Big[\Big(x - \frac{a+b}{2}\Big)t_1 + \frac{a+b}{2}\Big] + \beta\Big[\Big(x - \frac{a+b}{2}\Big)t_2 + \frac{a+b}{2}\Big]\Big\}$$

$$\Big((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\Big) dx.$$

Since f is convex, we get

$$f\Big\{(1-\beta)\Big[\Big(x - \frac{a+b}{2}\Big)t_1 + \frac{a+b}{2}\Big] + \beta\Big[\Big(x - \frac{a+b}{2}\Big)t_2 + \frac{a+b}{2}\Big]\Big\}$$

$$\leq (1-\beta)f\Big[\Big(x - \frac{a+b}{2}\Big)t_1 + \frac{a+b}{2}\Big] + \beta f\Big[\Big(x - \frac{a+b}{2}\Big)t_2 + \frac{a+b}{2}\Big].$$

So

$$WH[(1 - \beta)t_1 + \beta t_2]$$

$$\leq \frac{\alpha}{2(b-a)^\alpha}(1 - \beta)\int_a^b f\left(\left(x - \frac{a+b}{2}\right)t_1 + \frac{a+b}{2}\right)\left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right)dx$$

$$+ \frac{\alpha}{2(b-a)^\alpha}\beta\int_a^b f\left(\left(x - \frac{a+b}{2}\right)t_2 + \frac{a+b}{2}\right)\left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right)dx$$

$$= (1 - \beta)WH(t_1) + \beta WH(t_2),$$

from which we get WH is convex on $[0, 1]$. Next, by elementary calculus, we have

$$WH(t) = \frac{\alpha}{2(b-a)^\alpha}\int_a^b f\left(tx + (1 - t)\frac{a+b}{2}\right)\left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right)dx$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}} f\left(tx + (1 - t)\frac{a+b}{2}\right)\left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right)dx$$

$$+ \frac{\alpha}{2(b-a)^\alpha}\int_{\frac{a+b}{2}}^b f\left(tx + (1 - t)\frac{a+b}{2}\right)\left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right)dx$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_0^{b-a} f\left(\left(\frac{a+b}{2}\right) - \frac{tx}{2}\right)\left(\left(\frac{b-a}{2}\right) + \frac{x}{2}\right)^{\alpha-1} + \left(\left(\frac{b-a}{2}\right) - \frac{x}{2}\right)^{\alpha-1}dx$$

$$+ \frac{\alpha}{2(b-a)^\alpha}\int_0^{b-a} f\left(\left(\frac{a+b}{2}\right) + \frac{tx}{2}\right)\left(\left(\frac{b-a}{2}\right) + \frac{x}{2}\right)^{\alpha-1} + \left(\left(\frac{b-a}{2}\right) - \frac{x}{2}\right)^{\alpha-1}dx$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_0^{b-a}\left[f\left(\left(\frac{a+b}{2}\right) - \frac{tx}{2}\right) + f\left(\left(\frac{a+b}{2}\right) + \frac{tx}{2}\right)\right]\left(\left(\frac{b-a}{2}\right) + \frac{x}{2}\right)^{\alpha-1}$$

$$+ \left(\left(\frac{b-a}{2}\right) - \frac{x}{2}\right)^{\alpha-1}dx.$$

It follows from Lemma 2.1 that $h(x) = \frac{1}{2}\left[f\left(\left(\frac{a+b}{2}\right) - \frac{x}{2}\right) + f\left(\left(\frac{a+b}{2}\right) + \frac{x}{2}\right)\right]$ is increasing on $[0, b-a]$. Since $\left(\left(\frac{b-a}{2}\right) + \frac{x}{2}\right)^{\alpha-1} + \left(\left(\frac{b-a}{2}\right) - \frac{x}{2}\right)^{\alpha-1}\right)$ is nonnegative, hence $WH(t)$ is increasing on $[0, 1]$. Finally, from

$$f\left(\frac{a+b}{2}\right) = WH(0)$$

and

$$WH(1) = \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)],$$

we have completed the proof. $\square$

Similarly, we have the following theorem:

Theorem 2.3. *Let f be defined as in Theorem 2.2, then WP is convex and monotonically increasing on $[0, 1]$ and*

$$\frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] = WP(0) \leq WP(t) \leq WP(1)$$

$$= \frac{f(a) + f(b)}{2},$$

with $\alpha > 0$, where

$$WP(t) = \frac{\alpha}{4(b-a)^\alpha} \int_a^b \Big[f\Big(\Big(\frac{1+t}{2}\Big)a + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{2b-a-x}{2}\Big)^{\alpha-1} + \Big(\frac{x-a}{2}\Big)^{\alpha-1}\Big)$$
$$+ f\Big(\Big(\frac{1+t}{2}\Big)b + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{b-x}{2}\Big)^{\alpha-1} + \Big(\frac{x+b-2a}{2}\Big)^{\alpha-1}\Big)\Big] dx.$$

Proof. We note that if f is convex and g is linear, then the composition $f \circ g$ is convex. Also we note that a positive constant multiple of a convex function and a sum of two convex functions are convex, hence $f\Big(\Big(\frac{1+t}{2}\Big)a + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{2b-a-x}{2}\Big)^{\alpha-1} + \Big(\frac{x-a}{2}\Big)^{\alpha-1}\Big)$ and $f\Big(\Big(\frac{1+t}{2}\Big)b + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{b-x}{2}\Big)^{\alpha-1} + \Big(\frac{x+b-2a}{2}\Big)^{\alpha-1}\Big)$ are convex, from which we get that $WP(t)$ is convex. Next, by elementary calculus, we have

$$WP(t) = \frac{\alpha}{4(b-a)^\alpha} \int_a^b \Big[f\Big(\Big(\frac{1+t}{2}\Big)a + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{2b-a-x}{2}\Big)^{\alpha-1} + \Big(\frac{x-a}{2}\Big)^{\alpha-1}\Big)$$
$$+ f\Big(\Big(\frac{1+t}{2}\Big)b + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{b-x}{2}\Big)^{\alpha-1} + \Big(\frac{x+b-2a}{2}\Big)^{\alpha-1}\Big)\Big] dx$$
$$= \frac{\alpha}{4(b-a)^\alpha} \int_0^{b-a} \Big[f\Big(a + \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{2b-2a-x}{2}\Big)^{\alpha-1} + \Big(\frac{x}{2}\Big)^{\alpha-1}\Big)$$
$$+ f\Big(b - \Big(\frac{1-t}{2}\Big)x\Big)\Big(\Big(\frac{x}{2}\Big)^{\alpha-1} + \Big(\frac{2b-2a-x}{2}\Big)^{\alpha-1}\Big)\Big] dx$$
$$= \frac{\alpha}{4(b-a)^\alpha} \int_0^{b-a} \Big[f\Big(a + \Big(\frac{1-t}{2}\Big)x\Big) + f\Big(b - \Big(\frac{1-t}{2}\Big)x\Big)\Big]$$
$$\times \Big[\Big(\frac{2b-2a-x}{2}\Big)^{\alpha-1} + \Big(\frac{x}{2}\Big)^{\alpha-1}\Big] dx.$$

It follows from Lemma 2.1 that $h(t) = \frac{1}{2}\Big[f\Big(\Big(\frac{a+b}{2}\Big) - \frac{t}{2}\Big) + f\Big(\Big(\frac{a+b}{2}\Big) + \frac{t}{2}\Big)\Big]$ and $k(t) = b - a - (1-t)x$ are increasing on $[0, b-a]$ and $[0,1]$, respectively. Hence $h(k(t)) = f\Big(a + \Big(\frac{1-t}{2}\Big)x\Big) + f\Big(b - \Big(\frac{1-t}{2}\Big)x\Big)$ is increasing on $[0,1]$. Since $\Big(\frac{2b-2a-x}{2}\Big)^{\alpha-1} + \Big(\frac{x}{2}\Big)^{\alpha-1}$ is nonnegative, it follows that WP is monotonically increasing on $[0,1]$. Finally, from

$$WP(0) = \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)],$$

and

$$\frac{f(a) + f(b)}{2} = WP(1),$$

we get the desired result. $\qquad\qquad\square$

Corollary 2.4. *With assumptions in Theorem 2.2, if $\alpha = 1$, we get*

$$WH(t) = \frac{1}{(b-a)} \int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big) dx = H(t),$$

where $H(t)$ is defined as Theorem 1.1, which is just the result in Theorem 1.1.

Corollary 2.5. *With assumptions in Theorem 2.3, if $\alpha = 1$, we get*

$$WP(t) = \frac{1}{2(b-a)} \int_a^b \Big[f\Big(\Big(\frac{1+t}{2}\Big)a + \Big(\frac{1-t}{2}\Big)x\Big) + f\Big(\Big(\frac{1+t}{2}\Big)b + \Big(\frac{1-t}{2}\Big)x\Big)\Big] dx = P(t),$$

where $P(t)$ is defined as Theorem 1.2, which is just the result in Theorem 1.2.

3. Conclusion

In this note, we obtain some new Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals. We conclude that the results obtained in this work are the refinements of the earlier results. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequalities for convex functions on the co-ordinates via Riemann-Liouville fractional integrals.

Acknowledgment

References

1. S. Abramovich, G. Farid, and J. Pečarić, More about Hermite-Hadamard inequalities, Cauchy's means, and superquadracity, *J. Inequal. Appl.,* **2010** (2010), Article ID 102467.
2. M. Bessenyei and Z. Páles, Hadamard-type inequalities for generalized convex functions, *Math. Inequal. Appl.,* **6** (2003), 379-392.
3. F.X. Chen, On Hermite-Hadamard type inequalities for *s*-convex functions on the coordinates via Riemann-Liouville fractional integrals, *J. Appl. Math.,* **2014** (2014), Article ID 248710, 8 pages.
4. S.S. Dragomir, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.,* **167** (1992), 49-56.
5. S.S. Dragomir and G.H. Toader, Some inequalities for *m*-convex functions, *Stud. Univ. Babes-Bolyai Math.,* **38** (1993), 21-28.
6. S.S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for *s*-convex functions in the second sense, *Demonstratio Math.,* **32** (1999), 687-696.
7. A.E. Farissi, Simple proof and refinement of Hermite-Hadamard inequality, *J. Math. Inequal.,* **4** (2010), 365-369.
8. X. Gao, A note on the Hermite-Hadamard inequality, *J. Math. Inequal.,* **4** (2010), 587-591.
9. U.S. Kirmaci, M.K. Bakula, M.E. Özdemir, J. Pečarić, Hadamard-type inequalities for *s*-convex functions, *Appl. Math. Comput.,* **1** (2007), 26-35.
10. M.Z. Sarikaya et al, On some Hadamard-type inequalities for *h*-convex functions, *J. Math. Inequal.,* **2** (2008), 335-341.
11. M.Z. Sarikaya et al, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.,* **57** (2013), 2403-2407.
12. G.S. Yang and K.L. Tseng, On certain integral inequalities related to Hermite-Hadamard inequalities, *J. Math. Anal. Appl.,* **239** (1999), 180-187.
13. G.S. Yang and M.C. Hong, A note on Hadamard's inequality, *Tamkang J. Math.,* **28** (1997), 33-37.

Ruiyin Xiang received M.Sc. from Chongqing University. Since 1995 he has been teaching at Chongqing Three Gorges University. His research interests include integral inequalities on several kinds of convex functions and applications.

School of Mathematics and Statistics, Chongqing Three-Gorges University, Wanzhou, Chongqing, 404000, P.R.China.
`e-mail:  xiangry@163.com`