ESOR METHOD WITH DIAGONAL PRECONDITIONERS FOR SPD LINEAR SYSTEMS†

SEYOUNG OH, JAE HEON YUN*, KYOUM SUN KIM

ABSTRACT. In this paper, we propose an extended SOR (ESOR) method with diagonal preconditioners for solving symmetric positive definite linear systems, and then we provide convergence results of the ESOR method. Lastly, we provide numerical experiments to evaluate the performance of the ESOR method with diagonal preconditioners.

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1. Introduction

In this paper, we consider an iterative method for solving the following linear system

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a symmetric positive definite (SPD) matrix. The basic iterative method [6, 7] for solving the linear system (1) can be expressed as

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b, \ k = 0, 1, \dots, \tag{2}$$

where x_0 is an initial vector and $A = M - N$ is a splitting of A. To improve the convergence rate of the basic iterative method, the original linear system (1) is usually transformed into the following preconditioned linear system

$$PAx = Pb, \tag{3}$$

where P is called a preconditioner. Then the preconditioned iterative method [1, 2, 3, 5, 8] for solving the linear system (3) is

$$x_{k+1} = M_p^{-1}N_px_k + M_p^{-1}Pb, \ k = 0, 1, \dots, \tag{4}$$

where x_0 is an initial vector and $PA = M_p - N_p$ is a splitting of PA. It is well known that the necessary and sufficient condition for the iterative method (4) to converge for any x_0 is $\rho(M_p^{-1}N_p) < 1$ (see [6, 7]).

Throughout the paper, we assume that $A = D - L - U$, where $D = \mathrm{diag}(A)$ is the diagonal matrix, and L and U are strictly lower triangular and strictly upper triangular matrices, respectively. For a vector $x \in \mathbb{R}^n$, $\|x\|_2$ denotes ℓ_2-norm of x and x^* denotes the conjugate transpose of x. For a square matrix B, $\rho(B)$ denotes the spectral radius of B.

In this paper, we only consider diagonal preconditioners P which are diagonal matrices. Recently, Tarazaga and Cuellar [4] proposed two diagonal preconditioners which were obtained by minimizing the norm of the iteration matrix using the Frobenius norm and the infinity norm. The diagonal preconditioner P_F obtained by minimizing the Frobenius norm is given by

$$P_F = \mathrm{diag}\left(\frac{a_{11}}{\|a_1\|_2^2}, \frac{a_{22}}{\|a_2\|_2^2}, \cdots, \frac{a_{nn}}{\|a_n\|_2^2} \right), \tag{5}$$

where a_i stands for the ith row of the matrix A. The diagonal preconditioner P_I obtained by minimizing the infinity norm is given by $P_I = \alpha I$, where $\alpha = \frac{2}{\|A\|_\infty + \mathrm{sg}(A)}$ with $\mathrm{sg}(A) = \min_{1 \le i \le n} \left(|a_{ii}| - \sum_{i \ne j}^{n} |a_{ij}| \right)$ and I denotes the identity matrix of order n. It was shown in [4] that $\rho(I - P_F A) < 1$ and $\rho(I - P_I A) < 1$ when A is a strictly diagonal dominant matrix with positive diagonal elements.

We now propose an *extended SOR (ESOR) method* with diagonal preconditioner P for solving the preconditioned linear system (3), which is defined by

$$x_{k+1} = (I - \omega PL)^{-1}(I - \omega P(D - U))x_k + \omega(I - \omega PL)^{-1}Pb, \ k = 0, 1, \ldots, \tag{6}$$

where $\omega > 0$ is a relaxation parameter. If we rearrange the equation (6), then the ESOR method can be rewritten as

$$x_{k+1} = H_P x_k + B_P b, \ k = 0, 1, \ldots, \tag{7}$$

where $H_P = I - \omega(P^{-1} - \omega L)^{-1}A$ and $B_P = \omega(P^{-1} - \omega L)^{-1}$. The H_P is called the *iteration matrix* for the ESOR method with diagonal preconditioner P. It is easy to see that the ESOR method reduces to the SOR method if $P = D^{-1}$. In this respect, the ESOR method can be viewed as an extension of the SOR method.

This paper is organized as follows. In Section 2, we provide convergence results of the ESOR method with diagonal preconditioners including the P_F and P_I proposed in [4]. In Section 3, we provide numerical experiments to evaluate the performance of the ESOR method with diagonal preconditioners. Lastly, some conclusions are drawn.

2. Convergence results of the ESOR method

In this section, we consider convergence of the ESOR method with diagonal preconditioner for solving the preconditioned linear system (3).

Theorem 2.1. *Let $A = (a_{ij}) = D - L - U$ be a symmetric positive definite matrix and $P = (p_{ij})$ be a diagonal matrix with positive diagonal elements. If $0 < \omega < \min\limits_{i} \dfrac{2}{a_{ii}p_{ii}}$, then the ESOR method with the diagonal preconditioner P converges for any x_0.*

Proof. Notice that $a_{ii} > 0$ for all i, $U = L^T$ and
$$H_{P_F} = I - \omega(P^{-1} - \omega L)^{-1}A = (P^{-1} - \omega L)^{-1}(P^{-1} - \omega L - \omega A).$$
It is sufficient to show that $\rho(H_P) < 1$. Assume that $H_P x = \lambda x$, where x is a nonzero vector. Then, one obtains
$$-\omega A x = (\lambda - 1)(P^{-1} - \omega L)x. \tag{8}$$
By premultiplying x^* on both sides of equation (8),
$$-\omega x^* A x = (\lambda - 1)x^*(P^{-1} - \omega L)x. \tag{9}$$
Since A is positive definite, it can be easily shown that $\lambda \neq 1$. Taking the complex conjugate transpose on both sides of equation (9),
$$-\omega x^* A x = (\lambda^* - 1)x^*(P^{-1} - \omega U)x. \tag{10}$$
Adding two equations (9) and (10), one obtains
$$-\left(\frac{1}{\lambda - 1} + \frac{1}{\lambda^* - 1}\right) = \frac{x^*(\frac{2}{\omega}P^{-1} - D + A)x}{x^* A x}. \tag{11}$$
Let $E = \frac{2}{\omega}P^{-1} - D$. Then E is a diagonal matrix whose diagonal element is given by $\frac{2}{\omega p_{ii}}$ for every i. Since $0 < \omega < \frac{2}{a_{ii}p_{ii}}$ for all i, every diagonal element of E is positive and so E is positive definite. Hence, (11) implies that
$$-\left(\frac{1}{\lambda - 1} + \frac{1}{\lambda^* - 1}\right) > 1. \tag{12}$$
By simple calculation, one easily obtains $|\lambda| < 1$. Hence, the proof is complete.
$\square$

Since the ESOR method with $P = D^{-1}$ becomes the SOR method, the following well-known property is immediately obtained from Theorem 2.1.

Corollary 2.2 ([6]). *Let $A = D - L - U$ be a symmetric positive definite matrix. If $0 < \omega < 2$, then the SOR method converges for any x_0.*

Corollary 2.3. *Let $A = (a_{ij}) = D - L - U$ be a symmetric positive definite matrix. If $0 < \omega < 2\min\limits_{i} \dfrac{\|a_i\|_2^2}{a_{ii}^2}$, then the ESOR method with $P = P_F$ converges for any x_0.*

Proof. Since $p_{ii} = \frac{a_{ii}}{\|a_i\|_2^2}$, the proof is directly obtained from Theorem 2.1.   □

Corollary 2.4. *Let $A = (a_{ij})$ be a symmetric positive definite matrix. If $0 < \omega < 2$, then the ESOR method with $P = P_F$ converges for any x_0.*

Proof. Since $\|a_i\|_2^2 \geq a_{ii}^2$ for all i, $\min_i \frac{\|a_i\|_2^2}{a_{ii}^2} \geq 1$. Hence, this corollary follows from Corollary 2.3.   □

Corollary 2.5. *Let $A = (a_{ij})$ be a symmetric strictly diagonally dominant or irreducibly diagonally dominant matrix with positive diagonal elements. If $0 < \omega < 2\min_i \dfrac{\|a_i\|_2^2}{a_{ii}^2}$, then the ESOR method with $P = P_F$ converges for any x_0.*

Proof. Since a symmetric strictly diagonally dominant or irreducibly diagonally dominant matrix with positive diagonal elements is positive definite, this corollary follows from Corollary 2.3.   □

Corollary 2.6. *Let $A = (a_{ij}) = D - L - U$ be a symmetric positive definite matrix. If A is strictly diagonally dominant and $0 < \omega < \frac{\|A\|_\infty + \mathrm{sg}(A)}{\|D\|_\infty}$, then the ESOR method with $P = P_I$ converges for any x_0.*

Proof. Note that $p_{ii} = \frac{2}{\|A\|_\infty + \mathrm{sg}(A)}$. From Theorem 2.1, the ESOR method with $P = P_I$ converges when $0 < \omega < \min_i \dfrac{\|A\|_\infty + \mathrm{sg}(A)}{a_{ii}} = \dfrac{\|A\|_\infty + \mathrm{sg}(A)}{\max_i a_{ii}}$. Since $\|D\|_\infty = \max_i a_{ii}$, the proof is complete.   □

Since $\frac{\|A\|_\infty + \mathrm{sg}(A)}{\|D\|_\infty} > 1$, the upper bound of ω in Corollary 2.6 is greater than 1.

Corollary 2.7. *Let A be a symmetric strictly diagonally dominant matrix with positive diagonal elements. If $0 < \omega < \frac{\|A\|_\infty + \mathrm{sg}(A)}{\|D\|_\infty}$, then the ESOR method with $P = P_I$ converges for any x_0.*

Corollary 2.8. *Let $A = (a_{ij}) = D - L - U$ be a symmetric positive definite matrix with $D = \beta I$, where β is a positive constant. If A is strictly diagonally dominant, then the ESOR method with $P = P_I$ is the same as the SOR method which converges for any x_0 when $0 < \omega < 2$.*

Proof. Since $D = \beta I$, $\|A\|_\infty + \mathrm{sg}(A) = 2\beta$. It follows that $P_I = \beta^{-1}I = D^{-1}$. Thus, the ESOR method with $P = P_I$ is the same as the SOR method. From Corollary 2.2, the ESOR method with $P = P_I$ converges when $0 < \omega < 2$.   □

3. Numerical experiments

In this section, we provide numerical experiments to evaluate the performance of the ESOR method with diagonal preconditioners. All numerical experiments are carried out using MATLAB. In Tables 1 to 4, ESOR(P_F) and ESOR(P_I) stand

for the ESOR methods with diagonal preconditioners P_F and P_I, respectively. Bold numbers in Tables 1 to 4 refer to the optimal performances for 3 different iterative methods. The first example considers the SPD matrix with constant diagonal and nonpositive off-diagonal entries.

Example 3.1. Consider the two dimensional Poisson's equation

$$-\Delta u = f(x, y) \text{ in } \Omega = (0, 1) \times (0, 1) \tag{13}$$

with the Dirichlet boundary condition on $\partial\Omega$. When the central difference scheme on a uniform grid with $m \times m$ interior node is applied to the discretization of the equation (13), we obtain a linear system $Ax = b$ whose coefficient matrix $A \in \mathbb{R}^{n \times n}$ is given by

$$A = I_m \otimes P + Q \otimes I_m,$$

where $\otimes$ denotes the Kronecker product, $n = m^2$, $P = \text{tridiag}(-1, 4, -1)$ and $Q = \text{tridiag}(-1, 0, -1)$ are $m \times m$ tridiagonal matrices. Note that this matrix A is a symmetric irreducibly diagonally dominant matrix. It is easy to compute that $2 \min_i \frac{\|a_i\|_2^2}{a_{ii}^2} = 2.25$, $\frac{\|A\|_\infty + \text{sg}(A)}{\|D\|_\infty} = 2$, and $\alpha = 0.25$. Since A has a constant diagonal (i.e., $D = 4I$), ESOR method with P_I is the same as SOR method from Corollary 2.7. Numerical results for Example 3.1 with $n = 10^2$ or $n = 15^2$ are provided in Table 1.

TABLE 1. Spectral radii for iteration matrices of ESOR and SOR methods for Example 3.1.

n	ω	ESOR(P_F)	ESOR(P_I)	SOR	n	ω	ESOR(P_F)	ESOR(P_I)	SOR
100	0.5	0.9799	0.9733	0.9733	225	0.5	0.9904	0.9873	0.9873
	1.0	0.9467	0.9206	0.9206		1.0	0.9746	0.9619	0.9619
	1.2	0.9263	0.8803	0.8803		1.6	0.9317	0.8275	0.8275
	1.6	0.8556	**0.6000**	**0.6000**		1.7	0.9178	**0.7000**	**0.7000**
	1.8	0.7783	0.8000	0.8000		1.8	0.8991	0.8000	0.8000
	1.9	0.6949	0.9000	0.9000		1.9	0.8720	0.9000	0.9000
	2.0	**0.6598**	1.0000	1.0000		2.0	0.8264	1.0000	1.0000
	2.2	0.8256	1.2000	1.2000		2.1	**0.7237**	1.1000	1.1000
	2.3	0.9085	1.3000	1.3000		2.3	0.8877	1.3000	1.3000

The second example considers the randomly generated SPD matrix with negative off-diagonal entries.

Example 3.2. Consider the SPD matrix $A \in \mathbb{R}^{n \times n}$ which is generated by using the following MATLAB functions:

$$\text{rand('state', 0); } b = \text{rand}(n, 1); \ A = 8I + \frac{20}{n} bb^T; \ A = \text{gallery('compar', } A);$$

Notice that $2 \min_i \frac{\|a_i\|_2^2}{a_{ii}^2} = 2$, $\frac{\|A\|_\infty + \text{sg}(A)}{\|D\|_\infty} = 2$, and $\alpha \approx 0.1220$ for $n = 100$ or 0.1235 for $n = 200$. Numerical results for Example 3.2 with $n = 100$ or $n = 200$ are provided in Table 2.

TABLE 2. Spectral radii for iteration matrices of ESOR and SOR methods
for Example 3.2.

n	ω	ESOR(P_F)	ESOR(P_I)	SOR	n	ω	ESOR(P_F)	ESOR(P_I)	SOR
100	0.5	0.9162	0.9158	0.9148	200	0.5	0.8760	0.8760	0.8752
	0.8	0.8378	0.8370	0.8345		0.8	0.7631	0.7630	0.7611
	1.0	0.7623	0.7609	0.7566		1.0	0.6557	0.6554	0.6520
	1.2	0.6490	0.6462	0.6377		1.2	0.4906	0.4900	0.4830
	1.3	0.5617	0.5573	0.5435		1.3	**0.3427**	**0.3414**	**0.3258**
	1.4	**0.4093**	**0.3979**	**0.4059**		1.4	0.4000	0.3955	0.4028
	1.5	0.5000	0.4910	0.5051		1.5	0.5000	0.4947	0.5025
	1.8	0.8000	0.7855	0.8024		1.8	0.8000	0.7918	0.8012
	2.0	1.0000	0.9814	1.0000		2.0	1.0000	0.9897	1.0000

The third example considers the randomly generated SPD matrix with positive off-diagonal entries.

Example 3.3. Consider the SPD matrix $A \in \mathbb{R}^{n \times n}$ which is generated by using the following MATLAB functions:

$$\text{rand('state', 0); } b = \text{rand}(n, 1); \ A = I + \frac{20}{n}bb^T;$$

Notice that $2\min_i \frac{\|a_i\|_2^2}{a_{ii}^2} \approx 2.003$ for $n = 100$ or 2.0001 for $n = 200$, $\frac{\|A\|_\infty + \text{sg}(A)}{\|D\|_\infty} = 2$, and $\alpha \approx 0.8366$ for $n = 100$ or 0.9100 for $n = 200$. Numerical results for Example 3.3 with $n = 100$ or $n = 200$ are provided in Table 3.

TABLE 3. Spectral radii for iteration matrices of ESOR and SOR methods
for Example 3.3.

n	ω	ESOR(P_F)	ESOR(P_I)	SOR	n	ω	ESOR(P_F)	ESOR(P_I)	SOR
100	0.4	0.8226	0.7110	0.6988	200	0.4	0.7556	0.6878	0.6803
	0.5	0.7764	0.6775	0.6709		0.5	0.6926	0.6530	0.6491
	0.6	0.7293	**0.6680**	**0.6698**		0.6	0.6500	**0.6451**	**0.6467**
	0.7	0.6814	0.6788	0.6886		0.7	**0.6354**	0.6595	0.6661
	0.8	0.6555	0.7025	0.7180		0.8	0.6378	0.6873	0.6973
	0.9	**0.6490**	0.7322	0.7511		0.9	0.6529	0.7206	0.7328
	1.0	0.6529	0.7635	0.7843		1.0	0.6759	0.7550	0.7683
	1.5	0.7453	0.8967	0.9174		1.5	0.8100	0.8979	0.9109
	1.8	0.8072	0.9528	0.9715		1.8	0.8756	0.9574	0.9692
	2.0	0.9997	0.9826	1.0000		2.0	0.9999	0.9890	1.0000

The last example considers the randomly generated SPD matrix whose off-diagonal entries contain both positive and negative numbers. More specifically, each of positive and negative entries takes 50% of all off-diagonal entries.

Example 3.4. Consider the SPD matrix $A \in \mathbb{R}^{n \times n}$ which is generated by using the following MATLAB functions:

$$\text{rand('state', 0); } b = \text{rand}(n, 1); \ b(1 : 2 : n) = -b(1 : 2 : n); \ A = I + \frac{25}{n}bb^T;$$

Notice that $2\min_i \frac{\|a_i\|_2^2}{a_{ii}^2} \approx 2.004$ for $n = 100$ or 2.0002 for $n = 200$, $\frac{\|A\|_\infty + \text{sg}(A)}{\|D\|_\infty} = 2$, and $\alpha \approx 0.8037$ for $n = 100$ or 0.8900 for $n = 200$. Numerical results for Example 3.4 with $n = 100$ or $n = 200$ are provided in Table 4.

TABLE 4. Spectral radii for iteration matrices of ESOR and SOR methods for Example 3.4.

n	ω	ESOR(P_F)	ESOR(P_I)	SOR	n	ω	ESOR(P_F)	ESOR(P_I)	SOR
100	0.4	0.8599	0.7407	0.7308	200	0.4	0.7955	0.7117	0.7117
	0.5	0.8234	**0.7212**	**0.7189**		0.5	0.7428	**0.6993**	**0.6984**
	0.6	0.7862	0.7229	0.7291		0.6	0.6978	0.7045	0.7091
	0.8	0.7108	0.7629	0.7800		0.7	**0.6862**	0.7250	0.7337
	0.9	0.7010	0.7892	0.8086		0.8	0.6877	0.7525	0.7638
	1.0	**0.6994**	0.8154	0.8359		0.9	0.6991	0.7819	0.7945
	1.1	0.7044	0.8402	0.8609		1.0	0.7165	0.8106	0.8238
	1.5	0.7582	0.9200	0.9388		1.5	0.8217	0.9222	0.9342
	1.8	0.8058	0.9623	0.9791		1.8	0.8742	0.9667	0.9774
	2.0	0.9995	0.9845	1.0000		2.0	0.9998	0.9901	1.0000

4. Conclusions

In this paper, we proposed the ESOR method with diagonal preconditioners and provided its convergence results. Numerical results are in good agreement with the theoretical results provided in Section 2 (see Tables 1 to 4). For SPD matrices with negative or nonpositive off-diagonal entries, the ESOR method does not have advantages compared eith the SOR method (see Tables 1 and 2). For SPD matrices containing many positive off-diagonal entries, the ESOR method with P_F performs better than the other two methods (see Tables 3 to 4). These observations are similar to those of the extended Jacobi method studied in [4]. It was also observed that the optimal number of ω is greater than 1 for SPD matrices with negative or nonpositive off-diagonal entries, while the optimal number of ω is not greater than 1 for SPD matrices containing many positive off-diagonal entries. All of these observations have not been proved theoretically, so further research should be required to prove these observations and find out additional advantages of the ESOR method with diagonal preconditioners.

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SeYoung Oh received M.Sc. from Seoul National University and Ph.D at University of Minnesota. Since 1992 he has been at Chungnam National University. His research interests include numerical optimization and biological computation.

Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea
e-mail: soh@cnu.ac.kr

Jae Heon Yun received M.Sc. from Kyungpook National University, and Ph.D. from Iowa State University. He is currently a professor at Chungbuk National University since 1991. His research interests are computational mathematics, iterative method and parallel computation.

Department of Mathematics, College of Natural Sciences, Chungbuk National University, Cheongju 362-763, Korea
e-mail: gmjae@chungbuk.ac.kr

Kyoum Sun Kim is a graduate student in Chungbuk National University. Her research interests are computational mathematics, new homotopy methods for solving functional equations.

Department of Mathematics, Chungbuk National University, Cheongju, 361-763, Korea.
e-mail: giunsun@naver.com