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MULTIPLICATIVELY WEIGHTED HARARY INDICES OF GRAPH OPERATIONS

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Abstract. In this paper, we present exact formulae for the multiplicatively weighted Harary indices of join, tensor product and strong product of graphs in terms of other graph invariants including the Harary index, Zagreb indices and Zagreb coindices. Finally, We apply our result to compute the multiplicatively weighted Harary indices of fan graph, wheel graph and closed fence graph.

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1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. For two simple graphs G and H their *tensor product,* denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Note that if G and H are connected graphs, then $G \times H$ is connected only if at least one of the graph is nonbipartite. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$, see Fig.1.

The *join* $G + H$ of graphs G and H is obtained from the disjoint union of the graphs G and H, where each vertex of G is adjacent to each vertex of H.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and

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other properties of chemical compounds [11]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index; for other related topological indices see [31].

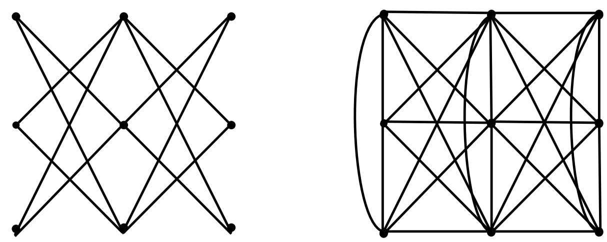

Fig.1. Tensor and strong product of C_3 and P_3

Let G be a connected graph. Then *Wiener index* of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,\, v \in V(G)} d_G(u,v)$ with the summation going over all pairs of distinct vertices of G. Similarly, the *Harary index* of G is defined as $H(G) = \frac{1}{2} \sum\limits_{u,\, v \in V(G)} \frac{1}{d_G(u,v)}$. The Harary index of a graph G has been introduced independently by Plavsic et al. [20] and by Ivanciuc et al. [16] in 1993. Its applications and mathematical properties are well studied in [**?**, 32, 19]. Zhou et al. [33] have obtained the lower and upper bounds of the Harary index of a connected graph. Very recently, Xu et al. [28] have obtained lower and upper bounds for the Harary index of a connected graph in relation to $\chi(G)$, chromatic number of G and $\omega(G)$, clique number of G. and characterized the extremal graphs that attain the lower and upper bounds of Harary index. Various topological indices on tensor product, Cartesian product and strong product have been studied various authors, see [2, 29, 30, 4, 21, 22, 23, 17, 13].

Dobrynin and Kochetova [5] and Gutman [12] independently proposed a vertex-degree-weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as $DD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. Hua and Zhang [14] introduced a new graph invariant named reciprocal degree distance, which can be seen as a degree-weight version of Harary index, that is, $RDD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$.

Similarly, the modified Schultz molecular topological index or Gutman index is defined as $DD_*(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} d_G(u) d_G(v) d_G(u,v)$. In Su et.al. [26] introduce the multiplicatively weighted Harary indices or reciprocal product-degree distance of graphs, which can be seen as a product -degree-weight version of Harray index $H_M(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)}$.

Hua and Zhang [14] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex and edge-connectivity. Pattabiraman and Vijayaragavan [24, 25] have obtained the exact expression for the reciprocal degree distance of join, tensor, strong and wreath product of graphs.

The *first Zagreb index* and *second Zagerb index* are defined as $M_1(G) = \sum\limits_{u \in V(G)} d_G(u)^2$ and $M_2(G) = \sum\limits_{uv \in E(G)} d_G(u)d_G(v)$. In fact, one can rewrite the first Zagreb index as $M_1(G) = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$. Similarly, the *first Zagreb coindex* and *second Zagerb coindex* are defined as $\overline{M}_1(G) = \sum\limits_{uv \notin E(G)} (d_G(u) + d_G(v))$ and $\overline{M}_2(G) = \sum\limits_{uv \notin E(G)} d_G(u)d_G(v)$. The Zagreb indices are found to have appilications in QSPR and QSAR studies as well, see [6]. For the survey on theory and application of Zagreb indices see [10]. Feng et al.[9] have given a sharp bounds for the Zagreb indices of graphs with a given matching number. Khalifeh et al. [18] have obtained the Zagreb indices of the Cartesian product, composition, join, disjunction and symmetric difference of graphs. Ashrafi et al. [3] determined the extremal values of Zagreb coindices over some special class of graphs. Hua and Zhang [15] have given some relations between Zagreb coindices and some other topolodical indices. Ashrafi et al. [1] have obtained the Zagreb indices of the Cartesian product, composition, join, disjunction and symmetric difference of graphs.

A path, cycle and complete graph on n vertices are denoted by P_n, C_n and K_n, respectively. We call C_3 a triangle. In this paper, we present exact formulae for the multiplicatively weighted Harary indices of join, tensor product and strong product of graphs in terms of other graph invariants including the Harary index, Zagreb indices and Zagreb coindices. Finally, We apply our result to compute the multiplicatively weighted Harary indices of fan graph, wheel graph and closed fence graph.

2. Multiplicatively weighted Harary index of $G_1 + G_2$

In this section, we compute the Multiplicatively Weighted Harary Index of join of two graphs.

Theorem 2.1. *Let G_1 and G_2 be graphs with n and m vertices p and q edges, respectively. Then* $H_M(G_1 + G_2) = M_2(G_1) + M_2(G_2) + mM_1(G_1) + nM_1(G_2) + \frac{1}{2}\Big(\overline{M}_2(G_1) + \overline{M}_2(G_2)\Big) + \frac{m}{2}\overline{M}_1(G_1) + \frac{n}{2}\overline{M}_1(G_2) + 4pq + 2mn(p+q) + \frac{1}{2}(m^2p + n^2q) + \frac{mn}{4}(6mn - m - n)$.

Proof. Set $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_m\}$. By definition of the join of two graphs, one can see that,

$$d_{G_1+G_2}(x) = \begin{cases} d_{G_1}(x) + |V(G_2)|, & \text{if } x \in V(G_1) \\ d_{G_2}(x) + |V(G_1)|, & \text{if } x \in V(G_2) \end{cases} \quad \text{and}$$

$$d_{G_1+G_2}(u,v) = \begin{cases} 0, & \text{if } u = v \\ 1, & \text{if } uv \in E(G_1) \text{ or } uv \in E(G_2) \text{ or } (u \in V(G_1) \text{ and } v \in V(G_2)) \\ 2, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} H_M(G_1 + G_2) &= \frac{1}{2} \sum_{u,v \in V(G_1+G_2)} \frac{d_{G_1+G_2}(u) d_{G_1+G_2}(v)}{d_{G_1+G_2}(u,v)} \\ &= \frac{1}{2}\Big(\sum_{uv \in E(G_1)} (d_{G_1}(u) + m)(d_{G_1}(v) + m) \\ &\quad + \frac{1}{2} \sum_{uv \notin E(G_1)} (d_{G_1}(u) + m)(d_{G_1}(v) + m) \\ &\quad + \sum_{uv \in E(G_2)} (d_{G_2}(u) + n)(d_{G_2}(v) + n) \\ &\quad + \frac{1}{2} \sum_{uv \notin E(G_2)} (d_{G_2}(u) + n)(d_{G_2}(v) + n) \\ &\quad + \sum_{u \in V(G_1),\ v \in V(G_2)} (d_{G_1}(u) + m)(d_{G_2}(v) + n)\Big) \\ &= M_2(G_1) + M_2(G_2) + mM_1(G_1) + nM_1(G_2) + \frac{1}{2}\Big(\overline{M}_2(G_1) + \overline{M}_2(G_2)\Big) \\ &\quad + \frac{m}{2}\overline{M}_1(G_1) + \frac{n}{2}\overline{M}_1(G_2) + 4pq + 2mn(p+q) + \frac{1}{2}(m^2 p + n^2 q) \\ &\quad + \frac{mn}{4}(6mn - m - n). \end{aligned}$$

□

Using Theorem 2.1, we have the following corollaries.

Corollary 2.2. *Let G be graph on n vertices and p edges. Then* $H_M(G+K_m) = M_2(G) + mM_1(G) + \frac{1}{2}\Big(\overline{M}_2(G) + m\overline{M}_1(G)\Big) + \frac{mp}{2}(4n+m) + \frac{mn}{4}(6mn - m - n) + \frac{1}{2}m(m-1)(n^2 + m^2 + 4p - 2n - 2m + 4mn + 1)$.

Let $K_{n,m}$ be the bipartite graph with two partitions having n and m vertices. Note that $K_{n,m} = \overline{K}_n + \overline{K}_m$.

Corollary 2.3. $H_M(K_{n,m}) = H_M(\overline{K}_n + \overline{K}_m) = \frac{nm}{4}(6nm - n - m)$.

One can observe that $M_1(C_n) = 4n,\ n \geq 3,\ M_1(P_1) = 0,\ M_1(P_n) = 4n - 6,\ n > 1$ and $M_1(K_n) = n(n-1)^2$. Similarly, $\overline{M_1}(K_n) = 0$, $\overline{M_1}(P_n) = 2(n-2)^2$

and $\overline{M_1}(C_n) = 2n(n-3)$. By direct calculations we obtain the second Zagreb indices and coindices of P_n and C_n. $M_2(P_n) = 4(n-2)$, $M_2(C_n) = 4n$, $\overline{M}_2(P_n) = 2n^2 - 10n + 13$, and $\overline{M}_2(C_n) = 2n(n-3)$.

Using Corollary 2.2, we compute the formulae for reciprocal degree distance of star, fan and wheel graphs, $K_1 + \overline{K}_m$,$P_n + K_1$ and $C_n + K_1$, see Fig.2.

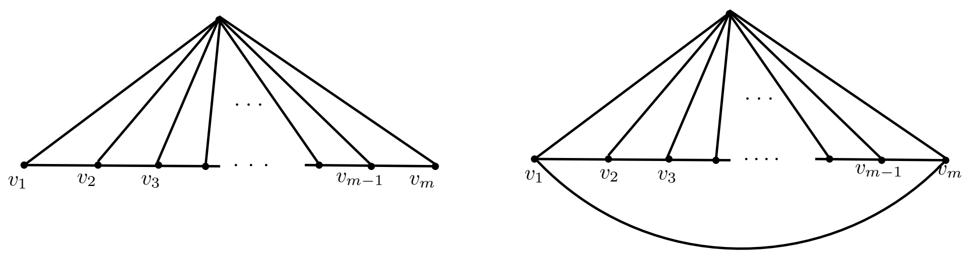

Fig. 2 Fan graph and wheel graph

Example 1.
(*i*) $H_M(K_1 + \overline{K}_m) = \frac{m(5m-1)}{4}$.
(*ii*) $H_M(P_n + K_1) = \frac{1}{4}(21n^2 - 11n - 16)$.
(*iii*) $H_M(C_n + K_1) = \frac{1}{4}n(21n^2 + 9)$.

3. Multiplicatively weighted Harary index of tensor product of graphs

In this section, we compute the Multiplicatively weighted Harary index of $G \times K_r$.

The proof of the following lemma follows easily from the properties and structure of $G \times K_r$. The lemma is used in the proof of the main theorem of this section.

Lemma 3.1. *Let G be a connected graph on $n \geq 2$ vertices. For any pair of vertices x_{ij}, $x_{kp} \in V(G \times K_r)$, $r \geq 3$, $i, k \in \{1, 2, \ldots, n\}$ $j, p \in \{1, 2, \ldots, r\}$. Then*
(*i*) *If $u_i u_k \in E(G)$, then*

$$d_{G\times K_r}(x_{ij}, x_{kp}) = \begin{cases} 1, & \text{if } j \neq p, \\ 2, & \text{if } j = p \text{ and } u_i u_k \text{ is on a triangle of } G, \\ 3, & \text{if } j = p \text{ and } u_i u_k \text{ is not on a triangle of } G. \end{cases}$$

(*ii*) *If $u_i u_k \notin E(G)$, then $d_{G\times K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.*
(*iii*) *$d_{G\times K_r}(x_{ij}, x_{ip}) = 2$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \times K_r$. We only prove the case when $u_i u_k \notin E(G)$, $i \neq k$ and $j = p$. The proofs for other cases are similar.

We may assume $j = 1$. Let $P = u_i u_{s_1} u_{s_2} \dots u_{s_p} u_k$ be the shortest path of length $p+1$ between u_i and u_k in G. From P we have a (x_{i1}, x_{k1})-path $P_1 = x_{i1}x_{s_1 2} \dots x_{s_{p-1}2}x_{s_p 3}x_{k1}$ if the length of P is odd, and $P_1 = x_{i1}x_{s_1 2} \dots x_{s_{p-1}2}x_{s_p 2}x_{k1}$ if the length of P is even.

Obviously, the length of P_1 is $p+1$, and thus $d_{G\times K_r}(x_{i1}, x_{k1}) \leq p+1 \leq d_G(u_i, u_k)$. If there were a (x_{i1}, x_{k1})-path in $G \times K_r$ that is shorter than $p+1$ then it is easy to find a (u_i, u_k)-path in G that is also shorter than $p+1$ in contrast to $d_G(u_i, u_k) = p+1$. □

Theorem 3.2. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then* $H_M(G \times K_r) = r(r-1)^2\Big(rH_M(G) + \frac{1}{4}(r-1)M_1(G) - \frac{1}{2}M_2(G) - \frac{1}{12}\sum\limits_{u_iu_k \in E_2} d_G(u_i)d_G(u_k)\Big)$, *where $r \geq 3$.*

Proof. Set $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(K_r) = \{v_1, v_2, \dots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \times K_r$. The degree of the vertex x_{ij} in $G \times K_r$ is $d_G(u_i)d_{K_r}(v_j)$, that is $d_{G\times K_r}(x_{ij}) = (r-1)d_G(u_i)$. By the definition of multiplicatively weighted Harary index

$$\begin{aligned}
H_M(G \times K_r) &= \frac{1}{2} \sum_{x_{ij},\, x_{kp} \in V(G\times K_r)} \frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij}, x_{kp})} \\
&= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij}, x_{ip})} \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \sum_{j=0}^{r-1} \frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij}, x_{kj})} \\
&\quad + \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij}, x_{kp})} \Bigg) \\
&= \frac{1}{2}\{A_1 + A_2 + A_3\}, \qquad (1)
\end{aligned}$$

where A_1 to A_3 are the sums of the above terms, in order.
We shall calculate A_1 to A_3 of (1) separately.

($\mathbf{A_1}$) First we compute $\sum\limits_{i=0}^{n-1} \sum\limits_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij}, x_{ip})}$.

$$\begin{aligned}
\sum_{i=0}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij}, x_{ip})} &= \sum_{i=0}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{(r-1)^2 d_G(u_i)^2}{2}, \text{ by Lemma 3.1} \\
&= \frac{1}{2} r(r-1)^3 M_1(G). \qquad (2)
\end{aligned}$$

($\mathbf{A_2}$) Next we compute $\sum\limits_{j=0}^{r-1}\sum\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}$.

Let $E_1=\{uv\in E(G)\,|\,uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2=E(G)-E_1$.

$$
\begin{aligned}
&\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}\\
=\;&\Big(\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1}+\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1}+\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1}\Big)\Big(\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}\Big)\\
=\;&\Bigg(\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1}\frac{(r-1)^2d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)}+\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1}\frac{(r-1)^2d_G(u_i)d_G(u_k)}{2}\\
&+\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1}\frac{(r-1)^2d_G(u_i)d_G(u_k)}{3}\Bigg),\text{ by Lemma 3.1}\\
=\;&(r-1)^2\Bigg\{\Bigg(\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)}+\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)}\\
&+\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)}\Bigg)-\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{2}-2\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{3}\Bigg\}\\
=\;&(r-1)^2\Bigg\{2H_M(G)-\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E(G)}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{2}-\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{6}\Bigg\}\\
=\;&(r-1)^2\Bigg\{2H_M(G)-M_2(G)-\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{6}\Bigg\},
\end{aligned}
\tag{3}
$$

Now summing (3) over $j=0,1,\ldots,r-1$, we get,

$$
\begin{aligned}
&\sum_{j=0}^{r-1}\Bigg(\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}\Bigg)\\
=\;&r(r-1)^2\Bigg\{2H_M(G)-M_2(G)-\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{6}\Bigg\}.
\end{aligned}
\tag{4}
$$

($\mathbf{A_3}$) Next we compute $\sum\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\left(\sum\limits_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij},x_{kp})}\right)$.

$$\begin{aligned}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kp}))}{d_{G\times K_r}(x_{ij},x_{kp}} &= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\frac{(r-1)^2 d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)},\\ &\quad \text{by Lemma 3.1}\\ &= r(r-1)^3\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)}\\ &= 2r(r-1)^3 H_M(G). \qquad (5)\end{aligned}$$

Using (1) and the sums $\mathbf{A_1}$,$\mathbf{A_2}$ and $\mathbf{A_3}$ in (2),(4) and (5), respectively, we have,

$$\begin{aligned}H_M(G\times K_r) &= r(r-1)^2\Big(rH_M(G)+\frac{1}{4}(r-1)M_1(G)-\frac{1}{2}M_2(G)\\ &\quad -\frac{1}{12}\sum_{u_iu_k\in E_2} d_G(u_i)d_G(u_k)\Big).\end{aligned}$$

□

Using Theorem 3.2, we have the following corollaries.

Corollary 3.3. *Let G be a connected graph on $n\geq 2$ vertices with m edges. If each edge of G is on a C_3, then $H_M(G\times K_r)=r(r-1)^2\Big(rH_M(G)+\frac{1}{4}(r-1)M_1(G)-\frac{1}{2}M_2(G)\Big)$, where $r\geq 3$.*

For a triangle free graph $\sum\limits_{u_iu_k\in E_2} d_G(u_i)d_G(u_k)=M_2(G)$.

Corollary 3.4. *If G is a connected triangle free graph on $n\geq 2$ vertices and m edges, then $H_M(G\times K_r)=r(r-1)^2\Big(rH_M(G)+\frac{1}{4}(r-1)M_1(G)-\frac{2}{3}M_2(G)\Big)$.*

By direct calculations we obtain expressions for the values of the Harary indices of K_n and C_n. $H(K_n)=\frac{n(n-1)}{2}$ and $H(C_n)=n\Big(\sum\limits_{i=1}^{\frac{n}{2}}\frac{1}{i}\Big)-1$ when n is even, and $n\Big(\sum\limits_{i=1}^{\frac{n-1}{2}}\frac{1}{i}\Big)$ otherwise. Similarly, $H_M(K_n)=\frac{n(n-1)^3}{2}$,$RDD(K_n)=n(n-1)^2$ and $H_M(C_n)=RDD(C_n)=4H(C_n)$.

Using Corollaries 3.3 and 3.4, we obtain the multiplicatively weighted Harary indices of the graphs $K_n\times K_r$ and $C_n\times K_r$.

Example 2. (i) $H_M(K_n \times K_r) = \frac{nr}{12}(n-1)^2(r-1)^2(6nr-4n-3r+1)$.

(ii) $H_M(C_n \times K_r) = \begin{cases} r(r-1)^2\Big(4rH(C_n)+n(r-3)\Big), & \text{if } n=3, \\ r(r-1)^2\Big(4rH(C_n)+\frac{n}{3}(3r-11)\Big), & \text{if } n>3. \end{cases}$

4. Multiplicatively weighted Harary index of strong product of graphs

In this section, we obtain the multiplicatively weighted Harary index of $G \boxtimes K_r$.

Theorem 4.1. *Let G be a connected graph with n vertices and m edges. Then* $H_M(G \boxtimes K_r) = r\Big(r^3 H_M(G) + r^2(r-1)RDD(G) + r(r-1)^2 H(G) + \frac{1}{2}r^2(r-1)^2 M_1(G) + \frac{1}{2}(r-1)^3 n + 2r(r-1)^2 m\Big)$.

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \boxtimes K_r$. The degree of the vertex x_{ij} in $G \boxtimes K_r$ is $d_G(u_i) + d_{K_r}(v_j) + d_G(u_i)d_{K_r}(v_j)$, that is $d_{G\boxtimes K_r}(x_{ij}) = rd_G(u_i) + (r-1)$. One can see that for any pair of vertices $x_{ij},\ x_{kp} \in V(G \boxtimes K_r)$, $d_{G\boxtimes K_r}(x_{ij}, x_{ip}) = 1$ and $d_{G\boxtimes K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.

$$\begin{aligned} H_M(G \boxtimes K_r) &= \frac{1}{2} \sum_{x_{ij},\, x_{kp} \in V(G\boxtimes K_r)} \frac{d_{G\boxtimes K_r}(x_{ij}) d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})} \\ &= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,\, p=0 \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij}, x_{ip})} \\ &\quad + \sum_{\substack{i,\, k=0 \\ i\neq k}}^{n-1} \sum_{j=0}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij}, x_{kj})} \\ &\quad + \sum_{\substack{i,\, k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,\, p=0 \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})} \Bigg) \\ &= \frac{1}{2}\{A_1 + A_2 + A_3\}, \end{aligned} \tag{6}$$

where A_1, A_2 and A_3 are the sums of the terms of the above expression, in order. We shall obtain A_1 to A_3 of (6), separately.

$$A_1 = \sum_{i=0}^{n-1} \sum_{\substack{j,\, p=0 \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij}, x_{ip})}$$

$$= \sum_{i=0}^{n-1} \sum_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \Big(rd_G(u_i) + (r-1))(rd_G(u_i) + (r-1)\Big)$$

$$= r(r-1)\Big(n(r-1)^2 + 4mr(r-1) + r^2 M_1(G)\Big). \qquad (7)$$

$$\begin{aligned} A_2 &= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{d_{G\boxtimes K_r}(x_{ij}) d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij}, x_{kj})} \\ &= \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{((r-1) + rd_G(u_i))((r-1) + rd_G(u_k))}{d_G(u_i, u_k)} \\ &= r^2 \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{d_G(u_i) d_G(u_k)}{d_G(u_i, u_k)} + \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{r(r-1)(d_G(u_i) + d_G(u_k))}{d_G(u_i, u_k)} \\ &\quad + \sum_{j=0}^{r-1} \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{(r-1)^2}{d_G(u_i, u_k)} \\ &= r\Big(2r^2 H_M(G) + 2r(r-1)RDD(G) + 2(r-1)^2 H(G)\Big). \qquad (8) \end{aligned}$$

$$\begin{aligned} A_3 &= \sum_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,\,p=0, \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})} \\ &= r(r-1)\Big(2r^2 H_M(G) + 2r(r-1)RDD(G) + 2(r-1)^2 H(G)\Big). \qquad (9) \end{aligned}$$

Using (7), (8) and (9) in (6), we have

$$\begin{aligned} H_M(G \boxtimes K_r) &= r^2\Big(r^2 H_M(G) + r(r-1)RDD(G) + (r-1)^2 H(G)\Big) \\ &\quad + r(r-1)\Big(\frac{r^2 M_1(G)}{2} + \frac{n(r-1)^2}{2} + 2mr(r-1)\Big). \end{aligned}$$

□

Using Theorem 4.1, we obtain the following corollary.

Corollary 4.2. $H_M(C_n \boxtimes K_r) = (9r^2 - 6r + 1)\Big(r^2 H(C_n) + \frac{nr(r-1)}{2}\Big)$.

As an application we present formula for multiplicatively weighted Harary index of closed fence graph, $C_n \boxtimes K_2$, see Fig. 3.

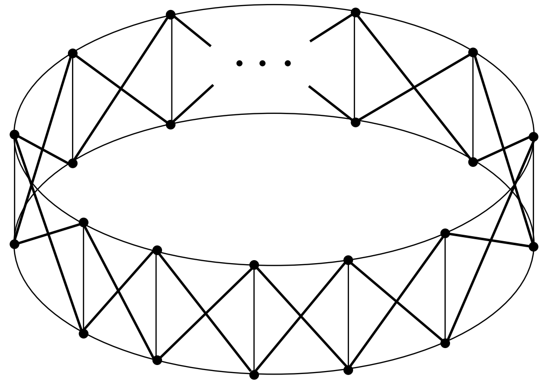

Fig. 3. Closed fence graph

Example 3. By Corolarry 4.2, we have

$$H_M(C_n \boxtimes K_2) = \begin{cases} 25\Big(n + n\sum_{i=1}^{\frac{n}{2}} \frac{1}{i} - 1\Big), \text{ if } n \text{ is even} \\ 25n\Big(1 + \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big), \text{ if } n \text{ is odd.} \end{cases}$$

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